Monografías de la Real Academia de Ciencias de Zaragoza. **29:** 1–8, (2006).

Two spectral parameter transfer matrix and alternating spin systems

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Abstract

A monodromy matrix depending on two spectral parameters that fulfils a Yang-Baxter equation is presented. The Bethe Ansatz method is used to determine the eigenvalues and eigenvectors of the associated transfer matrix. The relation with one-dimensional spin systems with alternating coupling is shown.

PACS: *75.10Jm, 05.50.+q and 02.10.Xm*

The physics of the one dimensional chains provides an excellent ground for a rigourous study of different properties of solvable models and their experimental verification in quantum magnetic systems. It is known that these models are useful analysing many body problems and recently, they have been applied to the study of 1D systems model of real atoms that have become to be the object of experiments.

One of the most popular model is the spin 1/2 chain

$$H = \sum_{i}^{2N} J \ \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} \ , \tag{1}$$

proposed by W. Heisenberg in 1928 [1] and solved by H. Bethe [2] using the well known Bethe Ansatz method.

Later, Faddeev and collaborators [3] solved that model by applying the Quantum Inverse Scattering Method (QISM). The generalization of the method gives a family of solvable models one of which is the Heisenberg model(1). A very good review of this method can be found in [4].

A solvable model is a system where a complete set of commuting operators can be found. In the QISM a monodromy matrix $T(\lambda)$ is defined in an auxiliary space, such matrix depends on the parameter λ which is known as the spectral parameter. Every element of this matrix is an operator acting in the space of the states of the system. The method shows that the model is solvable if there is a matrix $R(\mu)$ in the tensorial product of two auxiliary spaces that verifies the $Yang - Baxter$ equation

$$R(\lambda-\mu)(T(\lambda)\bigotimes T(\mu)) = (T(\mu)\bigotimes T(\lambda))R(\lambda-\mu) \ . \tag{2}$$

The Trace of $T(\lambda)$ in the auxiliary space gives the transfer matrix $F(\lambda)$ and the Hamiltonian is given by the derivative,

$$\left.\frac{d\ln F}{d\lambda}\right|_{\lambda=\frac{i}{2}} \ . \tag{3}$$

The successive derivatives give the set of commuting hermitian operators.

Recently it has been discovered a Spin-Pierls transition in the inorganic compounds $CuGeO_3$ [5] and $\alpha' - NaV_2O_5$ [6] which, in general, seems well described by an antiferromagnetic Heisenberg chain with dimerization. If we allow the spin-phonon interaction in the one dimensional $S = 1/2$ Heisenberg chain, it shows a spontaneous lattice dimerization where the strength of the antiferromagnetic coupling alternates between strong and weak values. To the study of those physical systems it has been applied several methods such as bosonization [7] and variatitonal methods between others. [8]

The hamiltonian which describes a chain with $2N$ sites and a nearest neighbour interaction of spins with alternating coupling constant can be writen

$$H = \sum_{i}^{2N} J(1+(-1)^i u)\vec{\sigma}_i \cdot \vec{\sigma}_{i+1} \ , \tag{4}$$

where σ^i are the Pauli matrices and we suppose periodic boundary conditions

$$\vec{\sigma}_{2N+1} = \vec{\sigma}_1 \ . \tag{5}$$

In the present paper we find a monodromy matrix depending on two spectral parameter from which the hamiltonian (4) is derived so we can apply a method equivalent to that of the Bethe Ansatz. Unfortunately, the final result shows that (4) is not solvable but the expressions obtained can be useful in other approaches.

