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Some Remarks on the Treatment of Quasi–Keplerian Systems

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Abstract

The unperturbed Kepler problem has been investigated in great detail, and solved in the light of very different formulations and methods. As expected and well known, the conclusion is drawn that the orbit equation corresponds to a conic section having a focus at the force centre. In this Note, some of those methods are confronted with a class of somewhat more general force models, namely, the so–called quasi–Keplerian systems and some of their possible generalizations.

Key words: Two–body motion; gravitational attraction; Keplerian, quasi–Keplerian and generalized quasi–Keplerian systems; central force field; perturbations.

1 The Why of This Paper

In the present paper we are specially concerned with the analytical treatment of certain (perhaps very simple) conservative, central–force fields –or, equivalently, the corresponding potentials– related to mechanical/dynamical systems which are very close to the standard Keplerian systems: roughly speaking, they *quantitatively* differ form the conventional, pure Kepler problem by the effect of perturbations of a *small or moderate amount*, in such a way that the geometrical and dynamical Keplerian aesthetics and picture characterizing the unperturbed, gravitational two–body motion are *slightly distorted*.

The type of systems of interest has been (and is still) important for the analytical and qualitative investigation of phenomena whose formulation and description can be modelled or approximated by means of adequate *perturbations* superimposed on the inverse–square–attraction law, specially within the framework of Analytical Mechanics and –in many particular cases– Celestial Mechanics and Astrodynamics.

The simplest, gravitational two–body problem (in the absence of any perturbation) has been thoroughly and extensively studied throughout several centuries, and a wealth of information can be found in the literature dealing with this subject: a plethora of general methods of different nature, special tricks and *ad hoc* techniques has been applied (or even deliberately devised) to formulate, reduce and solve this problem.

We are interested in studying the *performance of some of these methods* when confronted with certain *perturbed Keplerian systems.* In particular, we restrict ourselves to the consideration of some simple cases of perturbations which are represented by central–force fields with a special functional dependence: *powers of the separation distance* between the moving bodies, since (apart from physical considerations concerning the accuracy and practical usefulness of these models) special choices for the value of the exponent at issue will allow us to recognize –from a mathematical point of view– or establish some formal analogies with the conventional Keplerian reference solution and approach.

For these reasons, we concentrate on central–force laws of the type r^{-n}, more general than the inverse–square force law and the one giving rise to the so–called *quasi–Keplerian systems* (following the terminology of Deprit 1981, §4) and some of their possible variants and generalizations (Floría 1993).

As far as we know (and we have not found any deep reason for this fact, apart from some *technical intricacies of a mathematical nature*), many methods usually considered to reduce and solve the Kepler problem seem not to have been applied to perturbed Keplerian systems. We find it interesting to take this step, at least in order to check and reveal the *possibilities and limitations* of these methods when the force model becomes somewhat more general (and involved) than the inverse-square gravitational attraction. This approach and way of proceeding might be helpful while trying to detect and characterize cases in which a certain method could reveal some favourable performance.

Perturbations disturb motion, distort the reference solution obtained with the help of some technique. But they also damage (or, perhaps, even destroy) our ability to describe and interpret phenomena. Perturbations represent a challenge for our ability to describe and understand processes, they challenge our methods and techniques, and for this reason we are forced to modify the available methods, to adapt them to new problems . . . or to create new strategies to attack and overcome the difficulties raised by the incorporation of perturbations into our solvable (and solved) model problems.

Many other methods and approaches can be found in the literature. At the present level, our study is not exhaustive: in this Note we do not aim at the most perfect and final completeness and generality of our conclusions. We just draw attention to the fact that the potentiality of the said (and other) methods seems not to have been fully exploited yet. Other techniques, combined with other force models, might have been included in the study. We intend to complete these preliminary comments with some future results of our analysis of different methods and dynamical problems.

Apart from their possible interest in research, these comments might also become useful for teaching purposes, as special topics in subjects such as Differential Equations, Classical Mechanics, Celestial Mechanics and Astrodynamics.

