Monografías de la Real Academia de Ciencias de Zaragoza. **29:** 27–36, (2006).

Poisson coalgebras, symplectic realizations and integrable systems

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Abstract

The construction of integrable systems from symplectic realizations of Poisson coalgebras with Casimir elements is revisited. Several examples of Hamiltonians with either undeformed or 'quantum' coalgebra symmetry are given, and their Liouville integrability is discussed. The essential role of symplectic realizations in this context is emphasized.

1 Introduction

Poisson coalgebras (Poisson algebras endowed with a compatible comultiplication structure) have been shown to be relevant as the 'hidden' symmetries underlying the integrability properties of a wide class of N-dimensional (ND) (super)integrable classical Hamiltonian systems (see [1]–[5] and references therein). In this construction, once a symplectic realization of the coalgebra is given, their generators play the role of dynamical symmetries of the Hamiltonian, while the coproduct map is used to 'propagate' the integrability to arbitrary dimension.

From this coalgebra approach, several well-known (super)integrable systems have been recovered and some integrable deformations for them, as well as new integrable systems, have also been obtained. In particular, this integrability-preserving deformation procedure has been used to introduce both superintegrable and integrable free motions on spaces with either constant or variable curvature, and (super)integrable potential terms can also be considered on such spaces [6]. The aim of this contribution is to provide a summary of this approach to Hamiltonian integrability, where symplectic realizations of Poisson coalgebras play an essential role.

2 Hamiltonian systems on Poisson coalgebras

We recall that a coalgebra (A, Δ) is a (unital, associative) algebra A endowed with a coproduct map [7]:

$$\Delta : A \to A \otimes A,$$

which is coassociative

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta.$$

In addition, Δ has to be an algebra homomorphism from A to $A \otimes A$:

$$\Delta(a\, b) = \Delta(a)\, \Delta(b), \qquad \forall\, a, b \in A.$$

Moreover, if A is a Poisson algebra and

$$\Delta(\{a, b\}_A) = \{\Delta(a), \Delta(b)\}_{A\otimes A}\,, \qquad \forall a, b \in A,$$

we shall say that (A, Δ) is a Poisson coalgebra. The comultiplication Δ provides a 'two-fold way' for the definition of the objects on $A \otimes A \otimes A$, that will be essential as far as superintegrability is concerned.

Let us summarize the general construction of ref. [1]. Let (A, Δ) be a Poisson coalgebra with l generators X_i $(i = 1, \ldots, l)$, and r functionally independent Casimir functions $\mathcal{C}_j(X_1, \ldots, X_l)$ (with $j = 1, \ldots, r$). The coassociative coproduct $\Delta \equiv \Delta^{(2)}$ has to be a Poisson map with respect to the usual Poisson bracket on $A \otimes A$:

$$\{X_i \otimes X_j, X_r \otimes X_s\}_{A\otimes A} = \{X_i, X_r\}_A \otimes X_j X_s + X_i X_r \otimes \{X_j, X_s\}_A.$$

Then, the m-th coproduct map $\Delta^{(m)}(X_i)$

$$\Delta^{(m)} : A \to A \otimes A \otimes \ldots^{m)} \otimes A, \tag{1}$$

can be defined by applying recursively the coproduct $\Delta^{(2)}$ in the form

$$\Delta^{(m)} := (id \otimes id \otimes \ldots^{m-2)} \otimes id \otimes \Delta^{(2)}) \circ \Delta^{(m-1)}. \tag{2}$$

Such an induction ensures that $\Delta^{(m)}$ is also a Poisson map.

Table 1. Functions obtained by applying the coproduct map.

