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Central configuration in the planar $n + 1$ body problem with generalized forces.

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Abstract

In this paper we consider a polygonal configuration for the planar $(n + 1)$ body problem. When a newtonian field is considered, is well known that we have a central configuration. By introducing general functions that depends on distance, we prove that central configuration is preserved not only for a newtonian field but for any field which depends on the inverse of distances. The Manev-type and the Schwarzschild-type fields are particular cases of our study.

1 Introduction

In this paper we consider the planar motion of $(n+1)$ bodies in such a way that n bodies of equal mass are located at the vertices of a regular n-gon centered at the remaining body of mass m_0. This problem is usually referred to as the *ring problem*, since it was proposed by Maxwell [9] to study the stability of particles surrounding Saturn. But although the problem may be considered as a classical one, it attracted the interest of researchers in the last years because of the possibility of considering this kind of configuration to model some dynamical systems (formation flights, planets around a star,. . .) and many authors have studied this problem from different points of view [13, 10, 11, 5, 6, 7, 8, 12, 1]. Besides, the dynamics of a particle moving under the gravitational field of the ring is very rich, since there are several parameters, which give rise to bifurcations, families of periodic orbits, etc (see e.g. [1, 12]).

In [1] an extension of the problem is proposed, in such a way that the central body is an spheroid or a radiation source. In this paper, we go a step ahead considering that the force between any two bodies is a generalized force that is a function of the mutual distance.

It was proved by Scheeres [13] that, under Newtonian forces, the ring configuration remains self-similar along the time, that is to say, it is a central configuration. In this communication we prove that the configuration is central under the generalized forces above mentioned. As an illustrations, we present the case of a potential that is finite series of the inverse of the mutual distances; Newtonian, Manev and Schwarzchild problems are particular cases.

2 Equations of motion and central configuration

Let us consider the motion of $(n+1)$ bodies $(n \geq 2)$ in such a way that they attract each other by a generalized force that is proportional to a certain function of their mutual distance in the direction of the line joining them.

Denote by $\boldsymbol{r}_i$ the position vector of the i-th particle in a barycentric reference frame, then the potential function of the system can be expressed as

$$U = \mathcal{G} \sum_{0 \leq i < j \leq n} m_i\, m_j\, G_{ij}(1/r_{ij}) \tag{1}$$

where $\mathcal{G}$ is the gravitational constant, $\boldsymbol{r}_{ij} = \boldsymbol{r}_j - \boldsymbol{r}_i$ and $r_{ij} = \|\boldsymbol{r}_j - \boldsymbol{r}_i\|$. The functions G_{ij} depend on each specific case considered. For instance, in the Newtonian case, $G_{ij} = 1/r_{ij}$; more examples are given in Section 4.

The equations of motion are

$$m_i \frac{d^2\boldsymbol{r}_i}{d\,t^2} = \frac{\partial U}{\partial \boldsymbol{r}_i}, \qquad (i = 0, \ldots, n)$$

then, by introducing the function

$$g_{ij} = \frac{\partial G_{ij}}{\partial(1/r_{ij})}\,,$$

the gradient is

$$\frac{\partial U}{\partial \boldsymbol{r}_i} = \mathcal{G} m_i \sum_{j=0, j\neq i}^{n} m_j g_{ij} \frac{\boldsymbol{r}_{ij}}{r_{ij}^3}.$$

Let us define the moment of inertia I as $I = \sum_{i=0}^{n} m_i\, \boldsymbol{r}_i \cdot \boldsymbol{r}_i$. Then, according with Wintner [14, §355], the $(n+1)$ bodies are in central configuration if the condition

$$\frac{\partial U}{\partial \boldsymbol{r}_i} = \kappa \frac{\partial I}{\partial \boldsymbol{r}_i} \tag{2}$$

holds for $i = 0, \ldots, n$ and for some scalar κ which is independent of i. That is to say, the bodies are in central configuration if the force of gravitation acting on m_i is proportional to the mass m_i and to the barycentric position vector $\boldsymbol{r}_i$.