2 On Some Previous Studies on Quasi–Keplerian (and Related) Systems

There exist some previous analytical treatments of quasi–Keplerian systems in the literature. Some of them are based on the use of *reducing canonical transformations* within the framework of Hamiltonian Celestial Mechanics.

Deprit (1981, §4) considered *quasi–Delaunay transformations* and *torsions* for the simplest model of quasi–Keplerian systems. Deprit himself (1981, §7, §9) resorted to torsion–type transformations to study and solve the generalized quasi–Keplerian Hamiltonian represented by his radial intermediary for the Main Problem in Artificial Satellite Theory.

Floría (1994a) generalized Deprit's concept of quasi–Delaunay mappings, applied this kind of transformations to obtain generalized Delaunay–like canonical orbital elements, and reduced and solved the chain of radial intermediaries defined by Deprit (1981).

Other sets of generalized canonical orbital elements (of the Jacobi and of the Delaunay–Scheifele or Delaunay–Similar type), and the corresponding canonical solutions to generalized quasi–Keplerian Hamiltonians, were proposed by Floría (1993, 1994b).

Another approach has also been taken: Convert the classical equations of motion (derived from Newton's laws, or obtained in a more sophisticated Lagrangian or Hamiltonian formulation of the problem) into harmonic oscillator equations. Unperturbed, spatial Keplerian systems are exactly reduced to uncoupled and unperturbed harmonic oscillators (the only forcing term –if any– in one of the equations is just a constant term).

In this approach, one can make use of *regularization and linearization* techniques, by combining certain transformations introducing redundant dependent variables and reparametrizations of the motion with the help of an adequate fictitious time defined via a Sundman–type differential transformation . For instance, the results due to Ferrándiz & Fernández–Ferreirós (1991) concerning Deprit–type radial intermediaries, in terms of the so–called Burdet–Ferrándiz focal type variables (Ferrándiz 1988), were also analyzed in Aparicio & Floría (1996).

Deprit, Elipe & Ferrer (1994, §4) considered canonical and weakly canonical extensions of the point–transformation introducing focal variables, and applied them to the linearization of Keplerian systems. Aparicio & Floría (1998, 2000) have characterized types of perturbation potentials allowing exact linearization in terms of focal variables (whose coordinate segment can be interpreted as a set of homogeneous Cartesian coordinates in a projective space), and have presented the set of four second–order differential equations governing the resulting coupled and perturbed oscillators. As special instances, generalized quasi–Keplerian systems with potentials of Deprit type belong to the class of linearizable perturbations by means of the focal method.

3 Some Notations and Definitions

• Just to fix terminology and notations, let us consider the expression for the *Keplerian* potential, arising from the well–known Newton's Gravitation-Law, under the form:

$$V_\kappa(r) \;=\; -\frac{K}{r}\,, \quad \text{where } K \text{ is a positive constant} \quad \Longrightarrow \; \textit{Keplerian systems}\,.$$

Here r stands for the norm of the two–body relative position vector, and K is the Keplerian parameter (or gravitational coupling parameter) of the system.

On such systems, we shall *superimpose* the perturbing effects due to certain particular types of potentials.

• *Quasi–Keplerian system* (Deprit 1981, §4) are one-degree-of-freedom systems governed by central force laws derived from potentials of the form

$$\mathcal{V}_2(r) \;=\; V_\kappa + V_2(r) \;=\; -\frac{K}{r} + \frac{\mathcal{D}}{r^2}\;,$$

where $\mathcal{D}$ might be either an absolute constant (e.g., Manev's model) or some function of conserved quantities (e.g., Deprit–type radial intermediaries in Artificial Satellite Theory, as presented in Deprit 1981, §7 and §9).

This simple version of quasi–Keplerian systems includes the so–called *Manev potential* (Maneff, 1930, Bertrand, 1921), a model of gravitational potential that constitutes a nonrelativistic modification of Newton's Law of Gravitation; it has been used to accurately account for the motion of the apse line (i.e., the secular motion of the pericentre) of some celestial bodies, at least within the Solar System (e.g. the advance of the perihelion of the inner planets, or the motion of the perigee of the Moon), although Newton himself resorted to this model in his *"Principia"* (Deprit 1981, §4 and §9; Valluri, Wilson & Harper, 1997). Clairaut (see Aoki 1992) also used this correction to Newton's Universal Law.

