Monografías de la Real Academia de Ciencias de Zaragoza. **28:** 103–114, (2006).

Stationary solutions and their stabilities of a particular three-body problem

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Abstract

The equations of motion of a three-body problem made of a dumb-bell (two masses at fixed distance) moving around a central mass under gravitational attraction have been stablished. Linear and isosceles stationary solutions of these equations have been studied and sufficient conditions for the stability have been found in terms of Lyapunov's stability functions.

1 Introduction

In this paper we study the motion of a system made of three material points M_1, M_2 and M_3 interacting by Newtonian law, under the assumption that the distance between M_2 and M_3 is constant, i.e., points M_2 and M_3 form a dumb-bell. Particular cases of this problem are equivalent to the classical restricted three bodies problem or to the generalized two fixed centres [1].

The purpose of this paper is the study of the different stationary solutions of the problem for arbitrary masses of the bodies and arbitrary size of the dumbell. The interest of this study derives from the fact that it is the simplest problem about traslational-rotational motion of the a satellite in a gravitational field and gives the generic conections between the solution of this restricted three body problem and the classical one [2].

In the stationary solutions studied in the paper the mutual distances are constant and the triangle $M_1M_2M_3$ rotates, as a rigid body, about the Gz axis passing through the center of mass of the system.

It is shown that, when the points move on a fixed plane, there are linear solutions in which the points are on a rotating axis. When the mass of M_1 tends to zero, the stationary solution reduces to the linear Eulerian solution of the classical restricted three body problem [4, 5]. Necessary and sufficient conditions for stability of the solutions have been obtained.

Other solutions considered are the isosceles, in which the distances from M_1 to M_2 and M_3 are constant and equal. Now, the three points rotate around the Gz axis crossing orthogonaly to the plane of motion through the mutual center of mass and the points are permanently on isosceles relative position. Necessary and sufficient conditions for stability of the solutions have also been obtained [7].

Apart from its own dynamical properties, this model may be considered as an approximation for describing the motion of a binary small-body system, such as an asteroid or a Kuiper belt object. Indeed, one of the main features of asteroids is its irregular shape, and in particular its elongation, which at times is modelled by a finite straight segment [8, 9] or by the dumb-bell structure, among other choices.

2 Formulation of the problem

Let us consider three material points M_1, M_2 and M_3, of masses m_1, m_2 and m_3, mutually attracted by the Newtonian gravitational forces. Let us assume that points M_2 and M_3 are rigidly connected by a segment of constant length l and negligible mass, i.e., they form a dumb-bell.

Let C be the center of masses of the dumb-bell and l_2, l_3 the distances from M_2 and M_3 to C. We can define a rotating frame $\mathcal{B}(C,\, \boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)$ such that $\boldsymbol{b}_3$ is directed along the dumb-bell towards the point M_3 and $\boldsymbol{b}_1, \boldsymbol{b}_2$ are two orthonormal vectors, perpendicular to $\boldsymbol{b}_3$. In this frame, the principal moments of inertia (A, B, C) of the dumb-bell are

$$I_1 = I_2 = m_2 l_2^2 + m_3 l_3^2 = \frac{m_2 m_3}{m_2 + m_3} l^2, \quad I_3 = 0.$$

Let us introduce now an inertial reference frame $\mathcal{S}(C,\, \boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3)$. The attitude of the dumb-bell in $\mathcal{S}$ is given by two angles, namely nutation θ and precession ϕ. The angle $\theta \in [0, \pi)$ is such that $\cos\theta = \boldsymbol{s}_3 \cdot \boldsymbol{b}_3$. For the precession angle, we build the nodal vector $\boldsymbol{\ell}$ as $\boldsymbol{s}_3 \times \boldsymbol{b}_3 = \boldsymbol{\ell} \sin\theta$. Then, we define the precession angle ϕ as the longitude of the node $\boldsymbol{\ell}$ reckoned from $\boldsymbol{s}_1$ in the plane normal to $\boldsymbol{s}_3$, that is to say,

$$\boldsymbol{\ell} = \boldsymbol{s}_1 \cos\phi + \boldsymbol{s}_2 \sin\phi, \quad 0 \leq \phi < 2\pi.$$

