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On the relation between control systems and Lie systems

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In honor of José F. Cariñena for his 60th birthday. Thanks a lot Pepín!

1 Introduction

The aim of this paper is to analyze the proper relation existing between two types of dynamical systems: affine control systems and Lie systems, which arise in similar contexts and are receiving quite a lot of attention in the last years.

Lie systems are a special type of time dependent ordinary differential equations which enjoy very interesting properties . In particular, their solutions admit a nonlinear superposition principle, which makes them particularly interesting in many different areas [1, 2, 3, 4, 5, 7, 10, 11, 12].

On the other hand, control systems are used in almost any possible dynamical system whose dynamics can be externally driven, from spacecrafts to chemical plants. They have been studied extensively since the fifties and in particular the geometrical structures related to them have been object of deep research in the last twenty years. Our goal in this paper is to relate the time dependent systems of differential equations corresponding to Lie systems and control systems.

In the process, we will also study how is it possible to formulate the typical notion of feedback transformations from Control Theory in a context which allows a simple translation into the language of Lie systems. This allows us to generalize the concept of feedback linearizable systems and feedback nilpotentiable systems and introduce the notion of feedback transformations into system of Lie type.

2 Relation between control systems and Lie systems

The aim of this section is to prove the relationship between some control dynamical systems and Lie systems as introduced in [5, 7, 8, 10, 9, 13]. To begin with, let us characterize both systems independently, and proceed to link both concepts step by step.

2.1 Lie systems

Lie systems are, roughly speaking, systems of time dependent differential equations which are generated by vector fields which generate a finite dimensional Lie algebra. One of the main properties of such systems of equations is that the solutions admit a non-linear superposition principle, i.e. given two solutions, a certain function of them (in general a non-linear function) will define a new solution of the system.

In a more precise way, we can consider the following result:

Theorem 2.1 *Given a non-autonomous system of n first order differential equations, a necessary and sufficient condition for the existence of a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ such that the general solution is $x = \Phi(x_{(1)}, \ldots, x_{(m)}; k_1, \ldots, k_n)$, with $\{x_{(a)} \mid a = 1, \ldots, m\}$ being a set of particular solutions of the system and $k_1, \ldots, k_n$, n arbitrary constants,is that the system can be written as*

$$\frac{dx^i}{dt} = b_1(t)\xi^{1i}(x) + \cdots + b_r(t)\xi^{ri}(x)\,, \qquad i = 1, \ldots, n\,, \tag{1}$$

where $b_1, \ldots, b_r$, are r functions depending only on t, and $\xi^{\alpha i}$, $\alpha = 1, \ldots, r$, are functions of $x = (x^1, \ldots, x^n)$, such that the r vector fields in $\mathbb{R}^n$ given by

$$Y^{(\alpha)} \equiv \sum_{i=1}^{n} \xi^{\alpha i}(x^1, \ldots, x^n)\frac{\partial}{\partial} x^i\,, \qquad \alpha = 1, \ldots, r, \tag{2}$$

close on a real finite–dimensional Lie algebra, i.e. $\{Y^{(\alpha)}\}$ are linearly independent and there are r^3 real numbers, $f^{\alpha\beta}{}_{\gamma}$, such that

$$[Y^{(\alpha)}, Y^{(\beta)}] = \sum_{\gamma=1}^{r} f^{\alpha\beta}{}_{\gamma} Y^{(\gamma)}\ .$$

The number r satisfies $r \leq mn$.

Proof. For a geometric proof of this Theorem, see e.e. [5]. □

In this paper we shall focus mostly on the explicit time dependence of the dynamical system, and postpone for later the analysis of the Lie algebra condition.

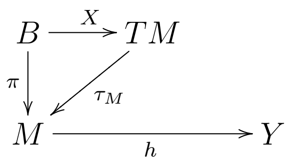
2.2 Control systems

On the other hand, control systems are dynamical systems which depend on a set of external parameters, which can be modified at will, aiming to force the system to behave in some desired way [21, 18]. In mathematical terms, we can consider the following definition:

Definition 2.1 *Consider a differentiable manifold M and a system defined on it. Roughly speaking, a control systems is a dynamical system which depends on a set of arbitrary control functions, u_i, which allow us to modify the behavior of the system. From a geometric point of view, we can describe the set of controls as a bundle B (usually a trivial vector bundle) on the state space manifold M, with a projection $\pi_B : B \to M$. The control dynamical system corresponds then to the integral curves of a vector field along the projection π_B,*