The hamiltonian (4) can be obtained from a transfer matrix. To do that, we first associate with the hamiltonian an auxiliary linear problem formulated in quantum terms. Let $L_i(\lambda, s)$ be a local matrix operator depending on two parameters, which acts on an auxiliary 2-dimensional space and on the two dimensional space of states for each site. Writing in components, it is

$$L_i(\lambda,s)_{(a,b),(\alpha_i,\beta_i)} = \lambda\delta_{a,b}\delta_{\alpha_i,\beta_i} + \frac{i}{2}s\sum_{l=1}^{3}\sigma^l_{a,b}\sigma^l_{\alpha_i,\beta_i} \ , \tag{6}$$

where the Latin indexes are in the auxiliary space and Greek indexes are for the space of states in the corresponding site. In terms of deltas and taking into account that

$$\vec{\sigma}_{a,b}\vec{\sigma}_{\alpha,\beta} = 2\delta_{a,\alpha}\delta_{b,\beta} - \delta_{a,b}\delta_{\alpha,\beta} \ , \tag{7}$$

L_i can be written as,

$$L_i(\lambda, s)_{(a,b),(\alpha_i,\beta_i)} = (\lambda - \frac{i}{2}s)\delta_{a,b}\delta_{\alpha_i,\beta_i} + is\delta_{a,\alpha_i}\delta_{b,\beta_i} \ . \tag{8}$$

We group by pairs the $2N$ sites and define the two sites operator by the product in the auxiliary space

$$M_i(\lambda_1, s_1, \lambda_2, s_2)) = L_{2i-1}(\lambda_1, s_1)L_{2i}(\lambda_2, s_2) \ , \tag{9}$$

and we proceed to consider the product of the M matrices in the auxiliary space

$$T(\lambda_1, s_1, \lambda_2, s_2) = \prod_i^N M_i(\lambda_1, s_1, \lambda_2, s_2) \ , \tag{10}$$

which is a 2×2 matrix in the auxiliary space and it is the monodromy matrix and can be written in the general form

$$T(\lambda_1, s_1, \lambda_2, s_2) = \begin{pmatrix} A(\lambda_1, s_1, \lambda_2, s_2) & B(\lambda_1, s_1, \lambda_2, s_2) \\ C(\lambda_1, s_1, \lambda_2, s_2) & D(\lambda_1, s_1, \lambda_2, s_2) \end{pmatrix} \ , \tag{11}$$

where A, B, C and D are operators acting on the space of states of the total chain. The hamiltonian belongs to this family of operators and is obtained from the trace F of this operator,

$$F(\lambda_1, s_1, \lambda_2, s_2) = \mathrm{trace}\,(T) \ = A(\lambda_1, s_1, \lambda_2, s_2) + B(\lambda_1, s_1, \lambda_2, s_2) \ , \tag{12}$$

from which we can obtain the main physical operators of the system. So, taking the definition of $F = \mathrm{trace}\,(T)$, we obtain for the indicated values of λ_1 and λ_2,

$$F(\lambda_1 = \frac{i}{2}s_1, s_1, \lambda_2 = \frac{i}{2}s_2, s_2) = (-s_1 s_2)^N \delta_{\alpha_1,\beta_2} \cdot \delta_{\alpha_2,\beta_3} \cdots \quad \cdot\delta_{\alpha_{2N-1},\beta_{2N}} \cdot \delta_{\alpha_{2N},\beta_1} \ , \tag{13}$$

that is the one-side translation operator which can be related to the momentum P by,

$$F(\lambda_1 = \frac{i}{2}s_1, s_1, \lambda_2 = \frac{i}{2}s_2, s_2) = (-s_1 s_2)^N \exp{(-iP)} \ . \tag{14}$$

.

The hamiltonian is obtained from the derivative of the function logarithm of F. Then we define the operators,

$$I_i = \delta_{a,b}\delta_{\alpha_i,\beta_i} \ , \qquad Q_i = \delta_{a,\beta_i}\delta_{\alpha_i,b} \ , \tag{15}$$

$$\left(\frac{\partial}{\partial\lambda_1}M_l + \frac{\partial}{\partial\lambda_2}M_l\right)\Big|_{\substack{\lambda_1=is_1/2\\ \lambda_2=is_2/2}} = is_2 I_{2l-1}\cdot Q_2 l + is_2 Q_{2l-1} I_{2l} \ , \tag{16}$$

where the $\cdot$ product is understood in the auxiliary space. In the same form we perform the derivative of the monodromy matrix T,

