Monografías de la Real Academia de Ciencias de Zaragoza. **29:** 9–16, (2006).

Symplectic Field Dynamics

Manuel Asorey

Departamento de Física Teórica, Facultad de Ciencias
Universidad de Zaragoza, 50009 Zaragoza, Spain
e-mail: asorey@unizar.es

Abstract

We analyse a new dynamical theory of symplectic connections with the most general second order lagrangian densities which only depends on the curvatures of symplectic connections and the symplectic structure itself. In spite of its space-time metric independence the theory is not topological but describes the dynamics of some elementary entities which are not relativistic massless particles. In this way the theory, which can be considered a generalisation of Maxwell and Einstein theories of Electromagnetism and Gravitation avoids the violation of known no-go theorems for helicity three fields

Keywords: *Higher spin fields, symplectic connections, gauge symmetries, massless particles*

1 Introduction

Symplectic manifolds became in the last decades the standard framework for the canonical description of classical mechanics [1, 2]. Although, the symplectic framework also seemed to be, in principle, very convenient for the description of quantum mechanics [3, 4] it was not easy to get a right picture [5, 6].

One of the essential ingredients of the Fedosov quantization by deformation program [5, 6] is the use symplectic connections[1]. A symplectic connection of a symplectic manifold (M, ω) is a linear connection in the tangent bundle TM which preserves the symplectic form, i.e.

$$\nabla\omega = 0\,.$$

The symplectic connection of a symplectic manifold is not unique even if we impose the torsionless condition. In this sense the symplectic geometry presents a behaviour very peculiar and different from that of Riemannian geometry. However, from a field theoretical point of view this fact is very interesting because permits the existence a new

[1] A further generalisation for Poisson manifolds is due by Konsevich [7]

type of field theories whose basic fields are symplectic connections of a given symplectic structure.

The similarity with the theory of Riemannian connections also suggest that symplectic connections might provide the clue for the formulation of a consistent theory of interacting relativistic massless helicity 3 particles in four dimensional manifolds.

The generalisation of the gauge principle for massless theories of higher spin has always been very elusive. Such a generalisation is, of course, possible for free particles in terms of 3-covariant symmetric tensors [8]. However, the introduction of any kind of interaction for higher helicity particles seem to be incompatible [9, 10, 11] with the gauge symmetry principles [12, 13, 14].

There had been many attempts to formulate, from the classical field theory viewpoint, consistent theories of self interacting higher helicity fields [15], either by modifying the field content of the theory or the gauge symmetry. But no consistent theory has been found involving a finite number of those massless fields in interaction with lower spin fields, including gravity [16, 17] or with themselves [12, 13, 14]. Partial progress has been, however, achieved by using infinite towers of massless fields [13, 14, 18, 19] or anti-de Sitter space-time backgrounds [20, 21, 22].

However, the theory of symplectic connections provides all required kinematic ingredients for a successful theory of helicity 3 fields. First, if the connections are torsionless they can be identified with symmetric 3-covariant tensors. On the other hand the fact that the fundamental fields are connections suggest the existence of a gauge principle similar to that of electromagnetic or gravitational fields [23].

2 Symplectic Connections and Massless Fields with helicity 3

Symplectic connections were introduced already in the forties [24, 25]. A more systematic global analysis was developed in the late fifties [26, 27] (see also Ref. [28]).

In recent years symplectic connections have appear playing a leading role in geometric quantization [3, 4, 29, 30], quantization deformations of symplectic structures [5, 6, 31, 32].