The system provides us with some information which is defined as a mapping between the manifold M and some manifold Y which contains the **outputs** *of the system.*

From a geometric point of view, control systems correspond to vector fields along the projection of a vector bundle



which in local coordinates reads:

$$\dot{x} = X(x, u)\,, \quad x \in M\,, \quad u \in B_x \quad y = h(x) \tag{3}$$

A special class of these systems are the control systems affine in the controls. Consider a basis of sections of the bundle B, $\{e^i\}_{i=1,\cdots,q}$ and a corresponding set of coordinates $\{u_i\}$ for the fiber of the bundle. We will assume, for simplicity, that the bundle is trivial, i.e. we can write $B = M \times U$ where U is a vector space. Assuming these coordinates for the control functions, we can write a general affine control system in the form:

$$\dot{x}(t) = X_d(x(t)) + \sum_{i=1}^{r} u_i(t)\, X_i(x(t)) \tag{4}$$

The control problem consists in finding a curve $(x(t), u(t))$ in the bundle B whose projection on M ($\pi_B(x(t), u(t) = x(t))$ is an integral curve of (4). The usual procedure consists in finding a suitable section $\sigma_B : M \to B$ in order to fulfill some requirement (to make a certain point x^* to be a stable fixed point, to force the system to follow a certain curve as its trajectory, etc).

Very interesting structural properties of the system are encoded in:

- The distribution $\mathfrak{C}$ spanned by $\{X_f, X_g^1, \cdots X_b^p\}$. It represent the set of directions of motion which can be imposed on the system.

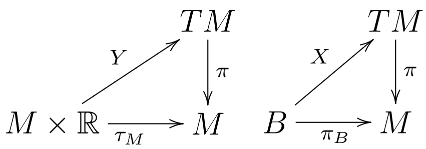
- The codistribution $\mathfrak{O}$ spanned by the set of one forms $\{dL_{X_g^1} \cdots L_{X_g^p} h_1, \cdots\}$. Loosely speaking, it represents the set points in M which can be distinguished by looking at the output manifold.

There are two different regimes in a control system in what regards the functional dependence of the control functions:

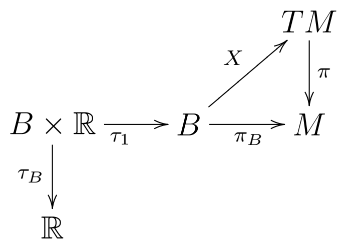
- Open loop: in this case, the control functions are supposed to be a function of the time, only, without explicit dependence on the point of M. This will be the case we study first.
- A second possibility consists in assuming a dependence of the control function in both the time and the point in M. Such a situation, usually called **feedback regime** will be analyzed later.

2.3 Comparing the time dependence structure of Lie systems and open-loop control systems

The open loop case exhibits of a control system exhibits a clear similarity with the structure of a Lie system. The explicit time dependence of the Lie case is defined through the control functions in the control one. In more detailed terms, let us recall the geometric definition we did of both structures:



Consider a trivial generalization of the control system in such a way that solutions are easily considered as suitable sections:

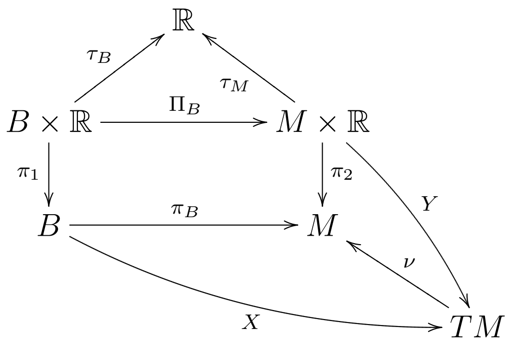


Solutions of a control systems will be defined by trivial sections of τ_B defined as

$$\gamma_B(t) = (x(t), u(t), t)$$

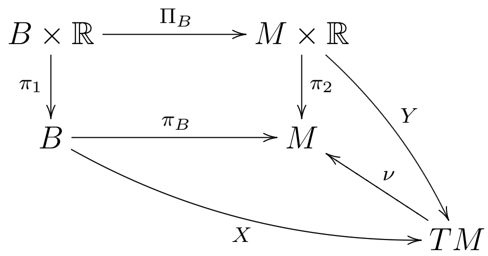
which render commutative the diagram above.