$$\begin{aligned}\left(\frac{\partial}{\partial\lambda_1}T + \frac{\partial}{\partial\lambda_2}T\right)\Big|_{\substack{\lambda_1=is_1/2\\ \lambda_2=is_2/2}} &= -i(s_1 s_2)^{N-1}\Big[\sum_h^N s_2 (\prod_{\cdot l=1}^{2h-2} Q_l \cdot I_{2h-1}\cdot \prod_{\cdot l=2h}^{2N} Q_l) \\ &\quad + \sum_h^N s_1 (\prod_{\cdot l=1}^{2h-1} Q_l \cdot I_{2h}\cdot \prod_{l=2h+1}^{2N} Q_l)\Big] \ , \end{aligned}\tag{17}$$

where again the $\cdot$ product is in the auxiliary space. Finally taking the *trace* and multiplying in the site spaces by F^{-1}, we obtain,

$$H = \frac{1}{F}\left(\frac{\partial}{\partial\lambda_1}F + \frac{\partial}{\partial\lambda_2}F\right)\Big|_{\substack{\lambda_1=is_1/2\\ \lambda_2=is_2/2}} = -i\sum_{h=1}^N \left(\frac{1}{s_1}\delta_{\alpha_{2h-1}\beta_{2h-2}}\delta_{\alpha_{2h-2}\beta_{2h-1}} + \frac{1}{s_2}\delta_{\alpha_{2h}\beta_{2h-1}}\delta_{\alpha_{2h-1}\beta_{2h}}\right) . \tag{18}$$

Using (7), we obtain the hamiltonian written in the habitual form.

$$H = \frac{i}{2}J\sum_l^{2N}(1+(-1)^l u)(\vec{\sigma}^l\vec{\sigma}^{l+1} - I) \ , \tag{19}$$

with

$$J = \frac{1}{2s_1} + \frac{1}{2s_2}, \quad u = \frac{1}{2J}\left(\frac{1}{s_1} - \frac{1}{s_2}\right) \ . \tag{20}$$

In order to find the eigenvalues and eigenstates of the transfer matrix we define first the reduced $\tilde{T}$ matrix.

$$\tilde{T}(\mu = \mu^1, x = \mu^1 - \mu^2) = T(\mu^1 = \frac{\lambda_1}{s_1}, 1, \mu^1 = \frac{\lambda_2}{s_2}, 1) = \frac{1}{(s_1 s_2)^N} T(\lambda_1, s_1, \lambda_2, s_2) \ . \tag{21}$$

If we define, in a similar way, the operator $\tilde{M}$,

$$\tilde{M}(\mu = \mu^1, x = \mu^1 - \mu^2) = M(\mu^1 = \frac{\lambda_1}{s_1}, 1, \mu^1 = \frac{\lambda_2}{s_2}, 1) \ , \tag{22}$$

then $\tilde{M}$ satisfies the Yang-Baxter equation,

$$R(\lambda-\mu)\cdot(\tilde{M}(\lambda,x)\otimes\tilde{M}(\mu,x)) = (\tilde{M}(\mu,x)\otimes\tilde{M}(\lambda,x))\cdot R(\lambda-\mu) \ , \tag{23}$$

where R is the 4×4 matrix in the space tensorial product of the two auxiliary spaces.

$$R(\lambda) = c(\lambda)Q + b(\lambda)I\otimes I \ , \tag{24}$$

being c and b the functions,

$$b(\lambda) = \frac{i}{\lambda + i} \ , \qquad c(\lambda) = \frac{\lambda}{\lambda + i} \ , \tag{25}$$

and the Q operator,

$$Q = \frac{1}{2}(I \otimes I + \vec{\sigma} \otimes \vec{\sigma}) \ , \tag{26}$$

is the same that in (15) but now it acts in the tensorial product of the two auxiliary spaces. The $\tilde{T}$ operator enjoys most of the properties of $\tilde{M}$, in particular the Yang-Baxter equation.

$$R(\lambda - \mu) \cdot (\tilde{T}(\lambda, x) \otimes \tilde{T}(\mu, x)) = (\tilde{T}(\mu, x) \otimes \tilde{T}(\lambda, x)) \cdot R(\lambda - \mu) \ , \tag{27}$$

that is the simple algebraic analog of the very difficult calculations in the Hamiltonian approach. From this relation (27) we derived the important commutation relations for the operators $\tilde{A}, \tilde{B}$ and $\tilde{D}$ in (11).