For any pair of symplectic connections ∇_1, ∇_2 of (M, ω) the difference

$$\Delta = \nabla_2 - \nabla_1 \tag{1}$$

defines a (1,2)-tensor field Δ such that for any pair of vector fields X, Y of M

$$\nabla_X Y = \nabla_X Y + \Delta_X Y. \tag{2}$$

If the torsion of ∇ vanishes

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0$$

the contraction of Δ with the symplectic form ω,

$$\widetilde{\Delta}(X, Y, Z) = \omega(X, \Delta_Y Z)$$

defines a symmetric (0,3) tensor $\widetilde{\Delta}$ with symmetric Young tableau

$$\ydiagram{3}\,.$$

Thus, if we fix a reference symplectic connection[2] ∇_* the space of torsionless symmetric connections [26, 27, 33, 34]

$$\mathcal{M} = \{\nabla; \nabla \text{ torsionless symplectic connection}\}$$

can be identified with the space of 3-covariant symmetric tensors

$$\mathcal{S} = \{\widetilde{\Delta}; \widetilde{\Delta}\ (0,3)\ \text{ symmetric tensor}\}.$$

The curvature tensor R of a torsionless symplectic connection defined by

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{3}$$

for any vector fields X, Y, Z defines a (1,3) tensor field which verifies the symmetry properties

$$\begin{gathered} R(X, Y, Z) = -R(Y, X, Z) \\ R(X, Y, Z) + R(Z, X, Y) + R(Y, Z, X) = 0 \\ \nabla_X R(Y, Z) + \nabla_Z R(X, Y) + \nabla_Y R(Z, X) = 0. \end{gathered} \tag{4}$$

The contraction of R with the symplectic form ω defines a (0,4) tensor

$$\widetilde{R}(W, X, Y, Z) = \omega(W, R(Y, Z)X) \tag{5}$$

with symmetry properties

$$\begin{gathered} \widetilde{R}(W, X, Y, Z) = -\widetilde{R}(W, X, Z, Y) \\ \widetilde{R}(W, X, Y, Z) = \widetilde{R}(X, W, Y, Z) \\ \widetilde{R}(W, X, Y, Z) + \widetilde{R}(Z, W, X, Y) + \widetilde{R}(Y, Z, W, X) + \widetilde{R}(X, Y, Z, W) = 0 \\ \nabla_V \widetilde{R}(W, X, Y, Z) + \nabla_Y \widetilde{R}(W, V, X, Z) + \nabla_X \widetilde{R}(W, Y, V, Z) = 0 \end{gathered} \tag{6}$$

which corresponds to a Young tableau

$$\ydiagram{3,1}$$

in contrast with that of the standard Riemann tensor

[2] If M is paracompact such connections always exists.

$$\ydiagram{2,2}\,.$$

A symplectic Ricci tensor can be defined in analogy with Riemannian Ricci tensor by tracing out the the curvature tensor

$$R_c(X,Y) = \operatorname{Tr} R(X,\cdot)Y.$$

In this case R_c it is dependent on the choice of the symplectic connection. But any symplectic Ricci tensor is also symmetric

$$R_c(X,Y) = R_c(Y,X)$$

with Young diagram

$$\ydiagram{2}\,.$$

Since R_c is a symmetric tensor it is not possible to define a non-trivial symplectic scalar curvature term

$$R_s = \operatorname{Tr}\operatorname{Tr}\omega^{-1}\otimes R_c = 0.$$

3 Symplectic Field Theory

The simplest non-trivial scalars which can be associated to symplectic connections are

$$\mathcal{L}_c(\nabla,\omega) = \operatorname{Tr}\operatorname{Tr} R_c^* R_c \qquad \mathcal{L}(\nabla,\omega) = \operatorname{Tr}\operatorname{Tr}\operatorname{Tr}\operatorname{Tr} R^* R, \tag{7}$$

where R^* and R_c^* are the dual (2,0) and (4,0) tensor fields of R and R^c with respect to the symplectic structure ω. In local coordinates, they are defined by

$$R^{ijkl} = \omega^{ii'}\omega^{jj'}\omega^{kk'}\omega^{ll'}R_{i'j'k'l'} \qquad R_c^{ij} = \omega^{ii'}\omega^{jj'}R_{ci'j'} \tag{8}$$