• *Generalized quasi-Keplerian systems* (Floría 1993, 1994b) are governed by potentials containing terms proportional to r^{-j} , $j = 0, 1, 2$:

$$\mathcal{V} \;=\; V_\kappa + V_0 + V_1 + V_2\,, \quad V_j = \mathcal{J}/r^j\,,$$

$\mathcal{J}$ being an absolute constant or some function of conserved quantities. Notice the *mnemonic use* of symbol $\mathcal{J}$ and the corresponding exponent j for the power of the reciprocal of the mutual distance. Formulated in adequate variables, the structure and functional dependence of these potentials is compatible with a Keplerian–like description of motion,

and solutions can be proposed in terms of circular (say, elementary trigonometric) functions. Other choices for the values of the exponents lead to expressions involving elliptic integrals and functions (that is, special functions are required in the representation).
• As a general notation, we shall also write

$$\mathcal{V}_n(r) \;=\; V_\kappa + V_n(r) \;=\; -\frac{K}{r} + \frac{\mathcal{N}}{r^n}\,,$$

where, as expected, this coefficient $\mathcal{N}$ can represent either an absolute (numerical) constant or some function of first integrals of the system.

Remember the mnemonic rule: $\mathcal{N} \longleftrightarrow$ power n.

Notice also that several perturbing terms of the V_n type can be added up all together and assembled to constitute a "compound" perturbing potential, which is then superimposed on the Keplerian term V_κ. In our subsequent developments, in order to study the joint effects of these perturbations, we can decompose the result and simply focus on the individual terms generated by each one of the V_n involved in the perturbation: this way of proceeding is justified by the linearity of the mathematical operations required at each step.

Many geometrical and dynamical properties of the standard Kepler problem also hold in the case of these types of perturbed Keplerian systems.

For future use throughout the rest of the paper, we introduce some additional notations. We are interested in the problem of motion of a particle in an attractive, central–force field. The motion is planar, and one may assume that the force centre is located at the origin $(0,0)$ of the coordinate system chosen in that plane.
• Let (x,y) denote the Cartesian coordinates of the moving particle,
• $r = \sqrt{x^2+y^2}$ stands for the (Euclidean) instantaneous distance from the origin.

For the motion of the particle under an inverse–square central attractive force (that is, force law for the Kepler problem) we have

$$V_\kappa(r) \;=\; -\frac{K}{r}\,, \quad \nabla_{(x,y)}V_\kappa \;=\; \frac{K}{r^2}\left(\frac{x}{r}\,,\,\frac{y}{r}\right) \;=\; \frac{K}{r^2}\,\hat{\mathbf{r}} \;=\; \frac{K}{r^3}\,\mathbf{r}\,,$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ denotes the unit vector along the radial direction. Newton's equations of motion read

$$\ddot{x} \;=\; -\frac{K}{r^3}\,x \;=\; \Phi\,x\,, \quad \ddot{y} \;=\; -\frac{K}{r^3}\,y \;=\; \Phi\,y\,,$$

that is

$$\ddot{\mathbf{r}} \;=\; -\frac{K}{r^3}\,\mathbf{r} = \Phi\,\mathbf{r}\,, \;\text{ with }\; \Phi \;=\; \Phi\,(r) \;=\; -\frac{K}{r^3}\,,$$

The problem possesses the integral of the angular-momentum-conservation:

$$\frac{d}{dt}\,(x\,\dot{y} - y\,\dot{x}) \;=\; 0 \;\Longrightarrow\; x\,\dot{y} - y\,\dot{x} \;=\; \mathcal{A} \;=\; \text{const.}$$

With these general notations, we shall review some methods to deal with the Kepler problem, and consider how they behave when applied to some perturbed Keplerian systems.