X_1	X_2	$\ldots$	X_l	$\mathcal{C}_1$	$\mathcal{C}_2$	$\ldots$	$\mathcal{C}_r$
$\Delta^{(2)}(X_1)$	$\Delta^{(2)}(X_2)$	$\ldots$	$\Delta^{(2)}(X_l)$	$\Delta^{(2)}(\mathcal{C}_1)$	$\Delta^{(2)}(\mathcal{C}_2)$	$\ldots$	$\Delta^{(2)}(\mathcal{C}_r)$
$\Delta^{(3)}(X_1)$	$\Delta^{(3)}(X_2)$	$\ldots$	$\Delta^{(3)}(X_l)$	$\Delta^{(3)}(\mathcal{C}_1)$	$\Delta^{(3)}(\mathcal{C}_2)$	$\ldots$	$\Delta^{(3)}(\mathcal{C}_r)$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Delta^{(N)}(X_1)$	$\Delta^{(N)}(X_2)$	$\ldots$	$\Delta^{(N)}(X_l)$	$\Delta^{(N)}(\mathcal{C}_1)$	$\Delta^{(N)}(\mathcal{C}_2)$	$\ldots$	$\Delta^{(N)}(\mathcal{C}_r)$

In this way, we can construct the set of functions shown in Table 1. From them, given a smooth function $\mathcal{H}(X_1,\ldots,X_l)$, the N-sites Hamiltonian is defined as the N-th coproduct of $\mathcal{H}$:

$$H^{(N)} := \Delta^{(N)}(\mathcal{H}(X_1,\ldots,X_l)) = \mathcal{H}(\Delta^{(N)}(X_1),\ldots,\Delta^{(N)}(X_l)). \tag{3}$$

From [1] it can be proven that the set of $r\cdot N$ functions $(m=1,\ldots,N; j=1,\ldots,r)$

$$C_j^{(m)} := \Delta^{(m)}(\mathcal{C}_j(X_1,\ldots,X_l)) = \mathcal{C}_j(\Delta^{(m)}(X_1),\ldots,\Delta^{(m)}(X_l)), \tag{4}$$

Poisson-commute with the Hamiltonian

$$\left\{C_j^{(m)}, H^{(N)}\right\}_{A\otimes A\otimes\ldots^{N)}\otimes A} = 0, \tag{5}$$

and is in involution:

$$\left\{C_i^{(m)}, C_j^{(n)}\right\}_{A\otimes A\otimes\ldots^{N)}\otimes A} = 0, \qquad m,n=1,\ldots,N, \quad i,j=1,\ldots,r. \tag{6}$$

Example 1. Lie–Poisson algebras g^* with generators X_i $(i=1,\ldots,l)$ and Casimir functions $\mathcal{C}_j(X_1,\ldots,X_l)$ $(j=1,\ldots,r)$, when endowed with the (primitive) coalgebra structure

$$\Delta(X_i) = X_i\otimes 1 + 1\otimes X_i, \tag{7}$$

are Poisson coalgebras. A very natural choice is to consider Hamiltonians $\mathcal{H}(X_1,\ldots,X_l)$ that, under the iterated application of the coproduct map, will generate dynamical systems on $g^*\otimes g^*\otimes\ldots^{N)}\otimes g^*$ with $N\cdot r$ constants of the motion in involution.

Example 2. The Poisson analogues of quantum algebras and groups [7] are also (deformed) coalgebras (A_z,Δ_z) (where z is the deformation parameter). Consequently, any function of the generators of a given 'quantum' Poisson algebra (with deformed Casimir elements $C_{z,j}$) will provide a deformation of the Hamiltonian described in the previous Example.

3 Symplectic realizations and integrability

Thus, for any Poisson coalgebra (A,Δ) and Hamiltonian function $\mathcal{H}$ we have constructed a Hamiltonian system on the Poisson manifold $A^{\otimes N}$ constructed as N-tensor copies of A. This is a 'cluster-type' dynamical system [4] with $l\cdot N$ dynamical variables whose evolution equations are

$$\dot{X}_{(i,m)} = \left\{X_{(i,m)}, H^{(N)}\right\}, \qquad i=1,\ldots,l, \quad m=1,\ldots,N,$$

where $X_{(i,m)}$ denotes the generator X_i living on the m-th copy of A. The r Casimir functions of A generate a maximum number of $r\cdot N$ integrals of the motion for $H^{(N)}$ (see

Table 1), but since $l - r \geq 2$ we have always less than $l \cdot N - 1$ integrals and, therefore, complete integrability for $\mathcal{H}$ cannot be reached in terms of the 'algebraic' dynamical variables $X_{(i,m)}$.