Since in this problem the masses m_i $(i = 1, \ldots, n)$ are equal, we can introduce a mass parameter μ such that $m_i = m_0\,\mu$ $(i = 1, \ldots, n)$. After that, the condition (2) can be split into the following two equations

$$\frac{2\kappa}{k^2 m_0}\boldsymbol{r}_0 + \mu \sum_{j=1}^{n} g_{j0}\frac{\boldsymbol{r}_{j0}}{r_{j0}^3} = 0, \tag{3}$$

and

$$\frac{2\kappa}{k^2 m_0}\boldsymbol{r}_i + g_{0i}\frac{\boldsymbol{r}_{0i}}{r_{0i}^3} + \mu \sum_{j=1, j\neq i}^{n} g_{ji}\frac{\boldsymbol{r}_{ji}}{r_{ji}^3} = 0, \qquad (i = 1, \ldots n) \tag{4}$$

3 The regular n-gon configuration

It is known [13] that under Newtonian potential, the configuration made of a regular n-gon with equal masses m_i on the vertices and m_0 on the center of the polygon is a central configuration. However, to the knowledge of the authors, for generalized potentials of type (1) this result has not been proved yet. Thus, we proposed ourselves to see if the configuration given by

$$\boldsymbol{r}_0, \qquad \boldsymbol{r}_i = \boldsymbol{r}_0 + \alpha \boldsymbol{r}_i^*, \qquad (i = 1, \ldots, n) \tag{5}$$

with α a positive scalar independent of the scrip i (but that may depend on time), and $\boldsymbol{r}_i^*$ vectors on the plane of primaries pointing towards the vertices of a regular n-gon centered at $\boldsymbol{r}_0$ can remain self-similar under the generalized forces. To achieve our goal, we have to replace this configuration into equations (3) and (4) and determine the value of proportionality constant κ in order both equations be fulfilled.

As usual, we choose a barycentric system, that is, the origin is placed at the center of mass of the system, then,

$$m_0 \boldsymbol{r}_0 + \sum_{i=1}^{n} m_i \boldsymbol{r}_i = 0. \tag{6}$$

Following Scheeres [13], let us point out some properties of the vectors involved in the regular polygonal configuration.

1. We can assume that vectors $\boldsymbol{r}_i^*$ are unit vectors, $\|\boldsymbol{r}_i^*\| = 1$.

2. There is an angle $\theta = \pi/n$, such that $\boldsymbol{r}_i^* \cdot \boldsymbol{r}_j^* = \cos 2\,\theta(j - i)$.

3. The distance between vertices i and j is $\|\boldsymbol{r}_{ij}^*\| = \|\boldsymbol{r}_j^* - \boldsymbol{r}_i^*\| = 2\,|\sin\theta\,(j - i)|$.

4. The vectors are periodic in their index with period n, i.e., $\boldsymbol{r}_i^* = \boldsymbol{r}_{n+i}^*$.

5. The sum $\sum_{i=1}^{n} \boldsymbol{r}_i^* = 0$, provided $n \geq 2$.

Associated to these vectors, Scheeres [13] defines a new set of unitary vectors $\boldsymbol{q}_i^*$ orthogonal to $\boldsymbol{r}_i^*$, lying on the same plane, and with the same basic properties, that is,

$$\|\boldsymbol{q}_i^*\| = 1, \qquad \boldsymbol{q}_i^* \cdot \boldsymbol{r}_i^* = 0 \quad \text{and} \qquad \boldsymbol{q}_i^* \times \boldsymbol{r}_i^* = \boldsymbol{z}.$$

Hence, there results that

$$\boldsymbol{q}_i^* \cdot \boldsymbol{q}_j^* = \cos 2\,\theta(j-i), \qquad \boldsymbol{q}_i^* \cdot \boldsymbol{r}_j^* = \sin 2\,\theta(j-i),$$

and every vector $\boldsymbol{r}_{i+k}^*$ can be decomposed as a linear combination of $\boldsymbol{r}_i^*$ and $\boldsymbol{q}_i^*$ as

$$\boldsymbol{r}_{i+k}^* = \cos 2\,\theta k\, \boldsymbol{r}_i^* + \sin 2\,\theta k\, \boldsymbol{q}_i^*, \tag{7}$$

expression that will be used later on.

