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Special Holonomy Manifolds in Physics

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Abstract

This is a pedagogical exposition of holonomy groups intended for physicists. After some pertinent definitions, we focus on special holonomy manifolds, two per division algebras, and comment upon several cases of interest in physics, associated with compactification from F-, M- and string theory, on manifolds of 8, 7 and 6 dimensions respectively.

1 Connections and Holonomy groups

Let (E, ∇) be a vector bundle with a *connection* $\nabla : P(M, G)$ is the principal bundle, and F supports a representation of G, so $E(M, F)$ is the associated bundle:

$$\nabla : \quad F \circ\!\!\rightarrow E \rightarrow M \tag{1}$$

∇ allows *covariant differentiation* of sections, e.g. for the tangent bundle $E = TM, \nabla_X Y = Z$ means: covariant derivation of vector field Y along (the flow of) X is the vector field Z. In the general case $\nabla_X \psi = \psi'$, where ψ, ψ' are sections. $\psi : M \rightarrow E$.

Connections allow also *parallel transport* along paths. E.g., a frame $\underline{e}$ at a point $\mathcal{P} \in M$ becomes $\underline{e'} = g \cdot \underline{e}$ also at $\mathcal{P}$, after a loop (closed path) γ from $\mathcal{P}$, where $g \in G$. Consider all the loops from $\mathcal{P}$ and write $Hol(\nabla) := \{g\}$; it is a (sub)group of G, called the *holonomy group* of the connection; it was invented by E. Cartan in 1925. For arcwise connected spaces, which is the case of manifolds, the holonomy group does not depend (up to equivalence) on the starting point $\mathcal{P}$.

Dedicado a José Cariñena en su 60 aniversario. Querido Pepín, parece increíble el largo camino recorrido por los dos en los últimos cuarenta años. Barcelona, Valladolid, Salamanca, Zaragoza... El jovencito en busca de un padrino de Tesis ha dado paso al respetable catedrático con un brillante historial. De nada me siento tan orgulloso como de que tú y otros discípulos hayais superado tan bien al maestro. Que sigas así, Pepín, pues aun esperamos muchos mucho de tí.

Let $Hol_0(\nabla)$ be the restriction to *contractible* loops. Clearly there is an *onto* map

$$\pi_1(M) \to Hol(\nabla)/Hol_0(\nabla) \tag{2}$$

The restricted holonomy group $Hol_0(\nabla)$ is naturally connected, whereas $Hol(\nabla)$ needs not to be. For *generic* vector bundles the holonomy group is expected to be as large as the structure group $GL(F)$. Two important theorems follow; first define

The *curvature* of a (vector bundle) connection ∇ is the operator on sections (e.g. vector fields)

$$R(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \tag{3}$$

The curvature is a *local* property, the holonomy a *global* one. But both are related be *the Ambrose-Singer theorem (1953)*:

"The Lie algebra of the holonomy group is generated by the curvature".

The other grand result is called the *reduction theorem*:

" The structure group can be reduced to the holonomy group".

That is, the total space of the bundle can be restricted by the holonomy loops.

If the curvature is zero, the connection is said *flat*; the *restricted* holonomy group $Hol_0(\nabla)$ is then $\{e\}$. Parallellelizable spaces (= trivial tangent bundle; they include S^7 and Lie groups) admit flat connections; just define the connection transport as translations in the (trivializable) tangent bundle.

We shall consider mainly connections in the tangent bundle of a manifold; then there is another tensor, the *torsion*, defined as

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \tag{4}$$

Of course, the same space might have several inequivalent connections (e.g. S^3 has the riemannian Levi-Civita connection, torsionless but curved, and the Lie-group connection ($S^3 \approx SU(2)$), flat but torsionful!).

We shall consider mainly *riemannian manifolds* $(\mathcal{V}, g)$; they enjoy the standard *Levi-Civita* connection $\nabla = \nabla^g$ in the tangent bundle, which is symmetric and isometric:

$$(symmetric :)Torsion(\nabla) = 0 = \nabla \cdot g(: isometric) \tag{5}$$

Let Isom $(\mathcal{V}, g)$ be the isometry group of the manifold: $* \in$ Isom, means $g* = g$. A *generic* riemanifold has no isometries, but the *generic* holonomy is the structure group, $O(n = dim\mathcal{V})$ or $SO(n)$. Spaces with maximal isometries have constant curvature; for example Isom $(S^n) = O(n+1)$, with constant curvature $K > 0$.

