

Reduction and projection of Dirac structures

Iván Calvo and Fernando Falceto

Departamento de Física Teórica
Universidad de Zaragoza
E-50009 Zaragoza, Spain
e-mail: icalvo@unizar.es, falceto@unizar.es

Abstract

Given a Dirac subbundle and an isotropic subbundle, we provide a canonical method to obtain a new Dirac subbundle. When the original Dirac subbundle is Courant involutive, this construction has interesting applications, unifying and generalizing some results on the reduction of Dirac structures previously found in the literature.

1 Introduction

The structure underlying the reduction of Poisson or symplectic manifolds is a Poisson algebra. Such an algebraic structure can be encoded in geometric terms through the concept of *Dirac structure*, which generalizes the Poisson and presymplectic geometries by embedding them in the framework of the geometry of $TM \oplus T^*M$. Dirac structures were introduced in a remarkable paper by T. Courant [3]. Therein, they are related to the Marsden-Weinstein reduction ([6]) and to the Dirac bracket ([4]) on a submanifold of a Poisson manifold. Recently, Dirac subbundles have been considered in connection to the reduction of implicit Hamiltonian systems (see [2],[1]). This simple but powerful construction allows to deal with mechanical situations in which we have both gauge symmetries and Casimir functions.

Dirac structures are special types of Lie algebroids contained in the vector bundle $TM \oplus T^*M$. We shall use the notation $\rho_1 : TM \oplus T^*M \rightarrow TM$, $\rho_2 : TM \oplus T^*M \rightarrow T^*M$ for the canonical projections.

The *Courant bracket* on sections of $TM \oplus T^*M$ is defined by

$$[(X, \xi), (X', \xi')] = ([X, Y], \iota(X)d\xi' - \iota(X')d\xi + \frac{1}{2}d(\iota(X)\xi' - \iota(X')\xi)).$$

It is bilinear and antisymmetric but not a Lie bracket, since it does not satisfy the Jacobi identity in general.

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Consider the natural symmetric bilinear form on $TM \oplus T^*M$,

$$\langle (X, \xi), (X', \xi') \rangle = \iota(X)\xi' + \iota(X')\xi. \quad (1.1)$$

Following the seminal paper [3], we define a *Dirac subbundle* as a subbundle $D \subset TM \oplus T^*M$ maximally isotropic with respect to (1.1). Maximal isotropy implies that $D^\perp = D$, where D^\perp stands for the orthogonal subspace of D . In particular, $\dim(D) = \dim(M)$.

A *Dirac structure* is a Dirac subbundle D such that its sections close under the Courant bracket. In this case, as shown in [3], the restriction to D of the Courant bracket fulfills the Jacobi identity and D with anchor $\rho_1|_D$ is a Lie algebroid.

The two basic examples of Dirac structures are:

Example 1.1. For any 2-form ω , the graph L_ω of $\omega^\flat : TM \rightarrow T^*M$ is a Dirac subbundle such that $\rho_1(L_\omega) = TM$ at every point of M . L_ω is a Dirac structure if and only if $d\omega = 0$.

Example 1.2. Let Π be a bivector field on M . The graph L_Π of the map $\Pi^\sharp : T^*M \rightarrow TM$ is always a Dirac subbundle. In this case ρ_2 maps L_Π onto T^*M . L_Π is a Dirac structure if and only if Π is a Poisson structure (i.e. if the bracket $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ defined by Π satisfies the Jacobi identity).

The concept of symmetry of a Dirac structure is a generalization of the definitions in the presymplectic and Poisson cases. Namely, a vector field Y is a *symmetry of the Dirac structure* D if $(\mathcal{L}_Y X, \mathcal{L}_Y \xi) \in \Gamma(D)$ for every $(X, \xi) \in \Gamma(D)$, where \mathcal{L} stands for the Lie derivative.

In this note, our main objective is to find a common description of the already known reduction and projection procedures for Dirac structures. Actually, our original motivation was to generalize for any Dirac structure the Marsden-Ratiu reduction of Poisson manifolds [5], which we describe next.

Let (M, Π) be a Poisson manifold. Consider a submanifold N and a distribution $E \subset TM|_N$ such that:

- (i) $E \cap TN$ is integrable.
- (ii) The foliation defined by $E \cap TN$ is regular, so that the space of leaves $N/(E \cap TN)$ is a manifold.
- (iii) E leaves invariant the Poisson bracket in the sense that if $f, g \in C^\infty(M)$ are invariant along E , then $\{f, g\}$ is also invariant along E .

Under these assumptions, the space of leaves $N/(E \cap TN)$ inherits a Poisson structure from (M, Π) if and only if

$$\Pi^\sharp(E^0) \subset TN + E \quad (1.2)$$

where E^0 is the annihilator of E , i.e. $E_p^0 = \{\alpha \in T_p^*M \mid \alpha(v) = 0, \forall v \in E_p\}$.