With these angles, the coordinates of points m_3 and m_2 in the space frame are

$$\begin{aligned} \boldsymbol{x}_3^c &= l_3 \boldsymbol{b}_3 = l_3(\boldsymbol{s}_1 \sin\theta \sin\phi - \boldsymbol{s}_2 \sin\theta \cos\phi + \boldsymbol{s}_3 \cos\theta), \\ \boldsymbol{x}_2^c &= -l_2 \boldsymbol{b}_3 = -l_2(\boldsymbol{s}_1 \sin\theta \sin\phi - \boldsymbol{s}_2 \sin\theta \cos\phi + \boldsymbol{s}_3 \cos\theta). \end{aligned}$$

According to Köning's theorem, the kinetic energy of the dumb-bell is the sum of the kinetic energy of the center of masses C, assuming the total mass $m_2 + m_3$ is on it, plus the kinetic energy of rotation. To find the total kinetic energy of the three bodies, we have to add the kinetic energy of the body M_1.

The frame $(C\,;\boldsymbol{\ell}, \boldsymbol{b}_3 \times \boldsymbol{\ell}, \boldsymbol{b}_3)$ (see figure 1) is made of principal axes of inertia. The angular velocity of rotation is

$$\boldsymbol{\Omega} = \dot{\phi}\boldsymbol{s}_3 + \dot{\theta}\boldsymbol{\ell} = \dot{\theta}\boldsymbol{\ell} + \dot{\phi}\sin\theta(\boldsymbol{b}_3 \times \boldsymbol{\ell}) + \dot{\phi}\cos\theta\boldsymbol{b}_3.$$

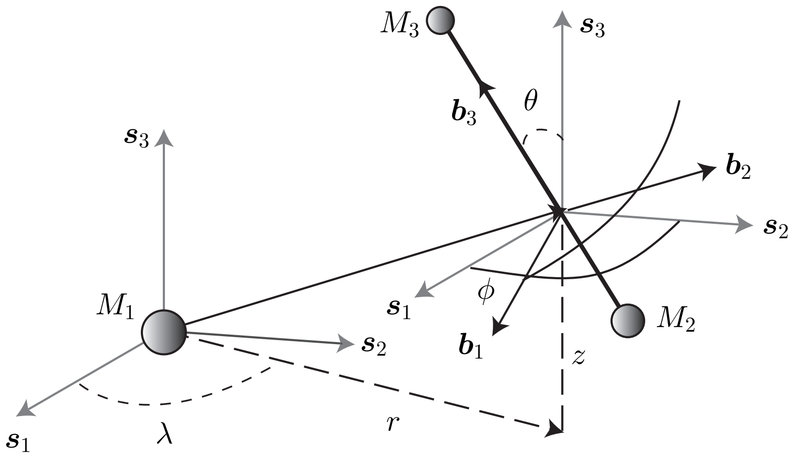

Figure 1.— The reference frames

We can reduce the order of the system by taking the so called *heliocentric* coordinates, that is, by referring the motion of M_2 and M_3 to the body M_1. Thus, we will refer the motion of the dumb-bell to the frame $(M_1, \boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3)$.

In these heliocentric coordinates, the total kinetic energy is (see e.g. Wintner [10])

$$T = \frac{1}{2}\frac{m_1(m_2+m_3)}{m_1+m_2+m_3}\dot{\boldsymbol{x}}_c^2 + \frac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}\boldsymbol{\Omega}.$$

But taking into account the values of the angular velocity of rotation $\boldsymbol{\Omega}$, and by using cylindrical coordinates (r, λ, z) for the orbital motion,

$$\boldsymbol{x}_c = r\cos\lambda\boldsymbol{s}_1 + r\sin\lambda\boldsymbol{s}_2 + z\boldsymbol{s}_3,$$

the kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\lambda}^2 + \dot{z}^2) + \frac{1}{2}A(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta), \tag{1}$$

where

$$m = \frac{m_1(m_2+m_3)}{m_1+m_2+m_3}, \quad \text{and} \quad A = \frac{m_2 m_3}{m_2+m_3}l^2 = \frac{m_1\,m_2\,m_3}{m\,(m_1+m_2+m_3)}\,l^2.$$

The potential function is

$$U = -\mathcal{G}m_1\left(\frac{m_2}{r_{12}} + \frac{m_3}{r_{13}}\right),$$

where mutual distances r_{1j}, for $j = 2, 3$, are

$$r_{1j}^2 = r^2 + z^2 + l_j^2 - (-1)^j \, 2 \, l_j \Big[z \cos\theta + r \sin\theta \sin(\phi - \lambda) \Big].$$