Comparing this diagram with the diagram expression of a Lie system (without the Lie algebra condition), it seems natural to consider the following extensions, by trivially considering a time extension in the bundle structure of the control bundle $\Pi_B = \pi_B \times \mathrm{Id} : B \times \mathbb{R} \to M \times \mathbb{R}$:



Please realize that we have just juxtaposed the three diagrams we had in the beginning, the one defining the control system, the same one extended to Π_B and finally the diagram defining the Lie system. From a mere inspection, it is simple to conclude that there is a natural relation between control vector fields and Lie systems:

Definition 2.2 *Consider a control system X defined on a manifold M as above. Then, we say that a Lie system Y is* **related to** *X if the following diagram commutes*



This implies that:

$$X \circ \pi_1 = Y \circ \Pi_B \tag{5}$$

Proposition 2.1 *Given a control system X and a section $\sigma_B : M \to B$, there exists a Lie system related to the control system and defined as:*

$$Y = X \circ \sigma_B \circ \pi_2. \tag{6}$$

PROOF. Trivial. □

It is important to realize that this is the usual way of solving a control system in open loop: we are replacing the control functions in the control system by some functions of time $\{u_i(t)\}_{i=1,\dots}$.

But then the problem of unicity arises naturally: consider we take two different sections of the control bundle σ_B^1 and σ_B^2. The construction above defines two different vector fields $Y_1 = X \circ \sigma_B^1 \circ \pi_2 \quad Y_2 = X \circ \sigma_B^2 \circ \pi_2$. The choice of the section implies a change in the Lie system the control is associated to. We shall see in the next section how this issue can be explained within the feedback transformation concept for certain cases.

Solutions to the control system are sections of the bundle τ_B which make the "left" part of the diagram commutative, while solutions to the Lie system are sections of the bundle τ_M which make the "right" part of the diagram commutative.

Assume now that the section σ_B is know, and we can then determine the suitable control function for any point $x \in M$. In such a case, the following result is trivial to prove:

Theorem 2.2 *Consider a control and a Lie system related as above. Then, any solution $\gamma_B(t)$ of the control system X determines a solution $\gamma_M(t) = \Pi_B(\gamma_B(t))$ of the Lie system Y.*

3 The Lie algebra condition

The following step is to consider the second condition of a Lie system: the existence of a set of vector fields which allow to write the differential equation as (1) and that generate a finite dimensional Lie algebra. Our task now is to translate this condition to the language of control systems.

As we saw above, given a control system X any choice of a section $\sigma_B : M \to B$ determines a differential equation which has the same time dependence structure as a Lie system. We are going to study now the conditions to be satisfied by X and the section σ_B for the resulting vector field Y to be a true Lie system. For simplicity, let us consider the case of affine control systems whose geometric structure has been described in the previous section. As we know that the set of vector fields $\{Y^\alpha\}$ of the Lie system associated to the affine control system X corresponds to $\{X_f, X_1, \cdots, X_r\}$, the Lie algebra condition of the Lie system is translated into:

Proposition 3.1 *Consider an affine control system X, an arbitrary section σ_B of the bundle π_B and the associated field $Y = X \circ \sigma_B \circ \pi_2$. Then, Y is a Lie system if the algebra $\mathfrak{C}$ corresponding to X is a real finite dimensional Lie algebra.*

Using the framework presented above, we are going to introduce now the concept which links Lie and control systems:

Definition 3.1 *Let X be a control system defined as above and $\sigma_B : M \to B$ a section of the control bundle. We say that (X, σ_B) is a* **control Lie system** *or a* **control system of Lie type** *if the associated time dependent vector field $Y = X \circ \sigma_B \circ \pi_2$ is a Lie system.*

4 Transforming systems: feedback transformations

4.1 Geometric definition of feedback

Feedback transformations have been used in many different contexts in Control Theory [21]. Roughly speaking, a feedback transformation maps a given system into a new one for which the control functions are obtained from the old ones via a linear transformation (in the controls), which depends on the state space coordinates. In principle, the control functions $\{u_i\}$ are supposed to depend on time (in the control theory terminology the system is then operating in *open loop*). It may happen, though, that in order to change the behaviour of the system it is necessary to make the controller depend on the state space coordinates, via a feedback transformation: $u(t,x) = \beta(x)v(t)$. This is the case of the phenomenon called **static feedback**. We can also consider a transformation which depends of a new dynamical system, this will be the case of the **dynamic feedback**. In both cases, we are actually defining a new control system, with control functions $\{v\}$ and whose behaviour can be completely different from the original one.