4 On an Approach due to Monsky

We sketch the procedure according to which Monsky (2004) solves the Kepler problem. In the preceding notations, the combination of terms $x\dot{y}-y\dot{x}$ is just the Wronskian $W[x,y]$ of x and y, two solutions to the second–order equation $\ddot{\varphi} = \Phi(t)\,\varphi$, where (during the motion) Φ is the function of time $-K/r^3$. One must distinguish between two cases:

- $\mathcal{A} = W[x,y] = 0$: rectilinear motion through the origin $(0,0)$.
- $\mathcal{A} \neq 0$: Since

$$\begin{aligned} r\,\dot{r} &= x\,\dot{x} + y\,\dot{y}\,, \qquad (r\,\dot{r})^2 + \mathcal{A}^2 = \left(x^2+y^2\right)\left(\dot{x}^2+\dot{y}^2\right)\,, \\ \dot{x}^2+\dot{y}^2 &= \dot{r}^2 + \frac{\mathcal{A}^2}{r^2}\,, \qquad r\,\ddot{r} = -\,\frac{K}{r} + \frac{\mathcal{A}^2}{r^2}\,, \end{aligned}$$

function r is seen to satisfy the ODE

$$\ddot{r} \;=\; -\,\frac{K}{r^3}\left(r - \frac{\mathcal{A}^2}{K}\right)\,.$$

Transform $r \longrightarrow u$ by means of

$$u \,=\, r - \frac{\mathcal{A}^2}{K} \;\Longrightarrow\; \ddot{u} \,=\, \ddot{r} \,=\, \frac{K}{r^3}\,u\,.$$

Then u, like x and y, is a solution of $\ddot{\varphi} = \Phi\,\varphi$. Given that $\mathcal{A} = W[x,y] \neq 0$, functions x and y are a basis of the solution space of this equation, whence $u = r - (\mathcal{A}^2/K)$ is a linear combination of x and y:

$$r - \left(\mathcal{A}^2/K\right) \,=\, C_1\,x + C_2\,y\,.$$

Rotating about the origin, one may assume that $C_2 = 0$, and r takes the form (for some constant D)

$$r \,=\, D\,x + \left(\mathcal{A}^2/K\right)\,.$$

When $D = 0$, this equation corresponds to a circumference.
If $D \neq 0$, this is the focus-directrix equation of a (branch of) a conic section with a focus at $(0,0)$ and directrix $x = -\mathcal{A}^2/DK$.

If we follow this procedure in the general case when

$$\begin{aligned} \mathcal{V}_n(r) &= V_\kappa + V_n(r) \,=\, -\,\frac{K}{r} + \frac{\mathcal{N}}{r^n}\,, \\ \nabla_{(x,y)}V_n &= -\,\frac{n\,\mathcal{N}}{r^{n+1}}\left(\frac{x}{r}\,,\,\frac{y}{r}\right) = -\,\frac{n\,\mathcal{N}}{r^{n+1}}\,\hat{\mathbf{r}}\,. \end{aligned}$$

the equations of motion read

$$\ddot{x} = \left[-\frac{K}{r^3} + \frac{n\,\mathcal{N}}{r^{n+2}} \right] x = \Phi_n\, x\,, \qquad \ddot{y} = \left[-\frac{K}{r^3} + \frac{n\,\mathcal{N}}{r^{n+2}} \right] y = \Phi_n\, y\,;$$

$$\ddot{\mathbf{r}} = \left[-\frac{K}{r^3} + \frac{n\,\mathcal{N}}{r^{n+2}} \right] \mathbf{r} = \Phi_n\, \mathbf{r}\,, \quad \text{where} \quad \Phi_n = \Phi_n(r) = -\frac{K}{r^3} + \frac{n\,\mathcal{N}}{r^{n+2}}\,;$$

and then

$$x\dot{y} - y\dot{x} = \mathcal{A} = \text{const.} \;\; \text{(central field)}\,, \qquad r\,\dot{r} = x\,\dot{x} + y\,\dot{y}\,,$$

$$(r\,\dot{r})^2 + \mathcal{A}^2 = \left(x^2 + y^2\right)\left(\dot{x}^2 + \dot{y}^2\right)\,, \qquad \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + \frac{\mathcal{A}^2}{r^2}\,,$$

$$r\,\ddot{r} = r^2\,\Phi_n + \frac{\mathcal{A}^2}{r^2} \quad \Longrightarrow \quad \ddot{r} = r\,\Phi_n + \frac{\mathcal{A}^2}{r^3}\,.$$

