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Some periodic orbits in the restricted three-body problem, for $\mu > 0$, from the $\mu = 0$ case

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Abstract

In the present work, we deal with horseshoe motion in the frame of the Restricted Three-Body Problem (RTBP) for different values of the mass parameter μ. On one hand, we study numerically families of periodic horseshoe orbits for μ small and how they are organised. We figure out the mechanism of the organisation of such families from the two-body problem ($\mu = 0$). On the other hand, we study the existence of horseshoe periodic orbits for other values of μ. We claim that the behaviour of the invariant manifolds associated to the equilibrium point L_3 as well as the existence of homoclinic orbits play an important role.

1 Introduction

Recently, with the discover of a new asteroid, called 2002 AA_{29}, moving in a co-orbital motion with respect to the Earth, the interest on this type of motion has come up. The most famous such motion in the Solar System is the co-orbital pair of Saturnian satellites Janus and Epimetheus, discovered in 1981. The dynamics of co-orbital satellites, where two bodies are in (almost) 1:1 mean motion resonance, have been studied by several authors using different approaches. One of the frameworks considered is the Restricted Three Body Problem, in which one of the satellites is taken as a test particle with zero mass. In this context, when one considers the co-orbital motion in the synodical (rotating) frame, the motion has a horseshoe shape, surrounding the Lagrangian equilibrium points L_3, L_4 and L_5.

From the analytical point of view, using singular perturbation theory the motion of co-orbital satellites may be approximated by two different solutions of a two-body problem when they are far apart (see Figure 3); however this approximation fails when the two satellites are close to each other (see for example [7] and [10]). In [3], the authors analyse the problem introducing small parameters in the three-body equations, truncating higher order terms and deriving dynamical information from the resulting equations. Numerically horseshoe periodic orbits have been explored as invariant objects using Hill's problem ([6]), the *planar* restricted three-body problem ([4] and [5]) and the *spatial* RTBP ([2]).

In the present work we consider the planar RTBP and we focus our attention to periodic symmetric solutions of horseshoe shape of the RTBP in a rotating system. The periodic horseshoe orbits (HPO from now on) are not isolated but embedded into one parametric families. In [2], a certain number of these families have been computed numerically for $\mu = 10^{-4}$. In order to understand how these families are organised, we consider the two-body problem (for $\mu = 0$). We see that the families of periodic orbits obtained from rotating circular and elliptical orbits in the two-body problem are responsible for the organisation and some properties of the families of HPO for μ small. We call such families for $\mu = 0$ *generating families of HPO* and we give their analytical expression.

2 Families of planar horseshoe periodic orbits in the RTBP

First, let us briefly recall the equations and some aspects of the planar restricted three body problem (for example, see Szebehely, [9], for more details). Let us consider a system of three bodies in an inertial reference system: two main bodies (the primaries), of masses $1-\mu$ and μ, $\mu \in (0, 1/2]$, describing circular orbits around their common centre of mass (the origin of coordinates) in a plane, and a particle of infinitesimal mass which moves in the same plane under the gravitational effect of the primaries but has negligible effect on their motion. We also consider a rotating (synodical) system of coordinates, such that the big and small primaries remain fixed at positions $(\mu, 0)$ and $(\mu - 1, 0)$ respectively. Then, with suitable units, the equations of motion are

$$x'' - 2y' = \frac{\partial \Omega}{\partial x}, \qquad y'' + 2x' = \frac{\partial \Omega}{\partial y}, \tag{1}$$

where

$$\Omega(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu),$$

$r_1^2 = (x-\mu)^2 + y^2$ and $r_2^2 = (x - \mu + 1)^2 + y^2$ are the distances between the particle and the big and small primaries respectively, and $'$ stands for d/dt.