From the expressions of the kinetic energy and the potential, we can derive the Hamiltonian

$$\mathcal{H} = \frac{1}{2m} \left(P_r^2 + \frac{P_\lambda^2}{r^2} + P_z^2 \right) + \frac{1}{2A} \left(\frac{P_\phi^2}{\sin^2\theta} + P_\theta^2 \right) + U(r, z, \phi - \lambda, \theta).$$

Since angles ϕ and λ appear only as the difference $(\phi - \lambda)$, we can reduce the order of the Hamiltonian by means of the following canonical transformation

$$\begin{aligned} \psi &= \phi - \lambda, \quad & P_\psi &= P_\phi, \\ \omega &= \lambda, & P_\omega &= P_\phi + P_\lambda. \end{aligned}$$

Indeed, with this transformation, the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2m} \left(P_r^2 + \frac{(P_\omega - P_\psi)^2}{r^2} + P_z^2 \right) + \frac{1}{2A} \left(\frac{P_\psi^2}{\sin^2\theta} + P_\theta^2 \right) + U(r, z, \psi, \theta),$$

that is, it is reduced to four degrees of freedom. Since angle ω is cyclic, its conjugate moment P_ω is an integral of the motion. The Hamiltonian itself is another integral.

The equations of motion are

$$\begin{aligned} \dot{r} &= \frac{P_r}{m}, & \dot{P}_r &= \frac{(P_\omega - P_\psi)^2}{mr^3} - \frac{\partial U}{\partial r}, \\ \dot{z} &= \frac{P_z}{m}, & \dot{P}_z &= -\frac{\partial U}{\partial z}, \\ \dot{\theta} &= \frac{P_\theta}{A}, & \dot{P}_\theta &= \frac{P_\psi^2 \cos\theta}{A \sin^3\theta} - \frac{\partial U}{\partial \theta}, \\ \dot{\psi} &= -\frac{P_\omega - P_\psi}{mr^2} + \frac{P_\psi}{A \sin^2\theta}, \quad & \dot{P}_\psi &= -\frac{\partial U}{\partial \psi}. \end{aligned} \tag{2}$$

Equilibria are found by zeroing this system. Thus, there results that

$$P_r = P_z = P_\theta = 0, \quad P_\omega - P_\psi = \frac{mr^2}{A \sin^2\theta} P_\psi,$$

and

$$\begin{aligned} \frac{\partial U}{\partial r} &= \frac{mr}{A^2 \sin^4\theta} P_\psi^2, \quad & \frac{\partial U}{\partial z} &= 0, \\ \frac{\partial U}{\partial \theta} &= \frac{A \sin\theta \cos\theta}{mr} \frac{\partial U}{\partial r}, & \frac{\partial U}{\partial \psi} &= 0. \end{aligned}$$

We need to compute the partial derivatives of the potential U. Let us define firstly the shorcuts

$$F = \mathcal{G} \, m_1 \left(\frac{m_3}{r_{13}^3} + \frac{m_2}{r_{12}^3} \right), \quad G = \mathcal{G} \, m_1 \left(\frac{m_3 l_3}{r_{13}^3} - \frac{m_2 l_2}{r_{12}^3} \right). \tag{3}$$

Then, we have that the partial derivatives may be put as

$$\begin{aligned} \frac{\partial U}{\partial r} &= Fr + G \sin\theta \sin\psi, \quad & \frac{\partial U}{\partial z} &= Fz + G \cos\theta, \\ \frac{\partial U}{\partial \theta} &= G(-z \sin\theta + r \cos\theta \sin\psi), & \frac{\partial U}{\partial \psi} &= G \sin\theta \cos\psi, \end{aligned}$$

and equations for equilibria reduce to

$$P_\psi = \frac{A\,\sin^2\theta}{m\,r^2 + A\,\sin^2\theta}\,P_\omega \tag{4}$$

$$F\,r + G\sin\theta\sin\psi = \frac{m\,r}{A^2\,\sin^4\theta}\,P_\psi^2 \tag{5}$$

$$F\,z + G\cos\theta = 0, \tag{6}$$

$$A\sin\theta\cos\theta\,(Fr + G\sin\theta\sin\psi) - m\,r\,G\,(-z\sin\theta + r\cos\theta\sin\psi) = 0, \tag{7}$$

$$G\,r\,\sin\theta\cos\psi = 0. \tag{8}$$

The finding of general solution of this system is rather complicated, hence, we will look only for particular solutions. Cases $r = 0$ and $\theta = 0$ will be excluded since they correspond to singularities of the problem.