We can also rewrite

$$r\,\ddot{r} = \frac{\mathcal{A}^2}{r^2} + r^2 \left[-\frac{K}{r^3} + \frac{n\,\mathcal{N}}{r^{n+2}} \right] = \frac{\mathcal{A}^2}{r^2} + \left[-\frac{K}{r} + \frac{n\,\mathcal{N}}{r^n} \right]$$

$$\Longrightarrow \; \ddot{r} = \frac{\mathcal{A}^2}{r^3} - \frac{K}{r^2} + \frac{n\,\mathcal{N}}{r^{n+1}}\,.$$

And, finally,

$$\ddot{r} = -\frac{K}{r^3} \left[r - \frac{\mathcal{A}^2}{K} - \frac{n\,\mathcal{N}}{K}\,\frac{1}{r^{n-2}} \right]\,.$$

From this result we can analyze and discuss some interesting cases, according to values of exponent n:

• $n = 0$: We recover the conventional "Keplerian" description, and the expression in brackets reduces to $r - (\mathcal{A}^2/K)$. This holds true for any potential that differs from V_κ by an additive constant $\mathcal{N} = \mathcal{N}/r^0$, which is not significant when derivatives of the potential are considered. This fact is consistent with the arbitrariness in the choice of the reference level for the potential energy.

• $n = 1$: The expression in brackets becomes $r\,\{\,1 - (\mathcal{N}/K)\} - (\mathcal{A}^2/K)$; this case would correspond to a fictitious, auxiliary "Keplerian" system with a modified gravitational coupling parameter $K - \mathcal{N}$.

• $n = 2$: The quantity in brackets reads $r - [\,(\mathcal{A}^2 + 2\,\mathcal{D})\,/K\,]$, which would be related to a fictitious "Keplerian" motion with a modified angular momentum.

These modifications in the original (unperturbed) Keplerian solution account for the variations of the orbit under the effect of the perturbing potential.

These conclusions are in full agreement with our results concerning generalized quasi–Keplerian systems compatible with the geometrical and dynamical "Keplerian–like" picture of motion, obtained in terms of generalized canonical elements of a Jacobi type (Floría 1993) and of a Delaunay–Similar type (Floría 1994b). Thanks to the use of universal functions, generalized Delaunay–Similar canonical orbital elements, uniformly applicable to any kind of (not necessarily bound) orbital motion, were derived on the basis of generalized quasi–Keplerian Hamiltonians (Floría 1999).

5 A Formulation in Complex Variables

Gauthier (2005) and Weinstock (1992, §IV) have obtained their solutions to the unperturbed, gravitational two–body problem after formulating it in terms of complex variables. We now review the approach proposed by Weinstock.

Consider planar motion in the Oxy plane of coordinates, and opt for a formulation in complex variables in the z–plane, with the usual choice: x the real axis, and y the imaginary axis. Then we have the notations and conventions

- Complex position: $z \equiv x + iy = \|z\| \exp(i\phi)$,
- polar angle (from the x-axis): ϕ ,
- distance from the origin (centre of force): $\|z\| = r = \sqrt{x^2 + y^2}$.
- conjugate complex: $z^* = x - iy$, from which $\|z\|^2 = \|z^*\|^2 = z\, z^* = x^2 + y^2 = r^2$,
- complex velocity: $\dot{z} = dz/dt$.