In physics both groups, isometry and holonomy, are important; for example, in the *Kaluza-Klein* (de)construction, the *gauge* groups in the mundane space $\mathcal{V}_4$ come from the *isometry* group $U(1)$ of the compactification space S^1: that is why electromagnetism unfies with gravitation with a circle as fifth dimension, so in this case $\mathcal{V}_5 = S^1 \times \mathcal{V}_4$: the original Kaluza construction, 1919.

However, in the *supersymmetric* situation, it is the holonomy group of the compactifiation space which fixes the number of supersymmetries; for example, Calabi-Yau 3-folds (CY_3, real dimension 6) are favoured for the dim $10 \to 4$ compactation of the Heterotic Exceptional string, because the holonomy group, $SU(3)$, allows just $\mathcal{N} = 1$ Susy in our mundane, $4D$ space, as we want.

Isometries measure, of course, *symmetries*, whereas holonomy measures distance (obstruction) from *flatness*; no apparent relation exists, except opposite genericity (as stated above).

Simple examples in $D = 2$

The sphere S^2 has isometry $O(3)$, the torus T^2 has $U(1)^2$; other genus $g > 1$ surfaces have *no* isometries. In the nonorientable cases, RP^2 and Klein bottle are the only ones with isometries.

As for holonomy, the 2-Torus T^2 is the only CY_1 among surfaces, because is a group manifold, hence there is a connection with Hol $= \{e\}$ and $SU(1) \equiv \{e\}$. The other surfaces with genus $\neq 0$ have Hol $= U(1) = SO(2)$ (if orientable) and $O(2)$ (if not).

Simple examples in $D = 4$

The "round" sphere S^4 has $O(5)$ as isometry, and a connection with $SO(4)$ holonomy. The 4-Torus T^4 is flat, with isometry $U(1)^4$. Intermediate is the topologically unique $K3$ (complex) surface (see later), which is a Calabi-Yau$_2$ space, with $dim_R = 4$, with $SU(2)$

holonomy but no isometries. As for CP^2, it has $U(3)$ as isometry group, and $U(2)$ for holonomy; in fact, $CP^2 \approx SU(3)/U(2)$.

As introductory material, the first book on modern differential geometry is still the best [1].

Besides the original invention by E. Cartan (who did it in order to construct all symmetric spaces ca. 1925/26), and a short revival in the fifties (Berger, Lichnerowicz), the study of holonomy languished until resuscitation in the mid-eighties, in part by imposition of physics (as in so many other mathematical questions!). Then Bryant, Salamon and mainly Dominic Joyce (see book [2]) revitalized greatly the subject.

Finally, let us note that the holonomy groups come to the world with a particular action (representation) in the tangent space, so one should properly speak of the holonomy *representation*.

2 Special and Exceptional Holonomy Manifolds

What groups can appear as holonomy groups $Hol(g) \subset O(n)$ of riemanifolds $(\mathcal{V}_n, g)$? The issue was set and solved by M. Berger in 1955. To state precisely the problem, suppose $Hol(\cdot)$ acts *irreducibly* in the tangent space, and *symmetric* spaces G/H are excluded (because all are known (Cartan) and for them the subgroup H is the holonomy group). Berger found all possible candidate groups with these prescriptions by a hard case-by-case method.

Berger´s solution is best understood (Simons, 1962) as *the search for transitive groups over spheres*: with two exceptions, these are the special holonomy groups.