ω^{ij} being the inverse matrix of $\omega_{ij} = \omega(\partial_i,\partial_j)$. There is an extra scalar density associated to the $2n$–dimensional symplectic manifold, the volume form

$$\Omega = \overbrace{\omega\wedge\omega\wedge\cdots\wedge\omega}^{n}. \tag{9}$$

Both $\mathcal{L}$ or $\mathcal{L}_c$, can be used to define a field theory because they are quadratic in derivatives of symplectic connection fields. The corresponding actions

$$S(\nabla,\omega) = \int_M \Omega \operatorname{Tr}\operatorname{Tr}\operatorname{Tr}\operatorname{Tr} R^* R, \quad S_c(\nabla,\omega) = \int_M \Omega \operatorname{Tr}\operatorname{Tr} R_c^* R_c \tag{10}$$

are however not independent in four-dimensional symplectic manifolds M, because in that case there is combination of both terms

$$S(\nabla,\omega) - 2S_c(\nabla,\omega) \tag{11}$$

which is proportional to a topological invariant

$$P(\nabla,\omega)=\frac{1}{8\pi^2}\int_M \Omega\,\left(\mathrm{Tr}\,\mathrm{Tr}\,\mathrm{Tr}\,\mathrm{Tr}\,R^*R-2\,\mathrm{Tr}\,\mathrm{Tr}\,R^*_cR_c\right), \tag{12}$$

the first Poyntrjagin class of ∇. This means that the classical motion equations of the actions

$$\omega^{i'i}\nabla_{i'}\widetilde{R}_{ijkl}+\omega^{i'i}\nabla_{i'}\widetilde{R}_{iklj}+\omega^{i'i}\nabla_{i'}\widetilde{R}_{iljk}=0 \tag{13}$$

and

$$\nabla_i R^c_{jk}+\nabla_j R^c_{ki}+\nabla_k R^c_{ij}=0 \tag{14}$$

are equivalent [35] However, from a quantum point of view. the topological term might have some physical effects giving rise to a θ–vacuum term

$$S_q(\nabla,\omega)=\frac{1}{2\alpha_0}S(\nabla,\omega)+\frac{\theta}{2\pi}P(\nabla,\omega) \tag{15}$$

which in local coordinates reads

$$S_q(\nabla,\omega)=\frac{1}{2{\alpha_0}^2}\int_M R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}+\frac{\theta}{32\pi^2}\int_M\left[R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}-2\,R_{\mu\nu}R^{\mu\nu}\right] \tag{16}$$

and only involves the curvature tensors of the connections and the symplectic form. The metric independence of (16) implies that the dynamics of the symplectic fields is completely decoupled from the gravitation. The theory is invariant under symplectomorphisms, i.e. diffeomorphisms which preserve ω. In local coordinates symplectomorphisms are infinitesimally generated by a scalar function ϕ such that $\xi^\mu=\omega^{\mu\nu}\partial_\nu\phi$. Symplectic connections are however transformed in this approximation as

$$\Delta'_{\mu\nu\sigma}=\Delta_{\mu\nu\sigma}+\partial_\mu\partial_\nu\partial_\sigma\phi, \tag{17}$$

and the action (16) is invariant.

The symplectomorphic gauge symmetry indicates the existence of dynamical constraints. But, the Cauchy problem is more degenerate because of the existence of many zero modes in the quadratic terms which are not associated to any known gauge symmetry [23].

4 Conclusions

Although the above theory of symplectic connections is metric independent it is not a topological theory. The reason is that this theory is only invariant under symplectomorphisms but not under general diffeomorphisms. Therefore, the theory describes the dynamics of some field theoretical degrees of freedom which are not merely topological

On the other hand since the symplectic background form is not preserved by Poincaré transformations, the degrees of freedom of the theory do not carry a relativistic particle interpretation. The only way of associating a particle–like interpretation of symplectic connections is by introducing couplings to space-time pseudo-riemannian metrics and to consider the symplectic form ω itself a dynamical field. In this manner when the space-time metric background is the Minkowski metric the theory becomes Poincaré invariant [23, 36]. However, in this approach although symplectic connections can be classically related to massless particles, quantum effects generate masses for symplectic connections fields because there is no gauge symmetry matching principle.

Acknowledgements

I thank Jesús Clemente, David García-Álvarez and José María Muñoz-Castañeda for enlightening discussions on related subjects. It is a pleasure to contribute with this short note to the volume in honour of my friend Pepin Cariñena, who while I was undergraduate student at Valladolid University astutely introduced me to the charm and power of modern differential geometry and group theory. This work was partially supported by CICYT (grant FPA2004-02948) and DGIID-DGA (grant2005-E24/2).