4.1.1 Static feedback

The geometrical interpretation of this phenomenon is simple. Consider again the description of an affine control system that we saw above. Consider the control bundle $B \to M$ and a bundle isomorphism $\hat{\beta} : B \to B$.

The effect that this transformation has on the control system (4) can be written as follows. The new system reads, in local coordinates:

$$\dot{x} = X_d(x) + \sum_{i=1}^{r} \beta^i(x,u) X_i(x) \tag{7}$$

Assuming that we are considering affine control systems, we ask the mapping β to be affine in the fiber, i.e.:

$$\beta^i(x,u) = \beta_1(x) + \beta_2(x)u \equiv v \in B_x \tag{8}$$

v is again an element of the fiber of the control bundle and is assumed to be the control function for the transformed system.

A very simple case for this transformation would be the following. A change on the basis of sections $\{e^i\}$ which corresponds to the set of coordinates $\{u_i\}$ leads to a transformation at the level of coordinates of the fiber in the form $u(t,x) = \beta(x)v(t) \qquad \beta(x) \in G$ where G is the structural group of the bundle B. Hence, we can interpret the feedback transformation as a change of the basis of sections of B, if the transformation β belongs to the structural group of the bundle. If the bundle structure of B is not determined by external considerations, this condition can always be fulfilled.

We may also think in a slightly more general situation where the bundle isomorphism on the control bundle B is not defined on the identity mapping on the base, but on a general diffeomorphism, i.e.:

$$\begin{array}{ccc} B & \xrightarrow{\beta} & B \\ \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \pi_B} \\ M & \xrightarrow[\psi]{} & M \end{array} \tag{9}$$

In local coordinates, this implies that the original system:

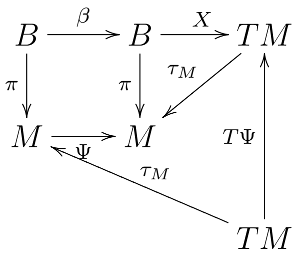
$$\begin{cases} \dot{x}(t) = X_d(x(t)) + \sum_i u_i(t) X_i(x(t)) \\ x(0) = x_0 \end{cases}$$

is transformed into:

$$\begin{cases} \dot{z}(t) = \psi_*(X_d)(z(t)) + \sum_i \beta(x(t), u(t)) \psi_*(X_i)(z(t)) \\ z(0) = \psi(x(0)) \end{cases} \tag{10}$$

This is the framework presented in [19].

From a geometric point of view, the pair of mappings (β, Ψ) does modify the vector field X, which becomes a new one denoted as $X^\beta = X \circ \beta$. It corresponds to the following diagram:



Hence, the resulting system is a vector field $X^\beta = T\psi^{-1}(X \circ \beta)$ along the projection $\pi^\Psi = \Psi \circ \pi_B$.

We may even consider a more drastic transformation, mapping the original state-space manifold onto a new one:

$$\Psi : M \to M'$$

In such a case, the vector field is transformed in the usual way.

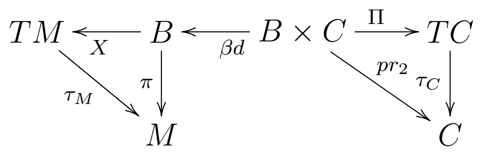
In any of these situations, the distribution $\mathfrak{C}$ generated by the control vector fields of the system does change in the transformation. We will denote as $\beta_{\mathfrak{C}}$ the corresponding transformation, i.e.

$$\mathfrak{C} \to \beta_{\mathfrak{C}} \mathfrak{C}$$

where we denote as $\mathfrak{C}^\beta$ the resulting distribution.

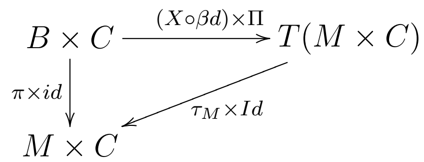
4.1.2 Dynamic feedback

For completeness, we can consider also in this framework the case of dynamic feedback. This case is slightly more involved than the previous one but still simple to describe. Consider then the control bundle $B \to M$ as before and a transformation defined as



We add then a new manifold and a new dynamical system (defined by Π, which is a vector field along the natural projection $pr_2 : B \times C \to C$) whose solutions determine (dynamically, hence the name) the feedback transformation of our original system.

