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On the dynamics of charged particles around rotating magnetic planets: perturbed Keplerian motions

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Abstract

We study the dynamics of charged particles around a rotating magnetic planet. The system is modelled by the Hamiltonian of the two–body problem to which we attach an axially–symmetric function which goes to infinity as soon as the particle approaches the planet. This perturbation consists in a magnetic dipole field and a corotational electric field. When it is weak compared to the Keplerian part of the Hamiltonian we average the system with respect to the mean anomaly. After truncating higher–order terms we use invariant theory to reduce the averaged system by virtue of its continuous and discrete symmetries, determining also the successive reduced phase spaces. Thence, we analyse the flow of the resulting system in the most reduced phase space, describing the equilibria, stability, as well as the different classes of bifurcations.

Key words and expressions: Planetary magnetospheres, perturbed Kepler problems, averaging and reduction, equilibria, stability and bifurcation.

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1 Introduction

The theoretical study of the motion of a charged particle in planetary magnetospheres has attracted the attention of physicists and astronomers since the second middle of the last century. The pioneering model goes back to Størmer's work in 1907 (see the paper [12] and the monography [13]), where the motion of a charge in a pure magnetic dipole field (the *Størmer model*) is considered. This model provides satisfactory results in the explanation of the dynamics of light particles (ions or electrons) which are present in the radiation belts surrounding magnetized planets [5]. However, when charged dust grains are considered, the ratio between the charge and the mass of the particle is small and the purely magnetic model has to be improved. The reason is that one has to take into account the gravitational field created by the planet, as well as the corotational electric field due to the rotation of the planet. This is the so–called *generalised Størmer* model, which will be denoted by the acronym GS.

In a recent series of papers, the GS model has been revisited by Horányi, Howard and coworkers [6, 7]. In these papers, the authors use a GS model that includes Keplerian gravity, a magnetic dipole aligned along the axis of rotation of the planet and a corotational electric field. In this framework, due to the axial symmetry of the system, the third component of the angular momentum is an integral, and the dynamics of the charged dust grain is governed by a two–dimensional effective potential. The shape of this effective potential has been intensively explored. Specifically the above–mentioned authors achieve the following results: i) the global stability conditions of the dust grain are obtained as a function of the parameters [7] and; ii) the existence of non–equatorial halo orbits [6, 7] for the dust grain is predicted.

The global dynamics of the problem is highly nonlinear and it is extremely difficult to state a global analytic model that explains the complete motion of the dust grain. Roughly speaking, in the GS problem the dust grain is subjected to gravitational and electromagnetic forces, which are in competition. The result of this fight depends on the charge–mass ratio of the dust grain. In this sense, the dynamics of the dust grain can be either gravitationally or electromagnetically dominated. If the dynamics of the grain is electromagnetically dominated, the Keplerian (also called the two–body) term in the effective potential can be taken as a perturbation of the electromagnetic terms. The effective potential presents non–equatorial potential wells where halo orbits survive [7]. On the other hand, if the Keplerian gravity dominates, we can argue the existence of a perturbed Keplerian potential well where the dust grain is trapped. This is the situation we deal with in this paper.

Our approach is analytical. We consider the Hamiltonian representing the GS problem as a sum of a pure Keplerian part, and a perturbation describing the magnetic dipole field and the corotational electric field. The basic idea is to transform our original system into an equivalent one, which is defined through an integrable Hamiltonian function and is, therefore, easier to be studied. Moreover, the simplified system contains the main features of the original one. Thus, we can extract dynamical information of the original system from the integrable Hamiltonian.

We achieve the transformation to the new dynamical system in three steps. First, by assuming that the perturbation is weak compared to the Keplerian term, we apply the Delaunay normalisation [4] up to first order. Thus, we obtain the averaged (or *normalised*) Hamiltonian with a new formal integral L (the *Keplerian symmetry*) representing the positive square root of the semi–major axis of the perturbed Keplerian ellipses, and where only two degrees of freedom remain in the Hamiltonian. Secondly, the axial symmetry of the problem allows one to reduce to one the degrees of freedom of the system. This symmetry is also used to obtain the two–dimensional phase space (the so–called *twice–reduced phase space*) related to the new system. Third, we exploit the finite symmetries of the original Hamiltonian in order to simplify the appearance of the equations and the shape of the two–dimensional phase space as much as possible. This step is achieved through a reduction mapping which gives rise to a new system, the so–called *fully–reduced Hamilton function* defined in a new phase space, called the *fully–reduced phase space*. That system is of one degree of freedom. Next we analyse the dynamical features of this system, calculating their equilibria and bifurcations.