References

[1] J.M. Souriau, *Structure des Systémes Dynamiques*, Dunod, Paris (1969)

[2] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Addison-Wesley, Reading, Massachussets (1978)

[3] A. Lichnerowicz, In:*Quantum theory, Groups, Fields and Particles.* (Barut, A.O., ed.), Reidel, Dordrecht (1983)

[4] H. Hess, In *Differential Geometric Methods in Math. Physics*, Lect. Notes in Math. **836**, Springer-Verlag (1980) 153

[5] B. V. Fedosov, J. Diff. Geom. **40** (1994) 213

[6] B. V. Fedosov, *Deformation quantization and index theory* , Akademie Verlag, Berlin (1996)

[7] M. Konsevich, Lett. Math. Phys. **66** (2003) 157-216

[8] M. Asorey, L. J. Boya and J. F. Cariñena, Rep. Math. Phys. **21** (1985) 391

[9] S. R. Coleman and J. Mandula, Phys. Rev. **159** (1967) 1251

[10] S. Weinberg, Phys. Rev. **135** (1964) B1049

[11] R. Haag, J. T. Lopuszanski and M. Sohnius, Nucl. Phys. **B 88** (1975) 257

[12] A. K. H. Bengtsson and I. Bengtsson, Class. Quant. Grav. **3** (1986) 927

[13] A. K. H. Bengtsson, Phys.Rev. **D32**(1985)2031

[14] F. A. Berends, G. J. H. Burgers and H. Van Dam, Z. Phys. **C 24** (1984) 247; Nucl. Phys. **B260** (1985) 295

[15] C. Fronsdal, Phys. Rev. **D18** (1978) 3624

[16] C. Aragone and S. Deser, Phys. Lett. **B 86** (1979) 161

[17] F. Berends, J. van Holten, P. van Nieuwenhuizen and B. de Wit , J. Phys. **A 13** (1980) 1643

[18] B. de Wit and D. Z. Freedman, Phys. Rev. D **21** (1980) 358

[19] J. M. F. Labastida, Nucl. Phys. **B322** (1989) 185

[20] E. S. Fradkin and M. A. Vasiliev, Nucl. Phys. **B 291** (1987) 141; Phys. Lett. **B 189** (1987) 89

[21] V. E. Lopatin and M. A. Vasiliev, Mod. Phys. Lett. A **3** (1988) 257

[22] M. A. Vasiliev, Nucl. Phys. **B 301** (1988) 26

[23] M. Asorey, D. García-Álvarez, J. M. Muñoz-Castañeda, Monog. SME (2006)

[24] H.-C. Lee, Amer. Journal Math. **65** (1943) 433; **67** (1945) 321

[25] P. Libermann , Ann. Mat. Pura Appl. **36** (1954) 27

[26] V. G. Lemlein, Doklady Akademii nauk SSSR **115** (1957), 655(in Russian)

[27] Ph. Tondeur, Comm.. Math. Helvetici **36** (1961) 234

[28] I. Vaisman, Monatshefte für Mathematik **100** (1985) 299

[29] B. Konstant,*Quantization and Unitary Representations* Lecture Notes in Mathematics, bf 170 (1970)87-208

[30] J. Śniatycki, *Geometric Quantization and Quantum Mechanics*, Springer-Verlag (1980).

[31] J. G. Clemente, Ph. D. dissertation (2000)

[32] B. Geyer and P. Lavrov, Int. J. Mod. Phys. **A 19** (2004) 3195

[33] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Ann. Phys. **111** (1978) 61

[34] I. Gelfand, V. Retakh and M. Shubin, Adv. Math. **136** (1998) 104

[35] P. Bourgeois and M. Cahen, J. Geom. Phys. **30** (1999) 233

[36] D. García-Álvarez, Ph. D. dissertation (2006)

[37] S. J. Chang, Phys. Rev. **D 9** (1967) 1306