However, we can get a specialization of the coalgebra formalism by working on the symplectic leaves of the initial Poisson coalgebra through suitable symplectic realizations. If A has rank r (i.e., A has r Casimir functions) a symplectic leaf of A (always even dimensional) will be denoted by $A_{(k_1,k_2,\ldots,k_r)}$, where the leaf is characterized by the set of constant values $(k_1, k_2, \ldots, k_r)$ for the Casimir functions. Then a symplectic realization D for $A_{(k_1,k_2,\ldots,k_r)}$ can be obtained in terms of s pairs (q_i, p_i) of canonical variables

$$D : x \to x(q_1, p_1, q_2, p_2, \ldots, q_s, p_s),$$

where x is any point on $A_{(k_1,k_2,\ldots,k_r)}$. Note that different symplectic leaves can be chosen for each copy of A within $A^{\otimes N}$, and for each of them a symplectic realization has to be constructed. In this way the symplectic realization of $H^{(N)}$ given by

$$H_D^{(N)} := (D \otimes D \otimes \ldots^{N)} \otimes D)(H^{(N)}),$$

is defined on the N-th tensor product of the symplectic leaves

$$A_{(k_1^{(1)},k_2^{(1)},\ldots,k_r^{(1)})} \otimes A_{(k_1^{(2)},k_2^{(2)},\ldots,k_r^{(2)})} \otimes \ldots \otimes A_{(k_1^{(N)},k_2^{(N)},\ldots,k_r^{(N)})},$$

where $k_i^{(m)}$ is the value of the i-th Casimir for the m-th symplectic leaf.

If we consider symplectic realizations with the same s for all the sites in the tensor product chain, $H_D^{(N)}$ turns out to be a function of $N \cdot s$ pairs of canonical variables, i.e., it defines a Hamiltonian system with $N \cdot s$ degrees of freedom. Therefore, we need a number of $(N \cdot s - 1)$ independent constants of the motion in involution to get complete integrability. Since the formalism provides at most $N \cdot r$ integrals, in order to have that

$$N \cdot s - 1 \leq N \cdot r,$$

we need that the chosen symplectic realization D of A fulfils

$$s \leq r + 1/N,$$

and we are led to the following necessary condition for complete integrability:

$$s \leq r.$$

Consequently, symplectic realizations with $s = 1, \ldots, r$ are candidates for the construction of completely integrable systems. Obviously, this condition is not sufficient since the functional independence of a $(N \cdot s - 1)$ dimensional subset of the $N \cdot r$ integrals coming from the coproduct has to be explicitly checked in each case. Indeed, for some symplectic realizations the number of independent integrals is less than $(N \cdot s - 1)$ and only partial integrability can be achieved.

Note that for a given rank r lower values of s are, in principle, preferred from the integrability viewpoint, since the number of degrees of freedom $N \cdot s$ decreases with respect to the (fixed) maximum number of integrals $N \cdot r$.

3.1 Superintegrability

Note that, instead of (2), another recursion relation for the m-th coproduct map can be defined:

$$\Delta_R^{(m)} := (\Delta^{(2)} \otimes id \otimes \ldots^{m-2)} \otimes id\,) \circ \Delta_R^{(m-1)}.$$

Due to the coassociativity property of the coproduct, this new expression will provide exactly the same expressions for the N-th coproduct of any generator. However, if we label from 1 to N the sites of the chain of N copies of A, lower dimensional coproducts $\Delta^{(m)}$ (with $m < N$) will be 'different' in the sense that $\Delta^{(m)}$ will contain objects living on the tensor product space $1 \otimes 2 \otimes \ldots \otimes m$, whilst $\Delta_R^{(m)}$ will be defined on the sites $(N - m + 1) \otimes (N - m + 2) \otimes \ldots \otimes N$. Therefore, the coalgebra symmetry of a given Hamiltonian gives rise to two 'pyramidal' sets of $r \cdot N$ integrals of the motion in involution that Poisson-commute with $H^{(N)}$ [5]. This 'right set' of integrals characterizes the *quasi-maximal* superintegrability of the coalgebra-symmetric Hamiltonians, that is, since both sets have $\Delta^{(N)}(\mathcal{C}) \equiv \Delta_R^{(N)}(\mathcal{C})$ in common there remains *one* integral to ensure maximal superintegrability.