The paper is organised as follows: the problem is formulated in Section 2. In Section 3 we apply the Delaunay normalisation. In Section 4, the Keplerian and the axial symmetries as well as the finite symmetries allow us to determine the different reduced phase spaces of the normalised system. The dynamics of the reduced system in the associated reduced phase space is the subject of Section 5. This involves the determination of equilibria and bifurcations with the corresponding analysis of the stability. A complete description of the dynamics of the problem containing the relationship between the fully–reduced system and the original one appears in reference [8].

2 The Problem

We assume that a particle of mass m and charge q is orbiting around a rotating magnetic planet of mass M and radius R. The Hamiltonian of this particle in Gaussian units is:

$$\mathcal{H} = \frac{1}{2\,m}\left(\mathbf{P} - \frac{q}{c}\,\mathbf{A}\right)^2 + U(\mathbf{x}), \tag{1}$$

where c is the speed of the light in the vacuum, $\mathbf{x} = (x, y, z)$ corresponds to the Cartesian coordinates and $\mathbf{P} = (P_x, P_y, P_z)$ represents the conjugate momenta of $\mathbf{x}$. Besides, $\mathbf{A}$ represents the vector potential describing the magnetic forces and $U(\mathbf{x})$ is the scalar potential accounting for the electric and gravitational interactions. The magnetic field $\mathbf{B}$ of the planet is taken to be created by a perfect magnetic dipole of strength μ aligned along the north–south poles of the planet (the z–axis). Thus if $r = \sqrt{x^2 + y^2 + z^2}$ stands for the distance of the charged particle to the centre of mass of the planet, the vectors $\mathbf{A}$

and $\mathbf{B}$ are given by

$$\mathbf{A} = \frac{\mu}{r^3}(-y, x, 0), \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{2}$$

If we assume that the magnetosphere surrounding the planet is a rigid conducting plasma that rotates with the same angular velocity ω as the planet, the charge q is subjected to a corotational electric field $\mathbf{E}$ of the form:

$$\mathbf{E} = -\frac{1}{c}(\mathbf{\Omega} \times \mathbf{x}) \times \mathbf{B} = -\frac{\mu\,\omega}{c}\,\nabla\Psi \tag{3}$$

where

$$\Psi = \frac{x^2 + y^2}{r^3} \qquad \text{and} \qquad \mathbf{\Omega} = (0, 0, \omega).$$

The combined action of the Keplerian and electrostatic forces gives the potential $U(\mathbf{x})$:

$$U(\mathbf{x}) = -\frac{M\,m}{r} + \frac{q\,\mu\,\omega}{c}\,\Psi. \tag{4}$$

By introducing the expressions (2) and (4) into (1) we get the Hamilton function

$$\mathcal{H} = \frac{1}{2\,m}(P_x^2 + P_y^2 + P_z^2) - \frac{M\,m}{r} - \frac{\mu\,q}{m\,c}\,\frac{H}{r^3} + \frac{\mu^2\,q^2}{2\,m\,c^2}\,\frac{x^2 + y^2}{r^6} + \frac{q\,\mu\,\omega}{c}\,\Psi, \tag{5}$$

where $H = x\,P_y - y\,P_x$ is the z–component of the angular momentum.

Since $\mathcal{H}$ is invariant under rotations around the z–axis, H is an integral of motion and cylindrical coordinates (ρ, z, ϕ) arise in a natural way. Hence, Eq. (5) reads:

$$\mathcal{H} = \frac{1}{2\,m}\left(P_\rho^2 + P_z^2 + \frac{H^2}{\rho^2}\right) - \frac{M\,m}{r} - \omega_c\,R^3\,\frac{H}{r^3} + \frac{m\,\omega_c^2\,R^6}{2}\,\frac{\rho^2}{r^6} + \frac{m\,\omega\,\omega_c\,R^3}{c}\,\frac{\rho^2}{r^3}. \tag{6}$$

The parameter $\omega_c = (q\,B_o)/(m\,c)$ stands for the *cyclotron* frequency, where $B_o = \mu/(R^3\,c)$ designates the magnetic field strength at the planetary equator.