3 Planar motions

Let us consider that the three bodies move on the fixed plane $M_1\boldsymbol{s}_1\boldsymbol{s}_2$. In that case, $z = 0$ and $\theta = \pi/2$. If the bodies are at equilibrium position, then $r = r_0 =$ constant. Thus, equations (6)-(8) are only reduced to

$$G\cos\psi = 0,$$

that is fulfilled either when $\psi = \pi/2,\ 3\pi/2$ or when $G = 0$, i.e., when $r_{12} = r_{13}$.

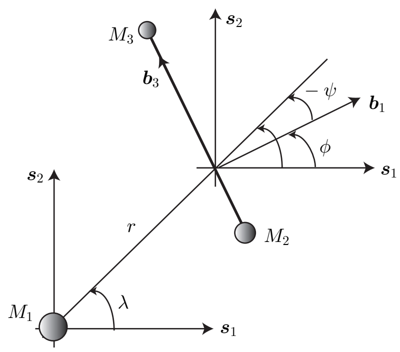

Figure 2.— Motion on the plane $\boldsymbol{s}_1\boldsymbol{s}_2$. It is achieved when $z = 0$ and $\theta = \pi/2$.

3.1 Linear solution

When $z = 0$, $r = r_0$, $\theta = \pi/2$ and $\psi = \pi/2, 3\pi/2$, the three points are on a line on the $\mathrm{M}_1\boldsymbol{s}_1\boldsymbol{s}_2$ plane that is rotating about the $\boldsymbol{s}_3$ with constant angular velocity, $n = \dot\omega$, given (taking into account (4)) by

$$\dot\omega = \frac{\partial\mathcal{H}}{\partial P_\omega} = \frac{1}{m\,r_0^2}\,(P_\omega - P_\psi) = \frac{1}{m\,r_0^2 + A}\,P_\omega \tag{9}$$

The equilibrium must fulfill equation (5), now written as

$$F\,r_0 \pm G = \frac{m\,r_0}{A^2}\,P_\psi^2 = \frac{m\,r_0}{(m\,r_0^2+A)^2}\,P_\omega^2 \geq 0.$$

Writing $\epsilon = \pm 1$ if $\psi = \pi/2$ and $r > l_2$ or $\psi = 3\pi/2$ and $r > l_3$, the previous equation particularize to

$$F\,r_0 + \epsilon\,G = \mathcal{G}\,m_1\left(\frac{m_3}{(r_0+\epsilon\,l_3)^2} + \frac{m_2}{(r_0-\epsilon\,l_2)^2}\right) = \frac{m\,r_0}{(m\,r_0^2+A)^2}\,P_\omega^2,$$

or, equivalently, if $\nu = m_3/m_2$ and $\rho = l/r_0$:

$$\frac{\nu}{(1+\nu+\epsilon\,\rho)^2} + \frac{1}{(1+\nu-\epsilon\,\nu\,\rho)^2} = C_0\,\frac{\rho}{(m+\dfrac{m_2\,m_3}{m_2+m_3}\,\rho^2)^2}, \qquad C_0 = \frac{m\,P_\omega^2}{\mathcal{G}\,m_1\,m_2\,l}.$$

This equation is equivalent to a polynomical one of degree six, hence it have six roots; for instance, for $\psi = \pi/2, m_1 = 10, m_2 = 2 = m_3, l = 1, C_0 \geq 4.4299$, two real roots (positives, less than $1+\nu$, so $r > l_2$) and four complex roots appear; if $C_0 \leq 4.4298$, only complex roots appear.

The position of the points in both cases, $\epsilon = \pm 1$, is shown in the figure 3.