3.1 First condition

The first equation (3) to check, let us recall, is

$$\frac{2\kappa}{k^2 m_0}\boldsymbol{r}_0 + \mu \sum_{j=1}^{n} g_{j0} \frac{\boldsymbol{r}_{j0}}{r_{j0}^3} = 0$$

Since the origin is at the center of masses (6), and taking into account expression (5), there results that

$$m_0 \left[(1+n\mu)\boldsymbol{r}_0 + \mu\alpha \sum_{i=1}^{n} \boldsymbol{r}_i^* \right] = 0,$$

then

$$\boldsymbol{r}_0 = \frac{-\mu\alpha}{1+n\mu} \sum_{i=1}^{n} \boldsymbol{r}_i^* = 0 \quad \Longrightarrow \quad \boldsymbol{r}_0 = 0.$$

So, we only have to check that

$$\mu \sum_{j=1}^{n} g_{j0} \frac{\boldsymbol{r}_{j0}}{r_{j0}^3} = 0.$$

Since $\boldsymbol{r}_{j0} = \boldsymbol{r}_0 - \boldsymbol{r}_j = -\boldsymbol{r}_j = -\alpha \boldsymbol{r}_j^*$, and $\|\boldsymbol{r}_i^*\| = 1$, there results that

$$r_{j0} = \alpha, \quad \text{and} \quad g_{j0} = g_{j0}(r_{j0}) = g_{j0}(\alpha\|\boldsymbol{r}_j^*\|) = g_{j0}(\alpha) = g_0(\alpha),$$

that is, g_{j0} is independent of the index j, then

$$\mu \sum_{j=1}^{n} g_{j0} \frac{\boldsymbol{r}_{j0}}{r_{j0}^3} = -\mu g_0(\alpha) \frac{1}{\alpha^2} \sum_{j=1}^{n} \boldsymbol{r}_j^* = 0$$

and the first condition (3) is fulfilled. Let us now to see the second condition (4).

3.2 Second condition

The second condition (4) to be satisfied is

$$\frac{2\kappa}{k^2 m_0}\boldsymbol{r}_i + g_{0i}\frac{\boldsymbol{r}_{0i}}{r_{0i}^3} + \mu \sum_{j=1, j\neq i}^{n} g_{ji}\frac{\boldsymbol{r}_{ji}}{r_{ji}^3} = 0.$$

We already proved that $\boldsymbol{r}_0 = 0$, thus, expression (5) reduces to $\boldsymbol{r}_i = \alpha \boldsymbol{r}_i^*$, and the above condition reads

$$\alpha \left[\left(g_{0i} + \frac{2\kappa\alpha^3}{k^2 m_0} \right) \boldsymbol{r}_i^* + \frac{\mu}{8}\sum_{j\neq i} g_{ji} |\csc\theta(j-i)|^3 (\boldsymbol{r}_i^* - \boldsymbol{r}_j^*) \right] = 0.$$

As proven before, g_{0i} is independent of i, whereas the function g_{ji} depends on $r_{ji} = 2\,\alpha\,|\sin\theta(j-i)|$, namely, on α and the angular distance from the origin between bodies m_i and m_j. Making a change of index $k = j - i$, the above expression converts into

$$\left(g_{0i}(\alpha) + \frac{2\kappa\alpha^3}{k^2 m_0} \right) \boldsymbol{r}_i^* + \frac{\mu}{8}\sum_{k=1}^{n-1} g_{k+i,i}(\alpha, k\theta)\,|\csc k\theta|^3 (\boldsymbol{r}_i^* - \boldsymbol{r}_{k+i}^*) = 0, \quad (i = 1, \ldots, n).$$