Then Newton's equation of motion for the Kepler motion reads

$$\ddot{z} \;=\; -\,\frac{K\, z}{\|z\|^3} \;=\; -\,\frac{K\, z}{z\, z^* \|z\|} \;=\; -\,\frac{K}{z^* \|z\|} \ .$$

Equivalently

$$(\, \ddot{z}\,)^* \;=\; \left[-\,\frac{K}{z^* \|z\|} \right]^* \Longrightarrow \ddot{z}^{\,*} \;=\; -\,\frac{K}{z \|z\|} \ ,$$

and the constancy of angular momentum takes on the form:

$$\frac{d}{dt}\,(\dot{z}z^* - z\dot{z}^*) = 0 \;\Rightarrow\; (\dot{z}z^* - z\dot{z}^*) \;=\; 2\, i\, \mathcal{A}\, ,$$

the integration constant $\mathcal{A}$ being real, because $z\dot{z}^*$ is the conjugate complex of $\dot{z}z^*$. In fact,

$$\mathcal{A} \;=\; x\,\dot{y} - y\,\dot{x} \;=\; \mathcal{I}m(\dot{z}z^*)\, , \quad \text{and} \quad 2\, i\, \mathcal{A} = (z^*)^2\, \frac{d}{dt}\left(\frac{z}{z^*}\right) \ .$$

Some helpful auxiliary formulae are:

$$\frac{\|z\|}{z^*} \;=\; \frac{\|z\| z}{z^* z} \;=\; \frac{\|z\| z}{\|z\|^2} \;=\; \frac{z}{\|z\|}\ , \qquad \frac{z}{z^*} = \frac{z}{z^*}\,\frac{z}{z} \;=\; \frac{z^2}{\|z\|^2} \;=\; \left(\frac{z}{\|z\|}\right)^2$$

$$\Rightarrow\; 2\, i\, \mathcal{A} \;=\; 2\, z^*\, \|z\|\, \frac{d}{dt}\left(\frac{z}{\|z\|}\right) \Longrightarrow 2\, i\, \mathcal{A}\, \ddot{z} \;=\; -\, 2\, z^*\, \|z\|\, \frac{d}{dt}\left(\frac{z}{\|z\|}\right) \frac{K}{\|z\| z^*}$$

$$\Rightarrow\; i\, \mathcal{A}\, \ddot{z} \;=\; -\, K\, \frac{d}{dt}\left(\frac{z}{\|z\|}\right) \Longrightarrow \frac{d}{dt}\left(\frac{z}{\|z\|} + \frac{i\, \mathcal{A}}{K}\, \dot{z}\right) = 0\ ,$$

which is integrated (in terms of real integration constants) to yield

$$\frac{z}{\|z\|} + \frac{i\, \mathcal{A}}{K}\, \dot{z} \;=\; A + i\, B \Longrightarrow \frac{z^*}{\|z\|} - \frac{i\, \mathcal{A}}{K}\, \dot{z}^* = A - i\, B\ .$$

From these expressions one obtains

$$2\,\|z\| \;+\; \frac{i\,\mathcal{A}}{K}\,[\,\dot{z}z^{*} - z\dot{z}^{*}\,] = A(z+z^{*}) \;-\; i\,B(z-z^{*})\;.$$

After restoring the original Cartesian variables,

$$\sqrt{x^2 \,+\, y^2} \;-\; \frac{\mathcal{A}^2}{K} \;=\; Ax \,+\, By\;,$$

which is the equation of a conic section with a focus at the coordinate origin (coinciding with the force centre).

In the general case when

$$\begin{aligned} \mathcal{V}_n(r) &= V_\kappa \,+\, V_n(r) \;=\; -\,\frac{K}{r} \,+\, \frac{\mathcal{N}}{r^n}\;, \\ \nabla_{(x,y)} V_n &= -\,\frac{n\,\mathcal{N}}{r^{n+1}} \left(\frac{x}{r}\,,\,\frac{y}{r}\right), \end{aligned}$$

the equations of motion

$$\ddot{x} \;=\; \Phi_n\,x\,, \quad \ddot{y} \;=\; \Phi_n\,y\,,$$

are rewritten in the form

$$\ddot{z} \;=\; \left[-\,\frac{K}{\|z\|^3} + \frac{n\,\mathcal{N}}{\|z\|^{n+2}}\right] z = \left[-\,\frac{K}{\|z\|} + \frac{n\,\mathcal{N}}{\|z\|^{n}}\right] \frac{z}{z\,z^{*}} = \frac{1}{z^{*}\|z\|}\left[-\,K + \frac{n\,\mathcal{N}}{\|z\|^{n-1}}\right]\;.$$