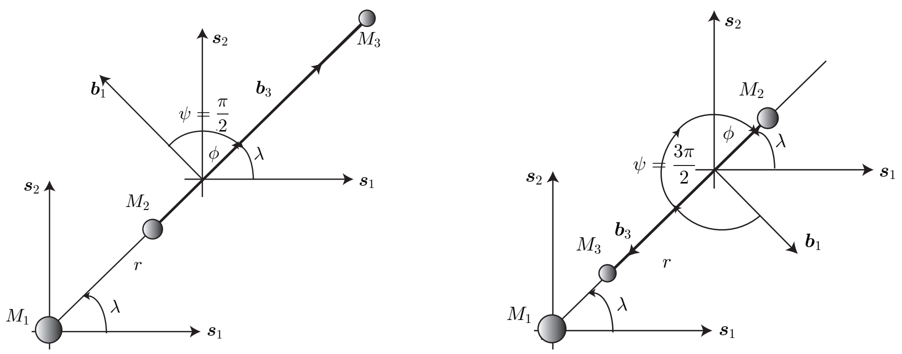

Figure 3.— Relative positions of the bodies in the collinear solution. Left: $\psi = \pi/2$. Right: $\psi = 3\pi/2$.

3.2 Isosceles stationary solutions.

These solutions are defined by

$$r = r_0, \quad z = 0, \quad \theta = \pi/2, \quad G = 0\,(\Longleftrightarrow r_{12} = r_{13}),$$

condition (5)

$$F\,r_0 + G\sin\theta\sin\psi = \frac{m\,r_0}{A^2\,\sin^4\theta}\,P_\psi^2,$$

and condition (4)

$$P_\psi = \frac{A\,\sin^2\theta}{m\,r^2 + A\,\sin^2\theta}\,P_\omega$$

Then

$$r_{12}^2 = r_0^2 - 2\,l_2\,r_0 \sin\psi + l_2^2 = r_{13}^2 = r_0^2 + 2\,l_3\,r \sin\psi + l_3^2 \Longrightarrow$$

$$\sin\psi = \frac{l_2 - l_3}{2\,r_0} = -\frac{1}{2}\,\frac{1-\nu}{1+\nu}\,\frac{l}{r_0}, \quad r_{12}^2 = r_{13}^2 = r_0^2 + l_2\,l_3 = r_0^2 + \frac{\nu}{(1+\nu)^2}\,l^2, \tag{10}$$

what it is only possible if

$$r_0 \geq \frac{1}{2}\,\frac{1-\nu}{1+\nu}\,l \quad \text{and so } r_{12} \geq \frac{l}{2}$$

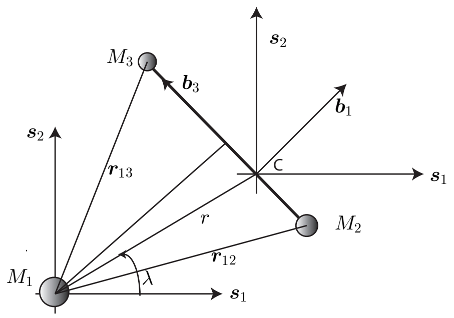

Figure 4.— The isosceles solution

In the figure 4, we can see the three points at the isosceles position.

Condition (5) is now written as

$$F = \mathcal{G}\,m_1 \frac{m_2 + m_3}{r_{12}^3} = \frac{m}{(m\,r_0^2 + A)^2}\,P_\omega^2 \tag{11}$$

what together to relations (10) define the values of r_0 and ψ_0 (and r_{12}) in the equilibria.

We can note that these two condition are equivalent to the following polynomical equation or order 4 in the variable r_{12}

$$(A + m\,(-l_2\,l_3 + r_{12}^2))^2 = D_0\,r_{12}^3, \qquad D_0 = \frac{P_\omega^2}{\mathcal{G}\,(m_1 + m_2 + m_3)},$$

or

$$m^2\,r_{12}^4 - D_0\,r_{12}^3 + 2\,m\,(A - l_2\,l_3\,m)\,r_{12}^2 + (A - l_2\,l_3\,m)^2 = 0 \tag{12}$$

Taking into account the coefficients of this polynomial, the Descartes and Huat theorems ([3]) allow one to assure that there exist two positive real roots or none and that there exist at least two complex conjugate roots. Lower and upper bounds for the positive real roots ([3]) are defined by the quantities

$$(1 + \frac{m\,P_\omega^2}{\mathcal{G}\,m_1\,((m_2 + m_3)\,(A - l_2\,l_3)^2})^{-1}, \qquad 1 + \frac{m\,P_\omega^2}{\mathcal{G}\,m^2\,m_1\,((m_2 + m_3)};$$

besides, $r_{12} \geq l/2$. Hence, the Bolzano mean value theorem will help us to finally conclude if there exist one valid positive root or none.