It is well known that these equations have the so called Jacobi first integral

$$x'^2 + y'^2 = 2\Omega(x, y) - C, \tag{2}$$

and 5 equilibrium points: the collinear points, L_1, L_2 and L_3, with positions $(x_i, 0)$, for $i = 1, 2, 3$, and the equilateral ones, L_4 and L_5, located at $(\mu - 1/2, \pm\sqrt{3}/2)$. It is also well known that the value of the Jacobi constant at the equilibrium points $C_i = C(L_i)$ for any value of $\mu \in (0, 1/2]$, satisfies $3 = C_4 = C_5 < C_3 \le C_1 < C_2$.

We have to take into account the Hill's regions in the (x, y) plane, where the motion of the particle is possible, which are bounded by the zero velocity curves (zvc) given by the equation

$$2\Omega(x, y) - C = 0$$

obtained from (2) (see Figure 1). It is clear from the zero velocity curves that the equi-

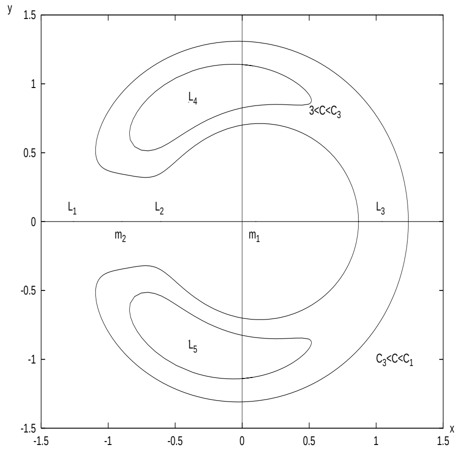

Figure 1: Zero velocity curves for $\mu > 0$. The motion is possible outside the region enclosed by the *zvc*. Horseshoe motion takes place for $C < C_1$

librium points play a role here and that horseshoe motion takes place for $C < C_1$.

According to [5], a horseshoe periodic orbit will be a periodic solution of (1) in which the particle follows a path which surrounds only the equilibrium points L_3, L_4 and L_5 and has two orthogonal crossings with the horizontal axis. We can consider (without loss of generality) that these two crossings occur at $t = 0$ and $t = T/2$, T being the period. Thus, the initial conditions at $t = 0$ are $(x_0, 0, 0, y_0')$ and at $t = T/2$ (called *final conditions*) $(x_f, 0, 0, y_f')$. Due to the symmetries of the equations, this is enough to ensure the periodicity of the solution and the symmetry with respect to the x axis.

In order to obtain numerically families of symmetrical horseshoe periodic orbits, we use that a family of periodic orbits with initial conditions $(x_0, 0, 0, y_0')$ is defined implicitly by the equation

$$x'(T/2, x_0, y_0') = 0$$

where $T = T(x_0, y_0')$ is given by the Poincaré section $y(T/2, x_0, y_0') = 0$. First, an initial seed of each family has been computed considering a fixed value of C and using a Bolzano argument to find zeros of $x'(T/2, x_0, y_0')$ as a function of x_0; then the numerical continuation of the family has been done using an arc step method to predict a new point on the curve and a modified Newton's method to refine it (see, for example, [1] for the details of the computation of these families or [8] for the methods used).

We show in Figure 2 the characteristic curves of the families of HPO computed for $\mu = 10^{-4}$ in the (x_0, C) plane (C obtained from x_0 and y_0') and the zvc (the continuous line curve on the left in both figures) given by equation

$$C = \frac{2(1-\mu)}{x_0 - \mu} + \frac{2\mu}{x_0 + 1 - \mu} + \mu(1-\mu) + x_0^2.$$

It is proved in [2] that the points (x_0, C) for which the same initial conditions has