4 Some examples

4.1 3D Lie–Poisson coalgebras

By following the known classifications summarized in [8] we consider the set of 9 non-isomorphic 3D ($l = 3$) real Lie algebras, all of them with rank $r = 1$ (note that the generators e_i in [8] are now written as J_i). Therefore, $s = 1$ and the 'one-particle' symplectic realizations given in Table 2 are candidates to provide ND completely integrable (and thus, quasi-maximally superintegrable) systems. The constant k is just the value of the Casimir C that fixes the symplectic leaf. Note that, in many cases, if $k = 0$ we would get a lower dimensional symplectic leaf, that we do not consider. We also emphasize that two symplectic realizations with the same value for k can always be related through a canonical transformation.

Many superintegrable Hamiltonians can be explicitly obtained by applying the coalgebra construction and the symplectic realizations given in Table 2. The only case in which the construction does not provide complete integrability is $A_{3,1}$ (the Heisenberg algebra), where the Casimir coincides with the central generator J_1 and its m-th coproducts are just numerical constants under the symplectic realization.

For the rest of the cases, once a symplectic realization is given, the Hamiltonian $H^{(N)}$ will be a function of N canonical pairs (q_i, p_i) and is completely integrable, since the functional independence of the $C^{(m)}$ functions is guaranteed by the fact that the m-th integral $C^{(m)}$ depends on the first m pairs (q_i, p_i) of canonical coordinates.

Example 3. For instance, the Calogero–Gaudin Hamiltonian [9, 10]

$$H^{(N)} = \sum_{i<j}^{N} 2\, p_i\, p_j\, (1 - \cos(q_i - q_j)),$$

comes from the $k = 0$ symplectic realization $(*)$ of $A_{3,8}$ (the $sl(2)$ algebra) by taking as the Hamiltonian the Casimir operator C [1]. In general, note that the choice of the symplectic realization drastically changes the 'shape' of the Hamiltonian. For instance, by using the Gelfan'd–Dyson symplectic map $(**)$ with $k = 0$ the very same Calogero–Gaudin system reads

$$H^{(N)} = \sum_{i<j}^{N} - p_i\, p_j\, (q_i - q_j)^2.$$

Table 2. Symplectic realizations for 3D Lie-Poisson algebras.

	J_1	J_2	J_3	C	
$A_{3,1}$	k	p	$-kq$	J_1	$k \neq 0$
$A_{3,2}$	$k e^{\frac{p}{k}}$	$p\, e^{\frac{p}{k}}$	$-kq$	$J_1\, e^{-\frac{J_2}{J_1}}$	$k \neq 0$
$A_{3,3}$	$\frac{p^2}{2}$	$k\frac{p^2}{2}$	$\frac{pq}{2}$	$\frac{J_2}{J_1}$	$k \neq 0$
$A_{3,4}$	ke^p	e^{-p}	$-q$	$J_1 J_2$	$k \neq 0$
$A^{\alpha}_{3,5}$	$e^{\frac{p}{\alpha}}$	ke^p	$-\alpha q$	$J_2 J_1^{-\alpha}$	$k \neq 0$
$A_{3,6}$	$\sqrt{k}\, \sin\, p$	$\sqrt{k}\, \cos\, p$	q	$J_1^2 + J_2^2$	$k \neq 0$
$A^{\alpha}_{3,7}$	$\sqrt{k}\, e^{\alpha p} \sin\, p$	$\sqrt{k}\, e^{\alpha p} \cos\, p$	q	$(J_1^2 + J_2^2) \left(\frac{J_1 + i\, J_2}{J_1 - i\, J_2} \right)^{i\alpha}$	$k \neq 0$
$A_{3,8}$	$\frac{e^q}{2}(k - 2p^2)$	p	e^{-q}	$2J_2^2 + J_1 J_3 + J_3 J_1$	$\forall k$
$(***)$	$\frac{q^2}{2}$	$\frac{pq}{2}$	$-\frac{p^2}{2} + \frac{k}{q^2}$		$\forall k, q \neq 0$
$(**)$	$-p\, q^2 + \sqrt{2\, k}\, q$	$p\, q - \sqrt{\frac{k}{2}}$	p		$\forall k$
	$-pe^q$	p	pe^{-q}		$k = 0$
$(*)$	$p \sin q + p$	$p \cos q$	$p \sin q - p$		$k = 0$
$A_{3,9}$	p	$\sqrt{k - p^2}\, \cos\, q$	$\sqrt{k - p^2}\, \sin\, q$	$J_1^2 + J_2^2 + J_3^2$	$\forall k$