We can combine both objects in the expression



The transformation of the controllability and observability distributions is similar to the static case. We will denote as $\beta d_{\mathfrak{C}}$ the corresponding transformation, i.e.

$$\mathfrak{C} \to \beta d_{\mathfrak{C}} \mathfrak{C}$$

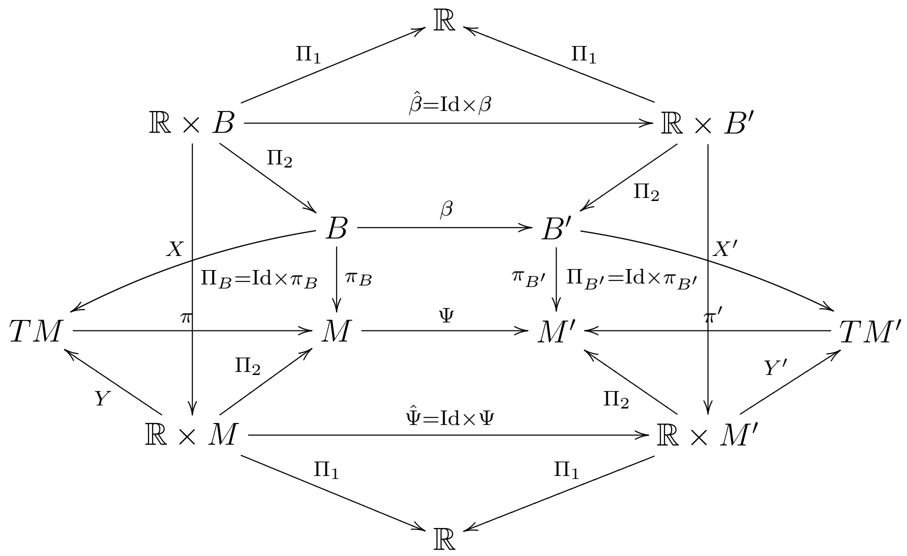
where we denote as $\mathfrak{C}^{\beta d}$ the resulting distribution.

4.1.3 The effect in our case

The aim of feedback transformations is to suitably modify the structure of the original control system in order to force it to fulfill some given requirement (to have a fixed point with suitable stability properties, for instance).

But it is also a fair question to study whether, given a control system which is not of Lie type, it is possible to transform it into a Lie system via a feedback transformation.

How does this look like on the complete setting? Let us restrict ourselves to the case of static feedback for simplicity, the dynamic case being very similar in nature. The complete picture is as depicted in the following commutative diagram:



The diagram above reflects the construction we have presented above in full generality. Of course, to consider the effect of feedback transformations on the corresponding Lie systems it is necessary to consider sections $\sigma_B : M \to B$ (and $\sigma_{B'} : M' \to B'$ via the feedback).

As we know the geometrical structure that the system must exhibit to be of Lie type, it seems reasonably simple to derive, formally, the conditions to be asked to the transformations. Please notice that this property is different from usual situations, where the aim is to obtain a linearized system, but it is similar to the concept introduced by Hermes and coworkers about feedback nilpotentiation [15]. In this last case, the aim is to obtain a Lie system corresponding to a nilpotent group:

4.2 A precedent: Hermes' nilpotentiable systems

In [15] the concept of feedback nilpotentiable systems is introduced in the following way:

Definition 4.1 *Given an affine control system:*

$$\dot{x} = \sum_k u_k(t) X_k \tag{11}$$

we say that it is nilpotentiable by feedback if we can find a feedback transformation: $u(t) = \beta_{ij}(t)v(t)$ *such that (11), once transformed, becomes a nilpotent system, i.e. a control systems whose control vector fields (completed, if needed, with some others obtained by commutation of these) span a nilpotent Lie algebra.*

In [15] the study of necessary properties for systems as (11) (and also with a drift term) defined on real analytic manifolds to be feedback equivalent to a nilpotent system is carried out. The main issue of that study is the possibility of defining the so called

nilpotent basis for the given control system, what in our language simply means to find out which is the action of the nilpotent group which defines the vector fields $\{X_k\}$ as fundamental. In that case, we know that we can transfer the study to the group, or following [15], to the algebra. The conclusions obtained are not many, since the approach is very general; hence, the only practical result contained in the paper is the following:

Theorem 4.1 *Let $D(x) = \mathrm{span}(Y_1(x), Y_2(x))$ be a two dimensional distribution on $\mathbb{R}^3$ and suppose that $Y_1(0)$, $Y_2(0)$ and $[Y_1, Y_2](0)$ are independent. Then D admits a nilpotent basis which generates a three dimensional nilpotent algebra.*

The interest of this type of results in our study is evident, since it is well known the Wei-Norman method [22, 23] allows us to integrate by quadratures the nilpotent Lie systems. We conclude thus:

Corollary 4.1 *Let $D(x) = \mathrm{span}(Y_1(x), Y_2(x))$ be a two dimensional distribution on $\mathbb{R}^3$ and suppose that $Y_1(0)$, $Y_2(0)$ and $[Y_1, Y_2](0)$ are independent. Then the corresponding control system is integrable by quadratures in a neighbourhood of $x = 0$.*

4.3 The general case: systems which are feedback equivalent to general Lie systems

Building on the previous example, it seems natural to define the analogous concept for general Lie systems: is it possible to consider a special type of feedback transformations which transforms a general affine control Lie system into a general Lie control system?