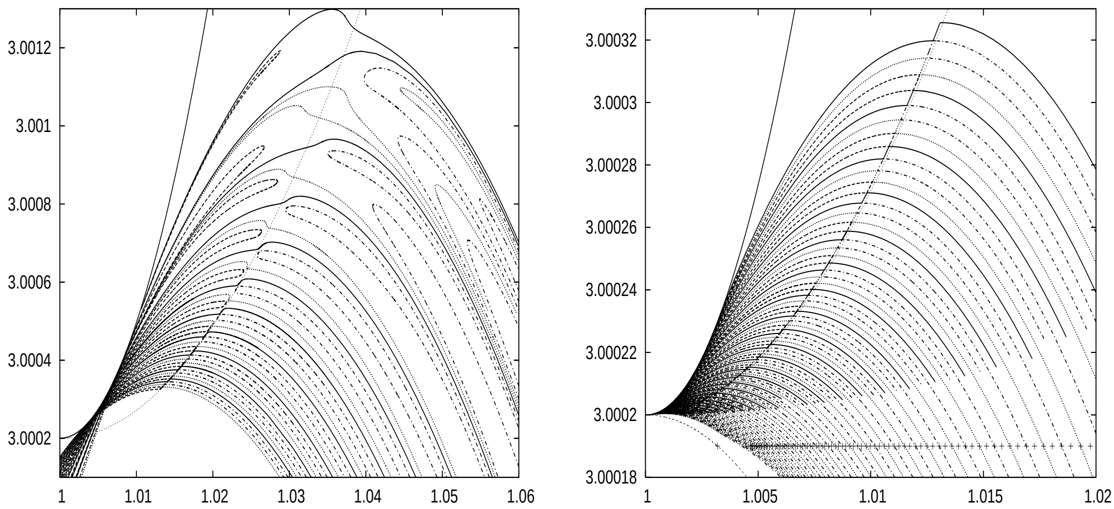

Figure 2: Characteristic curves od families of HPO for $3.00033 < C_{\rm m} < 3.0013$ (left) and $C_{\rm m} < 3.00033$ (right) in the (x_0, C) plane ($C_{\rm m}$ is the maximum value of the Jacobi's constant on each family). On the right, the separated dotted curve on the bottom left corresponds to the Lyapunov family of periodic (not horseshoe) orbits around L_3.

eccentricity (considered at the initial condition of a HPO) zero satisfy the equation

$$C = \frac{2(1-\mu)}{x_0 - \mu} + \frac{2\mu}{x_0 + 1 - \mu} + \mu(1-\mu) + 2\sqrt{x_0} - \frac{1}{x_0}.$$

This is the dot curve in Figure 2 (called skeleton in [2]) and corresponds to the generating family $l - i$ of circular orbits for $\mu = 0$.

We observe that the families follows some kind of organisation, specially for values of C close to $C(L_3)$. In order to understand this behaviour we study the families of rotating circular and elliptical orbits in the two-body problem. This is done in the next section.

3 Generating families of HPO for $\mu = 0$

It is clear that, for $\mu > 0$, the motion of the particle can be approximated by a two-body motion when the mass particle is far away from the small primary. For any HPO and $t \in [0, T/2]$, we will distinguish between the *outer* solution, the piece of the HPO from $t = 0$ to the returning point (close to the small primary), and the *inner* one from the returning point to $t = T/2$ (analogously for $t \in [T/2, T]$). Thus, the outer and inner solutions are approximated by two different solutions of the two-body problem (see Figure 3).

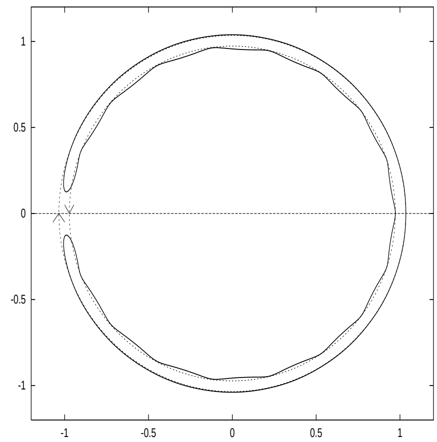

Figure 3: Example of a horseshoe periodic orbit (continuous line) for $\mu = 10^{-4}$ and the circular orbits (discontinuous line) for $\mu = 0$ approximating the outer and inner solutions.