The generic case is the orthogonal group acting *trans* on the sphere, $O(n)\circ \rightarrow S^{n-1}$, with isotopy $O(n-1)$, that is $S^{n-1} = O(n)/O(n-1)$. The cases of *trans* action on spheres coincident with special holonomy manifolds are (Berger´s list):

$$\begin{array}{llllll} R & O(n) & \quad or & SO(n) & acting\ \ on & S^{n-1} \\ C & U(n) & \quad or & SU(n) & acting\ \ on & S^{2n-1} \\ H & Sp(n)\cdot Sp(1) & \quad or & Sp(n) & acting\ \ on & S^{4n-1} \\ O & Spin(7) & on\ \ S^7 \quad or & G_2 & & on\ \ S^6 \end{array} \tag{6}$$

We exhibit the association with the four division algebras R, C, H, y O, which is obvious and remarkable. Recall also that the homology of compact simple Lie groups is given by that of the product of odd-dimensional spheres, see e.g. [3]. Then the real and complex cases are clear, for example $SU(3) \approx S^3 \times S^5$, as homology sphere product, so we have $S^5 = SU(3)/SU(2)$. $Sp(n)$ for us is the compact form of the C_n Cartan Lie algebra. Also there is a "nonunimodular" form

$$Sp(n) \cdot Sp(1) := Sp(n) \times_{/2} Sp(1) \tag{7}$$

As for the octonion cases, recall dim Spin(7) = 8, type (+1, real); in some sense which we do not elaborate, it could be said that Spin(7) "is" Oct(1), and G_2, defined as $Aut(O)$, is the "unimodular" form, $G_2 \approx SOct(1)$.

There are two more cases of *trans* actions on spheres

$$Sp(n) \cdot U(1) := Sp(n) \times_{/2} U(1) \;\; acting \;\; in \;\; S^{4n-1} \tag{8}$$

and

$$Spin(9) \;\; acting \;\; in \;\; S^{15} \tag{9}$$

which, however, do *not* give rise to new holonomy groups. $Spin(9)$ acts *trans* in S^{15} as $Spin(9) \approx S^3 \times S^7 \times S^{11} \times S^{15}$. In fact, $S^{15} = Spin(9)/Spin(7)$, equivalent, in some sense, to $S^{15} \approx$ "$Oct(2)$"/"$Oct(1)$". Spin(9) was really in Berger´s list, but the only space found was OP^2 (Moufang or octonionic plane), which is a *symmetric* space.

The sphere S^7 of unit octonions is singularized because there are *four* groups with *trans* actions, $O(8), U(4), Sp(2)$ and $Spin(7) =$ "$Oct(1)$"; similar for S^{15}, but no more.

Notice the next Spin case, $Spin(10)$: the action is *not trans* in the higher sphere, to wit, $dim \;\; Spin(10) = 16$, complex, so $Spin(10)$ acts on S^{31}, but the sphere homology product expansion for $O(10)$ is $S^3 \times S^7 \times S^{11} \times S^{15}$.

We expand now on the extant cases:

<u>Over the reals</u> we have the groups $O(n)$, generic holonomy, and $SO(n)$: clearly the second obtains when the space is orientable and the connection oriented: there is an obstructoin, the first Stiefel-Whitney class:

$$\mathcal{V} \quad orientable \quad manifold \leftrightarrow w_1 = 0, w_1 \in H^1(\mathcal{V}, Z_2) \tag{10}$$

Alternatively, the manifold $\mathcal{V}$ should have a global volume element (reduction $GL(n, R)$ $\leftarrow SL(n, R)$). In Berger´s classification, he took the manifolds as simply connected, which are then automatically orientable (if $\pi_1(\mathcal{V}) = 0$, all first order (co)homology vanishes, including w_1). Hence, $O(n)$ did not append in his list.

Over the complex we have complex manifolds, with structure group $U(n)$; but a generic hermitian metric $h = g + i\omega$ will allow in general a connection with holonomy $SO(2n)$, as $\nabla g = 0$ only, unless the complex structure J is also preserved: this is the case of Kähler manifolds, with $\nabla\omega(= d\omega) = 0$, where $\omega = g(J)$ is the symplectic form.