$$[\tilde{B}(\lambda, x), \tilde{B}(\mu, x)] = 0 \ , \tag{28}$$

$$\tilde{A}(\lambda, x)\, \tilde{B}(\mu, x) = \frac{1}{c(\mu - \lambda)} \tilde{B}(\mu, x)\tilde{A}(\lambda, x) - \frac{b(\mu - \lambda)}{c(\mu - \lambda)} \tilde{B}(\lambda, x)\tilde{A}(\mu, x) \ , \tag{29}$$

$$\tilde{D}(\lambda, x)\ \tilde{B}(\mu, x) = \frac{1}{c(\lambda - \mu)} \tilde{B}(\mu, x)\tilde{D}(\lambda, x) - \frac{b(\lambda - \mu)}{c(\lambda - \mu)} \tilde{B}(\lambda, x)\tilde{D}(\mu, x). \tag{30}$$

Now we present the properties of the eigenstates of the F operator and their associated eigenvalues. We begin with the state $|\Omega >$ that is defined as the state of the chain in which every site is with a z-spin component equals to $+1/2$,

$$|\Omega >= \prod_{\otimes i}^{2N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \ . \tag{31}$$

It is eigenstate of both $\tilde{A}(\lambda, x)$ and $\tilde{D}(\lambda, x)$ and consequently $|\Omega >$ is also eigenstate of $\tilde{F}(\lambda, x) = \tilde{A}(\lambda, x) + \tilde{D}(\lambda, x)$. Their values are

$$\tilde{A}(\lambda, x)|\Omega > = (\lambda + \frac{i}{2})^N (\lambda + x + \frac{i}{2})^N |\Omega > \ , \tag{32}$$

$$\tilde{D}(\lambda, x)|\Omega > = (\lambda - \frac{i}{2})^N (\lambda + x - \frac{i}{2})^N |\Omega > \ . \tag{33}$$

Following the habitual procedure, the $\tilde{B}(\lambda, x)$ can be interpreted from (29) as a kind of creation operator, then we want construct eigenstates of the form,

$$|\phi(\lambda_1, \cdots, \lambda_r, x) >= \tilde{B}(\lambda_1, x) \cdots \tilde{B}(\lambda_r, x)|\Omega > \ , \tag{34}$$

where x is fixed and $\lambda_1, \cdots \lambda_r$ are arbitrary parameters to be determinate later.

Now we apply $\tilde{A}(\lambda, x)$ to $|\phi(\lambda_1, \cdots, \lambda_r, x) >$. The action is found by pushing it to the right through the $\tilde{B}$'s using the commutation relation (29) r times.

$$\tilde{A}(\lambda, x)|\phi(\lambda_1, \cdots, \lambda_r, x) >= \prod_j^r \frac{1}{c(\lambda_j - \lambda)} \quad \tilde{B}(\lambda_1, x) \quad \cdots \tilde{B}(\lambda_r, x)\tilde{A}(\lambda, x)|\Omega > + \cdots$$
$$+\text{Unwanted} \quad \text{terms} \ldots \quad . \qquad (35)$$

Two type of terms arise when $\tilde{A}$ goes through the $\tilde{B}$'s, the first is the wanted term where $\tilde{A}$ and $\tilde{B}$ operators retain their respective arguments λ and λ_i that comes from the fist term of the right hand side of (29) . The other terms are called unwanted and they are characterized by having the arguments of the $\tilde{A}$ and $\tilde{B}$ operators interchanged. A typical unwanted term is,

$$-\frac{b(\lambda_k - \lambda)}{c(\lambda_k - \lambda)} \quad \prod_{j \neq k}^r \quad \frac{1}{c(\lambda_j - \lambda_k)} \tilde{B}(\lambda_1, x) \cdots$$
$$\tilde{B}(\lambda_{k-1}, x)\tilde{B}(\lambda, x)\tilde{B}(\lambda_{k+1}, x) \cdots \tilde{B}(\lambda_r, x)\tilde{A}(\lambda_k, x)|\Omega > \quad . \qquad (36)$$