From the discussion above, it seems quite simple to obtain a formal condition for this:

Definition 4.2 *Let X be an affine control system defined as above and a section $\sigma_B : M \to B$. Then, the system is said to be* **Lie feedback equivalent** *if there exists a feedback transformation $\beta : B \to B$ such that the transformed distribution $\mathfrak{C}^\beta$ at every point $x \in M$ spans a finite dimensional Lie algebra with the natural restriction of the vector field commutator.*

The theorem presented above for nilpotentiable systems is clearly a particular case of the definition above for the case of a nilpotent group.

The best idea seems to be to look for a geometric object which would serve as a test of the finite dimensionality of the Lie algebra structure associated to the control distribution. Is there anything which tells us that the object generated is finite dimensional as a Lie algebra? From an algebraic point of view there is indeed: the lower central series, which should be stable for the Lie algebra to be finite dimensional.

Let us consider then the set

$$\{\beta_{1j}X^j, \cdots, \beta_{kj}X^j\}$$

as a family of generators. The lower central series associated to this set will read as:

$$\{\beta_{1j}X^j, \cdots, \beta_{kj}X^j, [\beta_{1j}X^j, \beta_{2j}X^j], [\beta_{1j}X^j, \beta_{3j}X^j], \cdots, \cdots\}$$

where every element can be considered to be a certain differential operator acting on the tensor β (although β is a tensor defined on the bundle B, when we consider the image under the control vector field X , it becomes a tensor on M).

The condition of stability of the series implies that there is a certain step at which the terms obtained are a linear combination of the elements of the family. Hence, a certain PDE arises for β, depending on the initial family of vector fields and the particular final Lie algebra.

Let us consider a particular example to clarify the concept.

Example 4.1 *Consider for instance a control system defined as follows:*

the following feedback transformation:

$$\Psi(x_1, x_2, x_3) = (x_1, x_2, x_3) \qquad \beta(b_1, b_2) = (\frac{1}{\cos^2 x_3}, \cos x_3) \tag{12}$$

Of course, this change is only valid for $\cos x_3 \neq 0$. *We consider then the chart for* S^1 *defined as* $x_3 \in (-\pi/2, \pi/2)$ *or* $x_3 \in (\pi/2, 3\pi/2)$.

If we denote as (b_1, b_2) *the original coordinates in the control bundle* B *and as* (c_1, c_2) *the transformed ones, we can write the transformation as:*

$$c_1 = \frac{b_1}{\cos^2 x_3} \qquad c_2 = b_2 \cos x_3 \tag{13}$$

Then, we have $\Psi = \mathrm{Id}$ *what simplifies the corresponding expressions. The new distribution* $\mathfrak{C}_X^\beta$ *is spanned by the control vector fields:*

$$X_1^\beta = \cos^2 x_3 \frac{\partial}{\partial x_3} \qquad X_2^\beta = \tan x_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$$

The new system becomes thus:

$$\dot{x}_1 = c_2(t) \tan x_3 \quad \dot{x}_2 = c_2(t) \quad \dot{x}_3 = c_1(t) \cos^2 x_3$$

Let us compute the distribution $\mathfrak{C}_X^\beta$. *Computing the commutator* $[X_1^\beta, X_2^\beta]$ *we obtain:* $[X_1^\beta, X_2^\beta] = \frac{\partial}{\partial x_1} \equiv X_3^\beta$ *We obtain then:* $[X_1^\beta, X_2^\beta] = X_3^\beta \quad [X_1^\beta, X_3^\beta] = 0 = [X_2^\beta, X_3^\beta]$ *Hence, the distribution* $\mathfrak{C}_X^\beta$ *is isomorphic to the Heisenberg algebra* $\mathfrak{h}(3)$, *what implies that the transformed system is also of Lie type, but now the group is the Heisenberg group* $H(3)$.

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