The frequency of the motion of rotation about the Gz axis (see eq. (9)) is given by

$$n = \dot{\omega} = \frac{\partial \mathcal{H}}{\partial P_\omega} = \frac{1}{m\, r_0^2}\,(P_\omega - P_\psi) = \frac{1}{m\, r_0^2 + A}\, P_\omega$$

Taking into account expresion (10) and equation (11), we obtain

$$n^2 = \dot{\omega}^2 = \frac{\mathcal{G}\, m_1\,(m_2 + m_3)}{m}\,\frac{1}{r_{12}^3} = \mathcal{G}\,\frac{m_1 + m_2 + m_3}{r_0^3}\,\left(1 + \frac{\nu}{(1+\nu)^2}\,(\frac{l}{r_0})^2\right)^{-3/2},$$

or, equivalently,

$$\frac{n}{n_0} = \left(1 + \frac{\nu}{(1+\nu)^2}\,(\frac{l}{r_0})^2\right)^{-3/4}, \quad \text{where } n_0 = \mathcal{G}\,\frac{m_1 + m_2 + m_3}{r_0^3}$$

The relative frequency, n/n_0, versus ν and l/r_0 is shown in the figure **??**. Of course, when $r \to +\infty$, $n \to n_0$.

4 Sufficient conditions for stability of the stationary solutions.

The stationary solutions are defined by the following found values:

$$P_r^{(0)} = P_z^{(0)} = P_\theta^{(0)} = 0, P_\psi^{(0)}, r^{(0)}, z^{(0)}, \psi^{(0)}, \theta^{(0)}$$

Introducing the vector $\boldsymbol{v} = (y_1, y_2, y_3, y_4, x_1, x_2, x_3, x_4)$ of variations of the coordinates and momenta

$$y_1 = P_r, y_2 = P_z, y_3 = P_\psi - P_\psi^{(0)}, y_4 = P_\theta,$$
$$x_1 = r - r^{(0)}, x_2 = z - z^{(0)}, x_3 = \psi - \psi^{(0)}, x_4 = \theta - \theta^{(0)},$$

the Hamiltonian of the linearized perturbed problem [2] is, formaly, the same as the nonlinearized, but with coefficients evaluated at the equilibrium solution. Consequently, the quadratic part of the Hamiltonian of the linearized perturbed problem is the sum of a positive defined part, the kinetic energy, and the Hessian of the potential energy. This last part is

$$\mathcal{V}_2 = \tfrac{1}{2} \sum_{i,j=1}^{4} V_{ij}\, x_i\, x_j, \tag{13}$$

where V_{ij} are the following second derivatives of the potential evaluated at the equilibrium solution

$$V_{11} = \mathcal{U}_{rr} = F + r\, F_r + G_r \sin\psi \sin\theta, \qquad V_{12} = \mathcal{U}_{rz} = r\, F_z + G_z \sin\psi \sin\theta$$
$$V_{13} = \mathcal{U}_{r\psi} = r\, F_\psi + G \cos\psi \sin\theta + G_\psi \sin\psi \sin\theta,$$
$$V_{14} = \mathcal{U}_{r\theta} = r\, F_\theta + G \cos\theta \sin\psi + G_\theta \sin\psi \sin\theta$$
$$V_{22} = \mathcal{U}_{zz} = F + z\, F_z + G_z \cos\theta, \qquad V_{23} = \mathcal{U}_{z\psi} = z\, F_\psi + G_\psi \cos\theta$$
$$V_{24} = \mathcal{U}_{z\theta} = z\, F_\theta + G_\theta \cos\theta - G \sin\theta, \qquad V_{33} = \mathcal{U}_{\psi\psi} = r \sin\theta\,(G_\psi \cos\psi - G \sin\psi)$$
$$V_{34} = \mathcal{U}_{\psi\theta} = r \cos\psi\,(G \cos\theta + G_\theta \sin\theta)$$
$$V_{44} = \mathcal{U}_{\theta\theta} = -G(z \cos\theta + r\, \sin\psi \sin\theta) - G_\theta\,(z \sin\theta - r\, \cos\theta \sin\psi)$$