An interesting application of this construction is obtained by choosing $E = \Pi^\sharp(TN^0)$ with $\Pi^\sharp(TN^0) \cap TN = \{0\}$. Then, the inherited Poisson structure corresponds to the well-known *Dirac bracket* on N . Another particular case that we will discuss in the sequel is that of $N = M$.

2 Stretched Dirac structures

Take a Dirac subbundle D and a subbundle $S \subset TM \oplus T^*M$ isotropic with respect to the symmetric pairing (1.1), i.e. $S \subset S^\perp$. It is not difficult to show that we can ‘stretch’ D along S and obtain another Dirac subbundle, namely

$$D^S := (D \cap S^\perp) + S$$

(assuming it is a subbundle). We must show that it is maximally isotropic, but this is immediate:

$$\begin{aligned} (D^S)^\perp &= (D^\perp + S) \cap S^\perp \\ &= (D \cap S^\perp) + S = D^S \end{aligned}$$

where in the last line we have used that D is maximally isotropic and S is a subset of S^\perp . In a sense this construction is canonical, for D^S is the Dirac subbundle closest to D among those containing S , as stated in the following

Theorem 2.1. *Let D, S and D^S be as above and let D' be a Dirac subbundle such that $S \subset D'$. Then, $D' \cap D \subset D^S \cap D$. In addition, $D' \cap D = D^S \cap D$ if and only if $D' = D^S$.*

Proof:

From the isotropy of D' and given that $S \subset D'$ we deduce that $D' \subset S^\perp$. Hence,

$$D' \cap D \subset S^\perp \cap D = D^S \cap D.$$

If the equality $D' \cap D = D^S \cap D$ holds, then $D' \supset D' \cap D = S^\perp \cap D$. Since $S \subset D'$, we find that $D^S = (D \cap S^\perp) + S \subset D'$. But D^S and D' have the same dimension, so that they are equal. \square

In general even when D^S is a smooth subbundle its sections do not close under the Courant bracket. We discuss now some particularly interesting examples and applications of this construction.

Dirac bracket (or Dirac Dirac structure): Consider an integrable distribution $\Phi \subset TM$ and take $S = \Phi^0 \subset T^*M$. Then, for any Dirac structure D on M , D^S (if

smooth) is a Dirac structure such that $\rho_1(D^S)$ is everywhere tangent to the foliation. That is,

$$(D^S)_p = \{(X_p, \xi_p + \nu_p) | (X_p, \xi_p) \in D_p, X_p \in \Phi_p, \nu_p \in S_p\}.$$

Let us see that sections of D^S close under the Courant bracket:

$$\begin{aligned} [(X, \xi + \nu), (X', \xi' + \nu')] &= \\ &= ([X, X'], \iota(X)d\xi' - \iota(X')d\xi + \frac{1}{2}d(\iota(X)\xi' - \iota(X')\xi)) + \\ &+ (0, \iota(X)d\nu' - \iota(X')d\nu). \end{aligned} \quad (2.1)$$

$(X, \xi), (X', \xi')$ are sections of the Dirac structure and then so it is the first line of the right hand side of (2.1). X, X' are vector fields in the integrable distribution Φ and then $[X, X']$ is also a vector field of Φ . Finally, we have to show that the third line of (2.1) is a section of Φ^0 . Contracting with a vector field Y in Φ we obtain:

$$\iota(Y)(\iota(X)d\nu') = X(\iota(Y)\nu') - Y(\iota(X)\nu') - \iota([X, Y])\nu' = 0$$

where we have used that ν' is a section of $S = \Phi^0$ and X, Y are sections of Φ , which is an integrable distribution. Therefore, we have proven that D^S is a Dirac structure on M (assuming it is a subbundle).

D^S can be considered as a generalization of the Dirac bracket extended to the whole manifold for the case of Dirac structures. Recall that the Dirac structure D^S is the graph of a bivector field (which is a Poisson structure due to involutivity) if and only if $\rho_1(D^S) = T^*M$ at every point of M . This is equivalent to

$$D^S + TM = TM \oplus T^*M. \quad (2.2)$$

Taking the orthogonal of (2.2) we get the more familiar (and equivalent) condition

$$(D + \Phi^0) \cap \Phi = \{0\}. \quad (2.3)$$

If this holds, D^S corresponds to a Poisson bracket which should be called Dirac bracket. Indeed, it coincides with the standard Dirac bracket if D itself comes from a Poisson structure Π . In this case (2.3) can be rewritten as $\Pi^\sharp(\Phi^0) \cap \Phi = \{0\}$ (see Section 1).