Similar relation are obtained for the D operator

$$\tilde{D}(\lambda, x)|\phi(\lambda_1, \cdots, \lambda_r, x) >= (\lambda - \frac{i}{2})^N(\lambda + x - \frac{i}{2})^N \quad \prod_j^r \frac{1}{c(\lambda_j - \lambda)} |\phi(\lambda_1, \cdots, \lambda_r, x) > + \cdots$$
$$+ \qquad \text{Unwanted} \quad \text{terms} \ldots \quad . \qquad (37)$$

The eigenvalues for F are obtaining by adding (35) and (37) and requiring that the unwanted terms cancel term by term. This cancellation will lead to r algebraic equations involving the parameters $\lambda_1, \cdots, \lambda_r$, the Bethe ansatz equations (BAE) for our model. Then we have,

$$\tilde{F}(\lambda, x)|\phi(\lambda_1, \cdots, \lambda_r, x) >= \tilde{\Lambda}(\lambda, x, \lambda_1, \cdots, \lambda_r)|\phi(\lambda_1, \cdots, \lambda_r, x) > , \qquad (38)$$

with

$$\tilde{\Lambda}(\lambda, x, \lambda_1, \cdots, \lambda_r) \quad = \quad (\lambda + \frac{i}{2})^N(\lambda + x + \frac{i}{2})^N \prod_j^r \frac{1}{c(\lambda_j - \lambda)} +$$
$$+ (\lambda - \frac{i}{2})^N(\lambda + x - \frac{i}{2})^N \prod_j^r \frac{1}{c(\lambda - \lambda_j)} \quad . \qquad (39)$$

The BAE can be obtained by an equivalent method. We can note that the $\tilde{T}$ matrix is an analytic function of λ, in fact it is a polynomial of λ, and as such it cannot have a pole. The $c(\lambda_j - \lambda)$ function has a zero in $\lambda_j = \lambda$, then the residue in that point must be cero, that condition gives r equations,

$$(\lambda_j+\frac{i}{2})^N(\lambda_j+x+\frac{i}{2})^N\prod_{k\neq j}^{r}\frac{1}{c(\lambda_k-\lambda_j)}+(\lambda_j-\frac{i}{2})^N(\lambda_j+x-\frac{i}{2})^N\prod_{k\neq j}^{r}\frac{1}{c(\lambda_j-\lambda_k)}=0\ . \tag{40}$$

If we substitute the function $c(\lambda_j-\lambda_k)$ this expression becomes,

$$\frac{(\lambda_j-\frac{i}{2})^N(\lambda_j+x-\frac{i}{2})^N}{(\lambda_j+\frac{i}{2})^N(\lambda_j+x+\frac{i}{2})^N}=\prod_{k\neq j}^{r}\frac{\lambda_j-\lambda_k-i}{\lambda_j-\lambda_k+i}\quad,\qquad j=1,\ldots,r\ , \tag{41}$$

that are the BAE for our system. For finite N, this set of algebraic equations does not have analytic solutions, but for $N\to\infty$ the equations are simpler and explicit solutions can be obtained for fixed x, as can be found, for example in [4, 9]

The function $\ln(\tilde{F}(\lambda,x))$ has as eigenvalues the logarithm $\ln(\tilde{\Lambda}(\lambda,x,\lambda_1,\cdots,\lambda_r)$ and the same eigenstates. They are determined by the solutions of the BAE (41).

We can apply the operator derivative shown in equation (18) to eigenvalue equations (38) but the result is not an eigenvalue equation for the Hamiltonian (4), because it has one more term due to dependence in the parameter x of the eigenstate, which is related to the λ_2 parameter in the derivative.

It is known that the Hamiltonian (4) is not solvable as we have said in the beginning of this paper, but this method can relate this problem with other systems. One orientation of the problem can be to find, from the same monodromy matrix, other derivative combination independent of the one shown in (4) and look for a new relation of the operators. The study the properties of interplay between this possible set of operators is leaved as an open problem for a future work.

Acknowledgements

This work was partially supported by the Dirección General de Investigación Científica y Técnica, Grant No BFM2003/01300 and by the Diputación General de Aragón, E24/1.

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