and

$$
\begin{aligned}
F_r &= \frac{\partial F}{\partial r} = -3\,(F_5\, r + H_5 \sin\psi \sin\theta), & F_z &= \frac{\partial F}{\partial z} = -3\,(F_5\, z + H_5 \cos\theta),\\
F_\psi &= \frac{\partial F}{\partial \psi} = -3\, H_5\, r \cos\psi \sin\theta, & F_\theta &= \frac{\partial F}{\partial \theta} = -3\, H_5\, \left(r\cos\theta \sin\psi - z\sin\theta\right),\\
G_r &= \frac{\partial G}{\partial r} = 3\,(J_5\, r + K_5 \sin\psi \sin\theta), & G_z &= \frac{\partial G}{\partial z} = 3\,(J_5\, z + K_5 \cos\theta),\\
G_\psi &= \frac{\partial G}{\partial \psi} = 3\, K_5\, r \cos\psi \sin\theta, & G_\theta &= \frac{\partial G}{\partial \theta} = 3\, K_5\, \left(r\cos\theta \sin\psi - z\sin\theta\right)
\end{aligned}
$$

and we have adopted the following notation:

$$F_5 = \mathcal{G}\, m_1 \left(\frac{m_2}{r_{12}^5} + \frac{m_3}{r_{13}^5}\right), \quad H_5 = \mathcal{G}\, m_1 \left(-\frac{l_2\, m_2}{r_{12}^5} + \frac{l_3\, m_3}{r_{13}^5}\right), \tag{14}$$

$$J_5 = \mathcal{G}\, m_1 \left(\frac{l_2\, m_2}{r_{12}^5} + \frac{l_3\, m_3}{r_{13}^5}\right), \quad K_5 = \mathcal{G}\, m_1 \left(-\frac{l_2^2\, m_2}{r_{12}^5} + \frac{l_3^2\, m_3}{r_{13}^5}\right) \tag{15}$$

We will use this function as a Lyapunov function for our analysis of the statibility. In this way, the Lyapunov's stability of the stationary solutions follows from the fact that the quadratic form (13) be positively defined, i.e., in agreement with the Jacobi's criterium, if all the principal minors of the matrix which elements are (V_{ij}) have positive value.

4.1 Sufficient conditions for stability of stationary linear motions.

The matrix (V_{ij}) reduce, in this case, to

$$\begin{bmatrix} V_{11} & 0 & 0 & 0 \\ 0 & V_{22} & 0 & V_{24} \\ 0 & 0 & V_{33} & 0 \\ 0 & V_{24} & 0 & V_{44} \end{bmatrix},$$

where

$$V_{11} = F + r\, F_r + \epsilon\, G_r,$$

$$V_{22} = F, \qquad V_{33} = -\epsilon\, r\, G, \quad V_{44} = -\epsilon\, r\, G, \qquad V_{24} = -G,$$

and conditions for Lyapunov's stability become

$$V_{11} > 0, \quad V_{22} > 0, \quad V_{33} > 0, \quad V_{44}\, V_{22} - V_{24}^2 > 0 \tag{16}$$

Let us note that, in this case, $\epsilon\, G \leq 0$, so, from the definitions of quantities F and G in (3) and (5), we may assure that the tree first conditions are fulfilled; last condition (16) is now written as

$$V_{44}\, V_{22} - V_{24}^2 = -\epsilon\, r\, G\, F - G^2 = -\epsilon\, G\, (r\, F + \epsilon\, G),$$

that is also fulfilled.

4.2 Sufficient conditions for stability of the isosceles motions.

In this case, the only non vanishing elements of matrix V_{ij} are

$$V_{11} = F + r\,F_r + G_r\,\sin\psi, \quad V_{13} = 3\,r\,K_5\,\cos\psi\sin\psi,$$

$$V_{22} = F, \quad V_{33} = 3\,r^2\,K_5\,\cos^2\psi$$

and conditions for Lyapunov's stability become now

$$V_{11} > 0, \quad V_{22} > 0, \tag{17}$$

$$V_{33}\,V_{11} - V_{13}^2 > 0. \tag{18}$$

The second condition (17) is fulfilled, but the first one, that can be written as

$$V_{11} = \frac{\mathcal{G}\,m_1}{r_{12}^3}\left[(m_2+m_3)\,(1 - 3\,\frac{r^2}{r_{12}^2}) - 3\frac{l_2\,m_2\,(l_3-l_2)}{r_{12}^2}(1 - \frac{(l_3-l_2)^2}{4\,r^2})\right],$$

is equivalent to the following condition

$$3\,l_2\,m_2\,(l_2-l_3)^3 + 8\,l_2\,m_2\,(\frac{m_2}{m_3} - 2)\,r^2 + 8\,(m_2+m_3)\,r^4 < 0$$

that is never fulfilled as the signs of the coefficients explain.