Taking into account the decomposition (7) of vector $\boldsymbol{r}_{k+i}^*$, the above expression is a linear combination of two orthogonal vectors $\boldsymbol{r}_i^*$ and $\boldsymbol{q}_i^*$,

$$Q_i\,\boldsymbol{q}_i^* + R_i\,\boldsymbol{r}_i^* = 0,$$

where

$$Q_i = \frac{\mu}{8}\sum_{k=1}^{n-1} g_{k+i,i}(\alpha, |\sin k\theta|) |\csc k\theta|^3 \sin(2k\theta),$$

$$R_i = g_{0i}(\alpha) + \frac{2\kappa\alpha^3}{k^2 m_0} + \frac{\mu}{4}\sum_{k=1}^{n-1} g_{k+i,i}(\alpha, |\sin k\theta|) |\csc k\theta|.$$

In order this linear combination be zero, it is necessary that both coefficients Q_i and R_i be null.

Each term in Q_i is an odd function of the angle $\theta = \pi/n$, hence, the sum is zero for all i. In which respects to R_i, let us denote

$$\tilde{\omega} = g_{0i}(\alpha) + \frac{\mu}{4}\sum_{k=1}^{n-1} g_{k+i,i}(\alpha, \sin k\theta) |\csc k\theta|.$$

Since

$$g_{k+i,i}(r_{k+i,i}) = g_{k+i,i}(\alpha\, r_{k+i,i}^*) = g_{k+i,i}(\alpha\, 2|\sin k\theta|) = g_k(\alpha\, 2|\sin k\theta|),$$

there results that

$$\tilde{\omega} = g_0(\alpha) + \frac{\mu}{4}\sum_{k=1}^{n-1} g_k(\alpha, \sin k\theta) |\csc k\theta|.$$

In this way, we can choose the parameter κ as

$$\kappa = -\frac{m_0 k^2 \tilde{\omega}}{2\alpha^3},$$

which is independent of the index and also makes the coefficient of $\boldsymbol{r}_i^*$ null. Note that κ, in general, depends on n, μ and on time (through α).

In sum, we just proved that the regular n-gon configuration of $(n+1)$ bodies with generalized central forces is a central configuration.

4 Application

As an illustration, let us consider a problem with a general potential given by

$$U = \mathcal{G} \sum_{0 \le i < j \le n} m_i\, m_j \left(\frac{A_1}{r_{ji}} + \frac{A_2}{r_{ji}^2} + \frac{A_3}{r_{ji}^3} + \ldots + \frac{A_m}{r_{ji}^m} \right) =$$

$$= \mathcal{G} A_1 \sum_{0 \le i < j \le n} m_i\, m_j \left(\frac{1}{r_{ji}} + \frac{A_2^*}{r_{ji}^2} + \frac{A_3^*}{r_{ji}^3} + \ldots + \frac{A_m^*}{r_{ji}^m} \right).$$

These potentials are known as quasi-homogeneous potentials (see e.g. [2]), and classical potentials, like the Newtonian, Manev or Schwarzschild are but particular cases and will be considered below.

The g functions are

$$g_0(\alpha) = 1 + \frac{2A_2^*}{r_{0i}} + \frac{3A_3^*}{r_{0i}^2} + \ldots + \frac{nA_n^*}{r_{0i}^{n-1}} = 1 + \frac{2A_2^*}{\alpha} + \frac{3A_3^*}{\alpha^2} + \ldots + \frac{mA_m^*}{\alpha^{m-1}}$$

and

$$g_k(\alpha, \theta k) = 1 + \frac{2A_2^*}{2\alpha|\sin\theta k|} + \frac{3A_3^*}{2^2\alpha^2|\sin\theta k|^2} + \ldots + \frac{mA_m^*}{2^{m-1}\alpha^{m-1}|\sin\theta k|^{m-1}};$$

hence, $\tilde{\omega}$ is given by

$$\tilde{\omega} = 1 + \frac{2A_2^*}{\alpha} + \frac{3A_3^*}{\alpha^2} + \ldots + \frac{mA_m^*}{\alpha^{m-1}} +$$