Equivalently

$$\ddot{z}^{\,*} \;=\; \frac{1}{z\|z\|}\left[-\,K + \frac{n\,\mathcal{N}}{\|z\|^{n-1}}\right].$$

And one arrives at the expression:

$$i\,\mathcal{A}\,\ddot{z} \;=\; \left[-\,K + \frac{n\,\mathcal{N}}{\|z\|^{n-1}}\right]\frac{d}{dt}\left(\frac{z}{\|z\|}\right).$$

Completing the required calculations, a line of reasoning similar to the one followed at the end of the preceding section leads to analogous conclusions concerning solutions described as conic sections in precession.

6 An Approach to the Kepler Problem Due to Laplace

Weinstock (1992, §V) reports on one of the solutions to the inverse–square orbit problem proposed by Laplace. He simplifies the derivation, and adapts it to our present–day, modern terminology and notations.

Starting from the Newtonian equations of motion,

$$\begin{aligned} \ddot{x} &= -\,\frac{K}{r^3}\,x \;\Rightarrow\; r^3\,\ddot{x} = -Kx \Rightarrow \frac{d}{dt}(r^3\,\ddot{x}) = -K\dot{x}, \\ \ddot{y} &= -\,\frac{K}{r^3}\,y \;\Rightarrow\; r^3\,\ddot{y} = -Ky \Rightarrow \frac{d}{dt}(r^3\,\ddot{y}) = -K\dot{y}, \\ &\Rightarrow\; \frac{d}{dt}(r^3\,\ddot{r}) = -K\dot{r}. \end{aligned}$$

Accordingly, functions $x = x(t)$, $y = y(t)$ and $r = r(t)$ satisfy the same second–order, homogeneous linear ODE, namely

$$\frac{d}{dt}\left(\psi(t)\,\frac{dU}{dt}\right) \;=\; -K\,U\,, \quad \text{with} \quad \psi(t) \;=\; r^3 \;\text{ along the motion}\,.$$

Then one must have a linear relation of the type $\dot{r} = c_1\,\dot{x} + c_2\,\dot{y}$, from which (excluding rectilinear motion) one concludes that the orbit must be some conic–section given as $r = c_1\,x + c_2\,y + c_3$ with $r = \sqrt{x^2 + y^2}$.

Application of this approach to the case $\mathcal{V}_n = V_\kappa + V_n$ hinges on a set of intermediate formulae, starting from the equations of motion

$$\begin{aligned} \ddot{x} \;&=\; \left[-\frac{K}{r^3} + \frac{n\,\mathcal{N}}{r^{n+2}}\right] x \;=\; \Phi_n\,x\,, \quad \ddot{y} \;=\; \left[-\frac{K}{r^3} + \frac{n\,\mathcal{N}}{r^{n+2}}\right] y \;=\; \Phi_n\,y\,; \\ r^3\ddot{x} \;&=\; \left[-K + \frac{n\,\mathcal{N}}{r^{n-1}}\right] x\,, \quad r^3\ddot{y} = \left[-K + \frac{n\,\mathcal{N}}{r^{n-1}}\right] y \end{aligned}$$

$$\begin{aligned} \frac{d\,(r^3\ddot{x})}{dt} \;&=\; -K\,\dot{x} + \frac{n\mathcal{N}}{r^n}\left[\dot{x}\,r - (n-1)\,x\,\dot{r}\right], \\ \frac{d\,(r^3\ddot{y})}{dt} \;&=\; -K\,\dot{y} + \frac{n\mathcal{N}}{r^n}\left[\dot{y}\,r - (n-1)\,y\,\dot{r}\right], \\ \frac{d\,(r^3\ddot{r})}{dt} \;&=\; -K\,\dot{r} + \frac{n(2-n)\mathcal{N}}{r^{n-1}}\,\dot{r}\,. \end{aligned}$$

The discussion of values of the exponent n compatible with a Keplerian–like description of motion is similar to the considerations in preceding sections.

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