The "unimodular" restriction $SU(n)$ obtains when the associated bundle with group $U(1) = U(n)/SU(n)$ is trivial, which is measured by the first Chern class:

$$SU(n) \quad holonomy \quad \leftrightarrow 0 = c_1 \in H^2(\mathcal{V}, Z) \tag{11}$$

The natural name for these spaces would be "Special Kähler manifolds", but had become known instead as "Calabi-Yau spaces", after the conjecture of E. Calabi proven by S.T. Yau. As a bonus, these spaces have trivial Ricci tensor ($Ric = Tr\ Riem$, contraction of the Riemann tensor): define the Ricci 2-form $\rho := Ric(J)$; by the same token as above J parallel (=covariant constant) implies ρ closed, and it turns out that $[\rho] = 2\pi c_1(\mathcal{V})$. Therefore

$$SU(n) \quad holonomy \quad implies \quad Ricci \quad flat \quad manifolds, \quad Ric = 0. \tag{12}$$

In other words, Calabi-Yau spaces are candidates to Einstein spaces, solution of vacuum Einstein equations (without cosmological term). Also CY spaces have an holomorphic volume element.

Over the quaternions we have quaternionic manifolds, like HP^n, with "nonunimodular" holonomy $Sp(1) \cdot Sp(n)$, and hyperkähler manifolds, with holonomy $Sp(n)$. As clearly $Sp(n) \subset SU(2n)$, the later are also Ricci flat. In fact, the space $HP^n = Sp(n+1)/Sp(1) \cdot Sp(n)$, which a quaternionic space, is not valid as special holonomy manifold, because it is a symmetric space. By contrast, quaternionic manifolds have not to be even Kähler, as the example of $S^4 = HP^1$ shows; quaternionic and hyperkähler manifolds are not easy to come by.

Manifolds with *octonionic* holonomy : the two groups: $Spin(7) \subset SO(8)$, acting as holonomy groups on 8-dim manifolds, and $G_2 \subset SO(7)$, in 7-dim. manifolds, are groups associated to the octonions. $Spin(7) \approx$ "$Oct(1)$" and $G_2 = Aut(O) \approx$ "$SOct(1)$"; manifolds with these holonomy groups are called *exceptional holonomy* manifolds. We expect them to play some role in physcs, as for example the iternal spaces in $F-$ and $M-$theories have dimensions 8 and 7 bzw.

It is remarkable that, while O, U and Sp are isometry groups of symmetric regular positive bilinear forms (O), sesquilinear regular positive forms (U) and antisymmetric regular forms (Sp), the octonionic cases obtain from invariance of certain $3-$forms (G_2) and selfdual 4-forms ($Spin(7)$). This cannot go beyond dimension 8, because for example the dimension of 3-forms in 9-dimensions is $\binom{9}{3} = 84$, whereas dim $GL(9, R) = 81$.

In fact, the algebraic definition of $Spin(7)$ is the isotopy group of *certain* class of selfdual 4-forms in R^4: it preserves also orientation and euclidean metric, so $Spin(7) \subset SO(8)$. Notice the selfdual form is *not* generic, as $8^2 - \binom{8}{4}/2 = 29 > dim\ Spin(7) = 21$; the special selfdual four-form is called the *Cayley form* in the math literature. In any case $Spin(7)$ covers S^7 with isotopy G_2 : $21 - 7 = 14$.

The *algebraic* definition of G_2 is this: the stability group of the *generic* 3-form in R^7 as vector space: dim $GL(7, R) - dim \wedge^3 R^7 = \ \ dim\ G_2 : 49 - 35 = 14$. Of course, the original characterization of G_2 as $Aut(O)$ by Cartan is related to this: a 3-form becomes a T_2^1 tensor through a metric, and this is indicative of an algebra, i.e. a bilinear map $R^7 \times R^7 \to R^7$, given by the octonionic product (and restriction to the imaginary part).