Example 4. The following Hamiltonian

$$H^{(N)} = \sum_{i=1}^{N} \left(\frac{p_i^2}{2} - \frac{k^{(i)}}{q_i^2} \right) + \mathcal{F}\Big(\sum_{i=1}^{N} q_i^2 \Big), \tag{8}$$

is formed by the superposition of N 'centrifugal barriers' determined by the $k^{(i)}$-terms and a central potential through the arbitrary smooth function $\mathcal{F}$. This is also $A_{3,8}$ coalgebra-invariant under the symplectic realization $(***)$ [2] and the Hamiltonian is taken as

$$\mathcal{H} = -\, J_3 + \mathcal{F}(2\, J_1).$$

Therefore, as particular cases, this Hamiltonian reproduces the Smorodinsky–Winternitz system [11] for $\mathcal{F} = 2\omega J_1$ and provides a generalization of the Kepler potential when $\mathcal{F} = -\gamma/\sqrt{2J_1}$ (ω and γ are real constants).

Example 5. Given a Hamiltonian with certain coalgebra symmetry, any coalgebra deformation provides a superintegrable deformation of the initial Hamiltonian. In this way, the following superintegrable deformation of (8) has been obtained by making use of the non-standard quantum deformation of $A_{3,8}$ [2]:

$$H_z^{(N)} = \sum_{i=1}^{N} \Big(\frac{\sinh z q_i^2}{z q_i^2}\, \frac{p_i^2}{2} - \frac{z k^{(i)}}{\sinh z q_i^2} \Big) e^{z K_i^{(N)}(q^2)} + \mathcal{F}\Big(\sum_{i=1}^{N} q_i^2 \Big), \tag{9}$$

where the long-range K-functions are defined by $K_i^{(N)}(q^2) = -\sum_{s=1}^{i-1} q_s^2 + \sum_{l=i+1}^{N} q_l^2$. We stress that when all the constants $k^{(i)}$ vanish, Eq. (9) provides the Hamiltonian for the motion of a particle on an ND space of *non-constant curvature* under the action of an arbitrary 'central' (radial) potential $\mathcal{F}$ (see [6, 12] for the 2D and 3D spaces and potentials).

Example 6. Another interesting example of coalgebra-invariant system is the following analogue [1] of the Ruijsenaars-Schneider model [13]:

$$H_z^{(N)} = \sum_{i=1}^{N} \cosh\theta_i \, \exp\Big(-\frac{z}{2}\big(\sum_{j=1}^{i-1} q_j\big) + \frac{z}{2}\big(\sum_{k=i+1}^{N} q_k\big) \Big),$$

where (q_i, θ_i) are canonically conjugate coordinates. This completely integrable Hamiltonian was obtained by using the Poisson analogue of a quantum deformation of the symplectic realization of the (1+1)D Poincaré algebra $A_{3,4}$.