For $\mu = 0$ we consider the sidereal orbits described by the particle around the big primary: the circular ones or synodical orbits coming from rotating sidereal ellipses. These orbits are embedded into families which can be characterised in terms of the Jacobi constant, the semimajor axis and the eccentricity (see, for example, [9]) by the equation

$$e^2 + \frac{1}{4a}\left(C - \frac{1}{a}\right)^2 = 1. \tag{3}$$

For $e = 0$ we obtain the equation for circular orbits $C = \frac{1}{a} \pm 2\sqrt{a}$, with the plus sign (the $l - i$ families in the literature) for sidereal direct orbits and the minus (the $h - m$ families) for sidereal retrograde ones. In this case, in order to approximate a HPO, we are interested in the two circular orbits of families $l - i$ close to the zvc, for a value of $C > 3$ fixed: such two circular orbits will approximate the outer and inner solutions of a HPO for small $\mu > 0$ (see Figure 3).

Concerning the elliptic orbits, the outer and inner solutions of a HPO are approximated by a retrograde and a direct synodical solutions respectively. Let be n the sidereal mean motion and $n - 1$ the synodical one. By third Kepler's law $n^2a^3 = 1$, if $n < 0$, the orbit is retrograde in both systems of reference, while for $0 < n < 1$ ($a > 1$) the orbit is sidereal direct but synodical retrograde and for $n > 1$ ($a < 1$) direct in both systems.

Using that the initial condition is $x_0 = a(1 \pm e)$, equation (3) can be written as

$$\frac{(x-a)^2}{a^2} + \frac{1}{4a}\left(C - \frac{1}{a}\right)^2 = 1, \tag{4}$$

curve that, fixed a value of a, in the (x, C) plane is an ellipse itself with centre $(a, 1/a)$ and semimajor axis a and $2\sqrt{a}$. Thus, the ellipse of equation (4) give a family of elliptical orbits and each of its points represents an elliptic orbit with semimajor axis a and eccentricity $e = |x-a|/a$ (see Figure 4 left). If $x > a$ ($x < a$) the orthogonal crossing takes place at the apocentre (pericentre); if $x = a$ we have a circular orbit ($e = 0$), and if $x = 0$ or $x = 2a$, then the ellipse is rectilinear ($e = 1$). Also from equation (4) and for a fixed value of a, we shall distinguish between direct and retrograde orbits according to the value of the Jacobi constant: sidereal retrograde orbits if $C < 1/a$ or sidereal direct ones if $C > 1/a$ (see Figure 4 right).

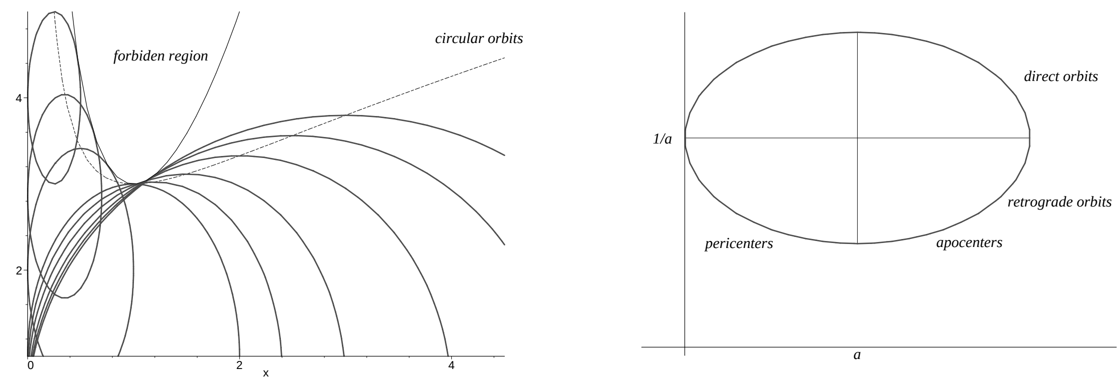

Figure 4: Left. Families of sidereal elliptical orbits in the plane (x, C). Right. One family given by equation (4) with a fixed a. Points $(a, 1/a \pm 2\sqrt{a})$ represent circular orbits.