Consequently, this function V is not an adequated Lyapunov function.

Nevertheless, necessary conditions for stability can be obtained by analizing the roots of the characteristic equation of the linearized equations of motion in the neighbourhood of the stationary solution. These equations, defined by the linearized Hamiltonian $\mathcal{H}$ at the equilibria, are

$$\begin{aligned}
\dot y_1 &= -V_{11}\,x_1 - V_{13}\,x_3, & \dot y_3 &= -V_{33}\,x_3 - V_{13}\,x_1,\\
\dot x_1 &= \frac{1}{m}\,y_1, & \dot x_3 &= \frac{1}{m\,r^2}\,y_3 - -\frac{P_\omega}{m\,r^2 + A},\\
\dot y_2 &= -V_{22}\,x_2, & \dot x_2 &= \frac{1}{m}\,y_2,\\
\dot y_4 &= -V_{44}\,x_4, & \dot x_4 &= \frac{1}{A}\,y_4
\end{aligned}$$

Hence, the charateristic equation can be separated into the following three equations:

$$\lambda^4 + a\,\lambda^2 + b = 0, \quad \lambda^2 + \frac{V_{22}}{m} = 0, \quad \lambda^2 + \frac{V_{44}}{A} = 0,$$

where

$$a = \frac{r^2\,V_{11} + V_{33}}{m\,r^2}, \qquad b = \frac{V_{11}\,V_{33} - V_{13}^2}{m^2\,r^2}$$

and stability follows if all the roots have vanishing real part.

Since

$$V_{22} > 0, \quad V_{44} \geq 0,$$

the last two inequalities are fulfilled. The first one is a biquadratic equation that define other four imaginaries roots if

$$a \geq 0, \quad b \geq 0, \quad a^2 - 4\,b \geq 0.$$

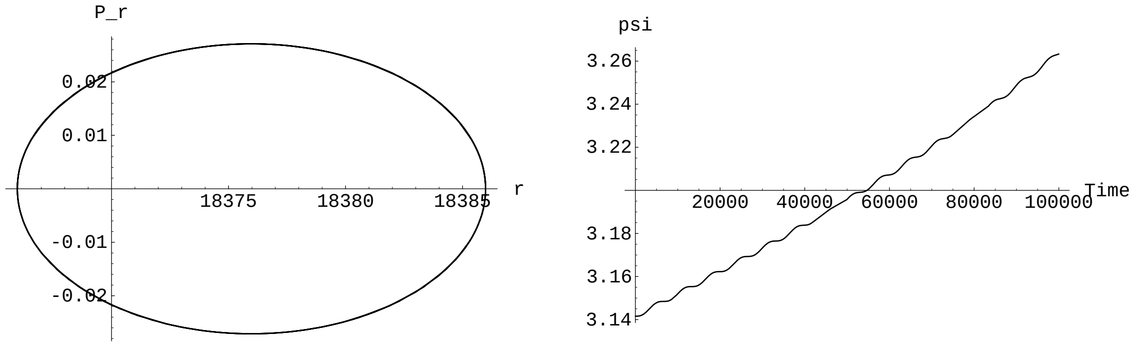

Figure 5.— A particular trajectory in the plane (r, P_r)

If, for instance, we take the following very particular values for the constants:

$$m_1 = 10, m_2 = m_3 = 2, l = 1, P_\omega = 914917., \mathcal{G} = 398585.28, \ (\Rightarrow r_0 = 18376.)$$

and the initial values

$$P_r = 0,\ P_z = 0,\ P_\psi = \frac{A\,P_\omega^2}{m\,r^2 + A}, P_\theta = 0,\ r = r_0 - 10,\ z = 0,\ \psi = \pi,\ \theta = \pi/2,$$

the trajectories are bounded, as we can see in figure 5. The distance follows the typical variation of the Keplerian motion, while the angular variable ψ grows almos linearly with time. This shows the instability of this equilibrium point. On the contrary, if we change the value of r to $r = 100$, the trajectories become unbounded.

5 Conclusions

The equations of motion of one three-body problem composed of a dumb-bell (two masses at fixed distance) moving around a central mass have been established. Several cases of stationary solutions of these equations have studied and sufficient conditions for stability has been found in terms of Lyapunov's stability functions.

Aknowledgements

Supported by the Spanish Ministry of Education and Science (Projects # ESP2005-07107 and # BFM2003-02137).

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