4.2 4D Lie-Poisson coalgebras

In this case the classification [8] provide a set of 12 non-isomorphic 4D ($l = 4$) real Lie algebras. Among them, 4 algebras have no Casimir functions (rank $r = 0$) and the remaining ones have rank $r = 2$. The former set of algebras do not has any constant of the motion coming from the coalgebra map, and therefore must be discarded in our approach.

For the remaining cases, the symplectic leaves of 4D algebras with rank 2 are 2D, so we can consider symplectic realizations with $s = 1$. In Table 3 we summarize some of them, and for the explicit expressions of the Casimir functions we refer to [8]. The functional independence of the integrals of the motion can be explicitly proven for all these cases. Note that the canonical transformation between both symplectic realizations of $A_{4,8}$ becomes evident.

Example 7. Let us now consider the oscillator algebra $A_{4,8}$ and the $\mathcal{H}$ function

$$\mathcal{H} = \lambda\, J_4 + \mu\, J_2\, J_3,$$

that gives rise under the realization $(*)$ with ($k_1 = 1$ and $k_2 = 0$) to the following integrable Hamiltonian

$$H^{(N)} = (\lambda + \mu) \sum_{i=1}^{N} p_i + 2\,\mu \sum_{i<j}^{N} \sqrt{p_i\, p_j}\, \cosh(q_i - q_j),$$

which is just the one introduced in [14]. The integrals of the motion in involution in the chosen realization read:

$$C^{(m)} = \sum_{i=1}^{m} p_i - \sum_{i<j}^{m} 2\, \sqrt{p_i\, p_j}\, \cosh(q_i - q_j).$$

Table 3. Symplectic realizations for some 4D Lie-Poisson algebras.

	J_1	J_2	J_3	J_4	
$A_{4,1}$	k_1	p	$\frac{p^2-k_2}{2k_1}$	$-k_1 q$	$k_1 \neq 0$
$A^{\alpha}_{4,2}$	$\frac{k_1^{\alpha}}{k_2} e^p$	$k_1 e^p$	$k_1 p\, e^p$	$-q$	$k_1 \neq 0$
$A_{4,3}$	e^p	k_1	$k_1\,(p - \log k_2)$	$-q$	$k_1 \neq 0$
$A_{4,4}$	$k_1 e^p$	$k_1 p\, e^p$	$\frac{k_1 e^p}{2}(p^2 + k_2)$	$-q$	$k_1 \neq 0$
$A^{a,b}_{4,5}$	$(k_1 a)^{\frac{1}{a}}\, e^p$	$a\, e^{ap}$	$\frac{(k_1 a)^{\frac{b}{a}}}{k_2}\, e^{bp}$	$-q$	$k_1, k_2 \neq 0$
$A_{4,8}$	k_1	q	$k_1\, p$	$-\frac{k_2}{2k_1} + \frac{(qp+pq)}{2}$	$k_1 \neq 0$
$(*)$	k_1	$\sqrt{p} e^q$	$k_1 \sqrt{p} e^{-q}$	$-\frac{k_2}{2k_1} + p$	$k_1 \neq 0$
$A_{4,10}$	k_1	$k_1 q$	p	$\frac{-p^2-k_1^2 q^2+k_2}{2k_1}$	$k_1 \neq 0$

Example 8. The algebra $A_{4,1}$ is the (1+1) extended Galilei Lie algebra, and their associated integrable systems have been constructed in [15], where their quantum deformations have also been analysed. Note that the problem of the classification of quantum deformations is only fully solved for all 3D Lie algebras and for some isolated cases in slightly higher dimensions (see [15] and references therein).

4.3 Higher dimensions

Non-isomorphic real Lie algebras of dimension 5 are also fully classified (we use the notation given in [8] for Mubarakzyanov results). There are 40 different Lie algebras with ranks $r = 1, 3$. Rank 3 cases admit one-body ($s = 1$) symplectic realizations (see some examples in Table 4) and lead to completely integrable systems [16]. The analysis of coalgebra systems coming from rank 2 algebras is more involved, since $s = 1, 2$ [16].