There is also a sense in which for each division algebra there is a *normal* form and am *unimodular* form:

$$\begin{array}{lll}
Reals & O(n) \approx S^0 \times S^3 \times S^7 ... & SO(n) \approx S^3 \times S^7 ... \\
R & Generic & Orientable, \quad w_1 = 0 \\
 & & \\
Complex & U(n) \approx S^1 \times S^3 \times S^5 ... & SU(n) \approx S^3 \times S^5 ... \\
C & K\ddot{a}hler & Calabi-Yau, \quad c_1 = 0 \\
 & & \\
Quaternions & Sp(n) \cdot Sp(1) \approx S^3 \times S^3 \times S^7 ... & Sp(n) \approx S^3 \times S^7 ... \\
H & Quaternionic & Hyperk\ddot{a}hler \\
 & & \\
Octonions & Spin(7) \approx S^3 \times S^7 \times S^{11} & G_2 \approx S^3 \times S^{11} \\
O & dim\ \ 8 & dim\ \ 7
\end{array} \tag{13}$$

<u>NOTES</u> 1). Today there are *compact* examples of all cases of special holonomy mani-

folds: big advances were made recently by Joyce [2], Salamon and others.

2). In dim 4, a remarkable case is the $K3$ manifold or Kummer surface (the name is due to A. Weil, 1953, for Kummer, Kähler and Kodaira). It is the only CY manifold in dimension four; it can be easily constructed ($R^4 \to T^4 \to \quad orbifold \to \quad blow \quad up$; [4]). For a long time was the paradigmatic example of $SU(n = 2)$ special holonomy.

3). Notice a generic complex n-manifold would have $SO(2n)$ holonomy inspite the structure group being $U(n)$!

4). The Calabi conjecture, proved by Yau, indicates the relation of the Ricci form with the Kähler structure.

5). In the 80s a big industry, led by Phil Candelas in Austin ([5]), was to find CY_3 manifolds for string compactifications. *Mirror* symmetry was discovered in this context; see later.

6). Except G_2, all special holonomy spaces are even dimensional.

7). One can show that G_2 and $Spin(7)$ holonomies are Ricci flat.

8). Although special holonomy representations are irreducible in the vector case, there might be p-forms which split under the holonomy subgroup. For example, for G_2, 3-forms split as $35 = 1+7+27$; the 7 irrep is justly the octonion product, and the 1 the invariant 3-form. As for $Spin(7)$, a self-4-form splits as $35 = 1 + 7 + 27$: it includes the invariant 4-form.

3 Cases in Physics: dimensions 6, 7 and 8

In 1983, just after the first studies in eleven dimensional supergravity (11-dim SuGra), Duff and Pope realized that it is the *holonomy* of the compactified space which determines the number of surviving Susy symmetries down to 4 dimensions. For spinor fields, as $S^7 = Spin(7)/G_2$, 7-manifolds with exceptional G_2 holonomy would have a surviving spinor, hence $\mathcal{N} = 1$ Susy down to 4-D. But after the String Revolution, 1984/85, the descent $10 \to 4$ took over, and the favourite spaces were CY 3-folds: the heterotic string has $\mathcal{N} = 1$ supersymmetry in 10$-$dim., which means $\mathcal{N} = 4$ down to earth; but it will be 1/4 of these after CY_3 compactation: the generic $SO(6)$ holonomy of any (orientable) dim-6 manifold would become $SU(4) = Spin(6)$ after imposing a (necessary) spin structure, and then if we want one spinor to survive the group descends to $SU(3)$. The 6-dim manifold has to be orientable, spin, complex, Kähler and Calabi-Yau.

With the advent of M-Theory (1995), P. Townsend resurrected the idea of 7-dim manifolds with G_2 holonomy. One can go even further to Spin(7), the largest exceptional holonomy group, by considering for example compactifications to 3-dim spaces (which seems natural; for example, the series of noncompact symmetries of supergravity includes E_7 in 4 dimensions, which is claiming for E_8 in three, which is of course the case). Another reason is F-Theory, which works in 12 dimensions with $(2, 10)$ signature, and where Spin(7) (perhaps in a nonpositive form) fits well.

Are such beasts as CY_3 spaces in abundance? Yes, you can produce them in assembly line, to the point of studying their Hodge numbers statistically! [5]. Another interesting phenomenological constraint in the "old-fashion" $10 \to 4$ descent, was the Euler number χ : it is related, via zeroes of the Dirac operator, to the number of generations, which is $\mid \chi \mid /2$.