Once we have defined the Dirac structure D^S we can restrict it to any leaf N of the foliation induced by Φ . Let $i : N \rightarrow M$ be the inclusion. The image of the bundle map

$$i_*^{-1} \oplus i^* : D^S|_N \rightarrow TN \oplus T^*N$$

defines a Dirac structure D_N^S in $TN \oplus T^*N$. The isotropy of D_N^S is obviously inherited from the isotropy of D^S . Now, using that $\text{Ker}(i_*^{-1} \oplus i^*) = \Phi^0|_N$ and that $\dim(D_N^S) = \dim(D^S) - \dim(\text{Ker}(i_*^{-1} \oplus i^*))$ we deduce that D_N^S is maximally isotropic in $TN \oplus T^*N$. The

proof of the closedness of the Courant bracket in D_N^S is just the proof of the closedness in D^S given above, recall at this point that we are assuming that D^S is a subbundle.

Projection along an integrable distribution: Now, let $S \subset TM$ be an integrable distribution. Assuming that both $D \cap S^\perp$ and D^S are subbundles, any section of D^S can be written as $(X + Y, \xi)$ where (X, ξ) is a section of $D \cap (TM \oplus S^0)$ and Y is a section of S .

D^S is not Courant involutive in general. This is not strange since one would expect to be able to define a Dirac structure only on M/S , the space of leaves of the foliation defined by S (assuming that M/S is a manifold). Objects on M which descend suitably to M/S will be said *projectable along S* . Functions on M/S , $C^\infty(M)_{\text{pr}}$, can be viewed as the set

$$C^\infty(M)_{\text{pr}} = \{f \in C^\infty(M) \mid X(f) = 0, \forall X \in \Gamma(S)\}.$$

Vector fields on M/S are then defined as derivations on $C^\infty(M)_{\text{pr}}$, i.e. $\mathfrak{X}(M)_{\text{pr}} = \{X \in \mathfrak{X}(M) \mid X(C^\infty(M)_{\text{pr}}) \subset C^\infty(M)_{\text{pr}}\}$. Notice that X belongs to $\mathfrak{X}(M)_{\text{pr}}$ if and only if $Z(X(f)) = 0, \forall f \in C^\infty(M)_{\text{pr}}, \forall Z \in \Gamma(S)$. Or equivalently, $[Z, X](f) = (\mathcal{L}_Z X)(f) = 0$ using that $Z(f) = 0$. Hence, we obtain the more useful characterization

$$\mathfrak{X}(M)_{\text{pr}} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_Z X \in \Gamma(S), \forall Z \in \Gamma(S)\}.$$

Analogously, 1-forms on M/S , $\Omega^1(M)_{\text{pr}}$, are linear maps from $\mathfrak{X}(M)_{\text{pr}}$ to $C^\infty(M)_{\text{pr}}$. An analogous argument to that followed for vector fields yields:

$$\Omega^1(M)_{\text{pr}} = \{\xi \in \Gamma(S^0) \mid \mathcal{L}_Z \xi = 0, \forall Z \in \Gamma(S)\}.$$

We are now ready to prove that sections $(X + Y, \xi)$ of D^S which are projectable along S , i.e.

- a) ξ is a section of S^0 ,
- b) $\forall Z \in \Gamma(S), (\mathcal{L}_Z(X + Y), \mathcal{L}_Z \xi) = (Z', 0)$, with $Z' \in \Gamma(S)$

are Courant involutive. Notice that the integrability of S implies that $\mathcal{L}_Z X = [Z, X]$ in condition b) must be a section of S . Hence, if $(X + Y, \xi)$ is a projectable section, X itself must be a projectable vector field.

Take two such sections $(X + Y, \xi)$ and $(X' + Y', \xi')$. First notice that

$$[X + Y, X' + Y'] - [X, X'] \in \Gamma(S)$$

because $[X, Y']$, $[Y, X']$ and $[Y, Y']$ are sections of S due to condition b) and the integrability of S . Therefore,

$$[X + Y, X' + Y'] = [X, X'] + W, \quad W \in \Gamma(S). \quad (2.4)$$

We must prove that $\mathcal{L}_Z[X + Y, X' + Y'] = [Z, [X + Y, X' + Y']]$ is a section of S for any $Z \in \Gamma(S)$. Using (2.4), this amounts to prove that $[Z, [X, X']] \in \Gamma(S)$. By the Jacobi identity,

$$[Z, [X, X']] = [[Z, X], X'] - [[Z, X'], X] \quad (2.5)$$

which is a section of S because X and X' are projectable vector fields.