In higher dimensions, classifications of real Lie algebras are partial and restricted to certain simple, solvable or nilpotent subclasses (see [17] for a very interesting Poisson approach to this problem). From the point of view of coalgebra integrability it is immediate to realize that the rank r of the algebra grows much more slowly than its dimension l, and we are forced to find a symplectic realization with small s in order to get complete integrability from the coalgebra approach. For instance, if we consider the compact real forms of the Cartan series A_l, the rank is just $r = l$ whilst the dimension of the algebra grows with l^2. In particular, for $su(3)$ we have that $r = 2$ and we need (at most) a two-particle ($s = 2$) symplectic realization [16].

Another interesting example is provided by the 'two-photon' algebra h_6, a 6D Lie algebra with $r = 2$ (therefore $s = 1, 2$) that admits an $s = 1$ symplectic realization [3] for which, among the $2N$ integrals provided by the coalgebra, only $2N - 5$ of them are functionally independent and $N - 2$ are in involution. Hence, any Hamiltonian $\mathcal{H}$ with h_6-coalgebra symmetry is 'almost' integrable (only one constant is left), and such a remaining integral does exist for some special choices of $\mathcal{H}$ which are connected with the subalgebras of h_6 [3].

Table 4. Symplectic realizations for some 5D Lie-Poisson algebras.

	J_1	J_2	J_3	J_4	J_5	
$A_{5,1}$	k_1	k_2	$p+\frac{k_3}{k_2}$	$\frac{k_2}{k_1}\,p$	$-k_1 q$	$k_1, k_2 \neq 0$
$A_{5,2}$	k_1	p	$\frac{1}{2k_1}\left(p^2-k_2\right)$	$\frac{1}{6k_1^2}\left(p^3-3k_2p+2k_3\right)$	$-k_1 q$	$k_1 \neq 0$
$A_{5,3}$	k_1	k_2	p	$-k_2q-\frac{k_3}{2k_1}$	$-k_1q-\frac{p^2}{2k_2}$	$k_1, k_2 \neq 0$
$A_{5,7}^{a,b,c}$	e^p	$\frac{1}{k_1}e^{ap}$	$\frac{1}{k_2}e^{bp}$	$\frac{1}{k_3}e^{cp}$	$-q$	$k_1, k_2, k_3 \neq 0$
$A_{5,8}^{c}$	k_1	$k_1 p$	$k_3 e^p$	$\frac{k_3^c}{k_2}e^{cp}$	$-q$	$k_1, k_2, k_3 \neq 0$
$A_{5,9}^{b,c}$	$k_3 e^p$	$k_3 p e^p$	$\frac{k_3^b}{k_1}e^{bp}$	$\frac{k_3^c}{k_2}e^{cp}$	$-q$	$k_1, k_2, k_3 \neq 0$
$A_{5,10}$	k_1	$k_1 p$	$\frac{k_1}{2}p^2-\frac{k_2}{2k_1}$	$k_3 e^p$	$-q$	$k_1, k_3 \neq 0$
$A_{5,11}^{c}$	$k_2 e^p$	$k_2 p e^p$	$\frac{k_2}{2}e^p\left(p^2+k_3\right)$	$\frac{k_2^c}{k_1}e^{cp}$	$-q$	$k_1, k_2 \neq 0$
$A_{5,12}$	$k_1 e^p$	$k_1 p e^p$	$\frac{k_1}{2}e^p\left(p^2+k_2\right)$	$\frac{k_1 e^p}{6}\left(p^3+3k_2p+2k_3\right)$	$-q$	$k_1 \neq 0$
$A_{5,15}$	e^p	$(p-\log k_2)\,e^p$	$\frac{e^{\alpha p}}{k_1}$	$\frac{e^{\alpha p}}{\alpha k_1}\left(\alpha p-\log k_1k_3\right)$	$-q$	$k_1, k_2, k_3 \neq 0$

Acknowledgements

This work was partially supported by Ministerio de Educación y Ciencia (Spain, Project FIS2004-07913), by Junta de Castilla y León (Spain, Project VA013C05) and by INFN-CICyT (Italy-Spain).

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