The solutions of equation (4) will be called the *generating families* of rotating ellipses. This equation (regard that for $e = 0$ it gives the equation of the circular families) represents the equations of the characteristic curves of the symmetrical periodic orbits for $\mu = 0$, since, for a fixed a, a point of the curve in the plane (x, C) gives rise to the synodical initial condition (orthogonal crossing) $(x, 0, 0, y')$, with $y' = \pm v - x$ (plus or minus sign if the sidereal orbit is direct or retrograde respectively), and $v = \sqrt{\frac{1 \pm e}{a(1 \mp e)}}$, $x = a(1 \mp e)$, where the up (down) sign is taken if the particle is at the pericentre (apocentre). Such characteristic curves allows us to understand better the families of HPO for small $\mu > 0$ presented in the past Section.

4 From generating families to families of HPO for $\mu > 0$

On the view of the results of the past section, we can comment some properties of the computed families of HPO in the (x_0, C) plane when compared with the generating families for $\mu = 0$:

1. Each continuous family reaches a maximum value of C, denoted by C_{m}. Each family has two branches, each one very close to one generating family (for $\mu = 0$) characterised by a particular value of a; we point out that, for $\mu > 0$ and small, the generating families that approach each branch do not coincide.

2. Let us consider one family of HPO and one branch of it. When we consider the (x, y)-projection of each periodic orbit, the shape of such projection will be almost the same for most orbits in the branch, since the outer solution of each HPO is near a rotating ellipse with the *same* semimajor axis a but with different eccentricity. The increase of the eccentricity is translated to bigger loops and therefore to an increasing number of crossings of the HPO with the horizontal axis (only two of them being orthogonal).

3. Each branch of a family can be approximated by a generating family given by an ellipse characterised by (4). This allows us to describe the evolution of the sign of y'_0, or to decide whether the particle is at an apocentre or pericentre. There can be four different situations: both branches on the left of the skeleton, both on the right or one branch on each side. Let us suppose that, following the curve as x_0 increases, the family has one branch on the left and the other on the right. The left branch will be close to the corresponding upper and left semiellipse of a generating family and that means that the initial position point of every HPO of this branch corresponds to a pericentre of the approximating direct rotating ellipse. In this case, when x_0 increases, there is a particular x_0 for which the family is tangent to the zvc, so $y'_0 = 0$ and from this value on, the sign changes, from positive values to negative ones. But still the initial point corresponds to a pericentre until the family crosses the skeleton, so there is a value of x_0 such that the approximating outer solution is circular. As x_0 increases, we have the right branch of the family approximated by another upper and right semiellipse of a generating family with different semimajor axis a. Therefore the initial point takes place at an apocentre (of the corresponding rotating ellipse) all along the right branch.

It is clear that the two-body problem explains the behaviour of the characteristic curves of the continuous families of HPO as well as many geometrical properties of each HPO. However, for any HPO given, what the two-body approximation does not explain is the intermediate piece of the solution between the outer and inner solutions. In fact, when C is close to, and less than, C_1, there is a thin neck region given by the zero velocity curve, which has a horseshoe shape itself, that allows the path from the outer region –where the outer solution lives– and the inner one –the oval shape region around the big primary m_1, where there is the inner solution– (see Figure 1); of course the small primary plays a key role there.

Finally, a natural question is how the diagram of the characteristic curves of such families varies with μ. The continuation of this diagram is not straightforward and will depend on the behaviour of the invariant manifolds of the collinear points, in particular of L_3. But this subject will be analysed elsewhere (see [1]).

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