For the cotangent part of the Courant bracket of projectable sections we have

$$\begin{aligned} d(\iota(X + Y)\xi') &= d(\iota(X)\xi'), \\ \iota(X + Y)d\xi' &= \iota(X)d\xi' + \mathcal{L}_Y\xi' - d\iota(Y)\xi' = \iota(X)d\xi' \end{aligned} \quad (2.6)$$

where we have used conditions a) and b) above. We want to show that the right-hand side of both equations in (2.6) are projectable sections, i.e. both are in $\Gamma(S^0)$ and have vanishing Lie derivative along any section $Z \in \Gamma(S)$. This is not difficult:

$$\begin{aligned} \iota(Z)(\iota(X)d\xi') &= -\iota(X)(\mathcal{L}_Z\xi' - d(\iota(Z)\xi')) = 0, \\ \iota(Z)d(\iota(X)\xi') &= \mathcal{L}_Z(\iota(X)\xi') = \iota(\mathcal{L}_ZX)\xi' + \iota(X)\mathcal{L}_Z\xi' = 0, \\ \mathcal{L}_Z(\iota(X)d\xi') &= \iota(\mathcal{L}_ZX)d\xi' + \iota(X)\mathcal{L}_Zd\xi' = \\ &= \mathcal{L}_{Z'}\xi' - d(\iota(Z')\xi') + \iota(X)d(\mathcal{L}_Z\xi') - \iota(X)d(\iota(Z)\xi') = 0, \\ \mathcal{L}_Z(d(\iota(X)\xi')) &= d(\iota(Z)d(\iota(X)\xi')) = 0, \end{aligned} \quad (2.7)$$

where $Z' = \mathcal{L}_ZX = [Z, X] \in \Gamma(S)$.

We say that S leaves D invariant if for any point $p \in M$ and any $(X_p, \xi_p) \in (D^S)_p$ there exists a (local) projectable section of D^S such that it coincides with (X_p, ξ_p) at p . This is the generalization of condition (iii) of the Marsden-Ratiu Poisson reduction presented in Section 1. One can show that if S leaves D invariant then S is a symmetry of D^S . In this situation we can project the Dirac structure onto M/S . Given the projection $\tau : M \rightarrow M/S$ the image of

$$\tau_* \oplus (\tau^*)^{-1} : D^S \rightarrow T(M/S) \oplus T^*(M/S)$$

correctly defines a Dirac structure on M/S .

Generalized Marsden-Ratiu reduction: The third example is a combination of the two previous ones and is inspired by the work of Marsden and Ratiu [5]. It consists in reducing a Dirac structure D to a submanifold N along a symmetry S of D^S . S is an integrable distribution as above and we view N as a leaf of another integrable distribution $\Phi \subset TM$.

As we have learnt D^S induces a Dirac structure in M/S . The latter can be reduced to $N/(S \cap TN)$ and is the generalization to any Dirac structure of the procedure of [5] for Poisson manifolds.

One may wonder when the reduced Dirac structure, which we denote by \hat{D} , is actually a Poisson structure on the orbit space $N/(S \cap TN)$. \hat{D} is Poisson if and only if

$$\hat{D} \cap T(N/(S \cap TN)) = \{0\}.$$

Equivalently, in terms of the original Dirac structure D , \hat{D} is Poisson if and only if for any $p \in N$ we have that

$$(X_p, \xi_p) \in D_p \text{ s.t. } \xi_p \in (S^0)_p \cap (TN^0)_p \Rightarrow X_p \in S.$$

This can be reformulated as

$$TM \oplus (S^0 \cap TN^0) \subset D \cap (S \oplus T^*M) + TM \quad (2.8)$$

and taking the orthogonal with respect to the symmetric pairing,

$$TM \cap (D + S^0) \subset S + TN \quad (2.9)$$

which can be phrased by saying that the (generalized) Poisson vector fields of S^0 are sections of $S + TN$. This is the natural generalization of condition (1.2) for an arbitrary Dirac structure D . Clearly, (2.9) and (1.2) coincide when D is Poisson.

Thus we have shown that Marsden and Ratiu reduction fits perfectly in the framework of Dirac structures in the sense that the reduction of an invariant Dirac structure always produces a Dirac structure (assuming the existence of the different subbundles) in the reduced space.

To summarize, the stretching of Dirac structures is an interesting type of deformation which, on the one hand, generalizes the Dirac bracket to any Dirac structure when the stretching is made along a subbundle of T^*M (concretely, the annihilator of an integrable distribution). On the other hand, if the stretching is performed along an integrable distribution which leaves invariant the Dirac structure, then our construction corresponds to its projection along the integrable distribution. It would be interesting to find more general applications and examples. In particular, it would be nice to understand (and interpret) the general case in which the stretching is made along a general isotropic subbundle of $TM \oplus T^*M$. It is also worthwhile to generalize the whole construction for the case of twisted Dirac structures. This will be the subject of further research.

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