As for the extension to F-Theory, we refer the reader to [7]. Besides some attempt to state the particle content, the theory is rather stagnant at this point (as is M-Theory in general). For a modern study of special holonomies with Lorentzian metrics, see [8].

4 Relation with Mirror Symmetry

Complex manifolds have a refinement on Betti numbers, as they separate in holo- or antiholomorphic. The full expression of them is called the the Hodge diamond. For the previous $K3$ manifold it is

$$\begin{array}{ccccc} & & h^{0,0}=1 & & \\ & h^{1,0}=0 & & h^{0,1}=0 & \\ h^{2,0}=1 & & h^{1,1}=20 & & h^{0,2}=1 \\ & h^{2,1}=0 & & h^{1,2}=0 & \\ & & h^{2,2}=1 & & \end{array} \tag{14}$$

with bettis $= 1, 0, 22, 0, 1$. For Calabi-Yau 3-folds the diamond is bigger, but still symmetric. A *mirror* pair X, Y of CY_3 are two such spaces with

$$h^{1,1}[X] \;\; and \; h^{2,1}[X] \quad equal \;\; h^{1,2}[Y] \;\; and \; h^{1,1}[Y] \tag{15}$$

There is no clear reason for this duality, but just another more example of a physics discovery on pure mathematics. These two numbers measure very different invariants, so mirror symmetry came up as a big surprise to mathematicians, when many conjectures by physicists seemed to be true.

Indeed, several of these conjectures *were* true. It is also true that string theory ”glosses over” orbifold singularities (i.e. quotienting manifolds by fix-point-action discrete groups), and the associated (quantum) conformal field theories make perfect sense.

We shall only comment on the interpretation of this mirror symmetry by M. Kontsevich [6]. For him, the crux of the matter is a trade between *complex* and *symplectic* geometry. In fact, complex structures preserve an imaginary unit, $J \in End\ \mathcal{V},\ \ J^2 = -1$, whereas a symplectic manifold enjoys a 2-form ω, whose matrix form is very similar to J!. In both cases, there is an extra condition:

In the complex case, $N(J) = 0$, where N stands for the Nihenhuis (obstruction); that makes up a complex, not only almost-complex, manifold (the Newlander-Nirenberg theorem). In the symplectic case, the 2-form is closed, or, alternatively, the (inverse) Poisson bracket satisfies Jacobi´s identity. Their isotropy groups, $GL(n, C)$ and $Sp(n, R)$ respectively, are of the same homotopy type, namely the homotopy of the intersection, $U(n)$. In any case, the relation hidden in Mirror Symmetry is an intrincate one. In the words of Dijkgraaf: ”Mirror symmetry is the claim that the generating function for certian invariants of the symplectic structures on the 2-Torus $S^1 \times S^1$ is a ”nice” function in the moduli space of complex structures in the same”: the 2-Torus is a *self-dual* manifold for Mirror Symmetry (MS).

The general case of MS is best understood in terms of toric varieties, which generalize projective spaces.

The main lesson of MS for physics seems to be this: certain topology changes (sometimes called “flops”) are compatible with the underlined string theory. That probably means that the complex-geometric description of strings is too fine... Perhaps it hints towards a new type of duality.

References

[1] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry*, 2 Vols. J. Wiley, New York 1963

[2] D. JOYCE, *Compact Manifolds with Special Holonomy.* Oxford U.P., 2000

[3] L. J. BOYA, "The Geometry of Compact Lie Groups", Rep. Math. Phys. **30** (1991), 149-162.

[4] C. GOMEZ in "Geometry and Physics" CIEMAT, 1998

[5] P. CANDELAS *et al.* "Calabi-Yau manifolds in weighted spaces", Nucl. Phys. **B341** (1990), 383-402.

[6] M. KONSEVICH, talk at the Int. Math. Cong., Zürich 1997

[7] L. J. BOYA, "Arguments for F-Theory", Mod. Phys. Lett. A **21** (2006), 287-304.

[8] R. L. BRYANT, "Pseudo-riemannian metrics with parallel spinor fields and vanishing Ricci tensor", arXiv:math.DG/0004073.