$$+ \frac{\mu}{4} \sum_{k=1}^{n-1} \left(1 + \frac{2A_2^*}{2\alpha|\sin\theta k|} + \frac{3A_3^*}{2^2\alpha^2|\sin\theta k|^2} + \ldots + \frac{mA_m^*}{2^{m-1}\alpha^{m-1}|\sin\theta k|^{m-1}} \right) |\csc\theta k| =$$

$$= 1 + \frac{\mu}{4} \sum_{k=1}^{n-1} |\csc\theta k| + \frac{2A_2^*}{\alpha} \left(1 + \frac{\mu}{2^3} \sum_{k=1}^{n-1} |\csc\theta k|^2 \right) +$$

$$+ \frac{3A_3^*}{\alpha^2} \left(1 + \frac{\mu}{2^4} \sum_{k=1}^{n-1} |\csc\theta k|^3 \right) + \ldots + \frac{mA_m^*}{\alpha^{m-1}} \left(1 + \frac{\mu}{2^{m+1}} \sum_{k=1}^{n-1} |\csc\theta k|^m \right).$$

4.1 Newtonian potential

In this case the constants $A_i, i = 2, \ldots, n$ are null and so, the functions $g_{ij} = 1$, $(i, j = 0, \ldots, n)$ and $\tilde{\omega}$ is given by

$$\tilde{\omega} = 1 + \frac{\mu}{4} \sum_{k=1}^{n-1} |\csc\theta k|,$$

which only depends on the number of primaries n and on the mass parameter μ, as it is well known.

4.2 Manev-type potential

The problem with a Manev potential has been studied by Mioc y Stavinschi [10]. They proved that for the planar symmetrical $(n+1)$ body problem with a potential of the type

$$U = \mathcal{G} \sum_{0 \le i < j \le n} m_i\, m_j\, (\frac{A_1}{r_{ji}} + \frac{A_2}{r_{ji}^2}) = \mathcal{G} A_1 \sum_{0 \le i < j \le n} m_i\, m_j\, (\frac{1}{r_{ji}} + \frac{A_2^*}{r_{ji}^2}),$$

the polygonal configuration is preserved all along the motion, but the n-gon has variable side and with variable rotation around the central mass.

But it is just a particular problem of our study. In fact, it is enough to obtain the corresponding functions g_{ij} for this potential.

For the Manev potential

$$\tilde{\omega} = 1 + \frac{\mu}{4} \sum_{k=1}^{n-1} |\csc\theta k| + \frac{2A_2^*}{\alpha} \left(1 + \frac{\mu}{2^3} \sum_{k=1}^{n-1} |\csc\theta k|^2\right)$$

4.3 Schwarzschild-type potential

The potential for this problem is

$$U = \mathcal{G} \sum_{0 \le i < j \le n} m_i\, m_j\, (\frac{A_1}{r_{ji}} + \frac{A_3}{r_{ji}^3}) = \mathcal{G} A_1 \sum_{0 \le i < j \le n} m_i\, m_j\, (\frac{1}{r_{ji}} + \frac{A_3^*}{r_{ji}^3})$$

The necessary functions in this case are

$$g_0(\alpha) = 1 + \frac{3A_3^*}{\alpha}, \qquad g_k(\alpha, \theta k) = 1 + \frac{3A_3^*}{2^2\alpha^2 |\sin\theta k|^2}$$

and the expression of $\tilde{\omega}$ that gives the proportionality parameter κ for the central configuration is:

$$\tilde{\omega} = 1 + \frac{\mu}{4} \sum_{k=1}^{n-1} |\csc\theta k| + \frac{3A_3^*}{\alpha^2} \left(1 + \frac{\mu}{2^4} \sum_{k=1}^{n-1} |\csc\theta k|^3\right).$$

5 Conclusions

The $n+1$ ring configuration with generalized forces depending on the mutual distances among the bodies is shown that is a central configuration.

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