In order to analyse the dynamics, it is convenient to use dimensionless coordinates and momenta. Firstly, we define the new coordinates as functions of the planet radius R, e.g. $\mathbf{x}' = \mathbf{x}/R$. As well we define a new (dimensionless) time $t' = \omega_K\,t$, where $\omega_K = \sqrt{M/R^3}$ is the *Keplerian frequency*. After introducing the above–mentioned transformations in Hamiltonian (6), and dropping primes in coordinates and momenta, we arrive at the following dimensionless Hamiltonian:

$$\mathcal{H}' = \frac{\mathcal{H}}{m\,R^2\,\omega_K^2} = \frac{1}{2}\left(P_\rho^2 + P_z^2 + \frac{H^2}{\rho^2}\right) - \frac{1}{r} - \delta\,\frac{H}{r^3} + \delta\,\beta\,\frac{\rho^2}{r^3} + \frac{\delta^2}{2}\,\frac{\rho^2}{r^6}. \tag{7}$$

The parameters δ and β of (7) are defined as $\delta = \omega_c/\omega_K$, $\beta = \omega/\omega_K$. The above parameters indicate, respectively, the ratio between the magnetic and the Keplerian interactions and the ratio between the electrostatic and Keplerian interactions. Note that for a given planet, B_o, ω and ω_K are constant and hence the Hamiltonian depends on three parameters; namely, on the one hand it depends on the internal parameters H and $\mathcal{H}' = E$ (the energy), and on the other hand it depends on the external parameter δ which indicates the charge–mass ratio q/m of the particle.

As stated in Section 1, the goal of this paper is to study the dynamics of the system when the main effect on the particle is assumed to be the Keplerian gravity. In other words, we are interested in those cases where the motion takes place inside of a Keplerian potential well. Moreover, this potential well must be located outside the planetary region in order to consider realistic orbits.

Under the above considerations, we introduce the effective potential U_{eff} from (7) as

$$U_{eff} = \frac{H^2}{2\,\rho^2} - \frac{1}{r} - \delta\,\frac{H}{r^3} + \delta\,\beta\,\frac{\rho^2}{r^3} + \frac{\delta^2}{2}\,\frac{\rho^2}{r^6}.$$

In the pure Keplerian case ($\delta = 0$), the function U_{eff} has a minimum at $z = 0$ and $\rho = H^2$ (for $H \neq 0$). The points r_+ and r_-, where the particles velocity is zero (the turning points), tend monotonically to$H^2/2$ and $+\infty$, respectively, as U_{eff} tends to 0. In this way, only values of $|H| > \sqrt{2}$ guarantee that r_- and r_+ are outside the planet. For a planet like Saturn, the spin rate is $\omega \approx 1.64 \times 14^{-4}$ rad/s, and the parameter $\beta \approx 0.4$. These values have been taken from the book by Murray and Dermott [10]. Hence, if $|\delta| \ll 1$ and $\beta < 1$, we can assume that the Keplerian potential well is only slightly affected by the terms depending on the electromagnetic interactions. This fact can be observed in Fig. 1, from which we infer that a deformed Keplerian well exists for $\delta \in [-0.01, 0.01]$. Other values of the parameters δ and β are used if we consider the magnetospheres of other giant planets.

In the rest of the paper we shall consider δ and β as parameters, so that our analysis could be used for other planets although we shall allow that δ varies in $[-0.01, 0.01]$ whereas β will be between 0 and 0.5, which includes Saturn's value. For more details on the ranges of validity of β and δ we address the reader to the book [10].

The critical points $(\rho_0,\ z_0)$ of U_{eff} are the roots of the following system of equations:

$$\begin{aligned}
\frac{\partial U_{eff}}{\partial \rho} &= -\frac{H^2}{r^3} + \frac{3\,\delta\,H\,r}{(\rho^2+z^2)^{5/2}} \\
&\quad + r\left[\frac{\delta^2\,(-2\,\rho^2+z^2)}{(\rho^2+z^2)^4} - \frac{\beta\,\delta\,(\rho^2-2\,z^2)}{(\rho^2+z^2)^{5/2}} + \frac{1}{(\rho^2+z^2)^{3/2}}\right] = 0, \\
\frac{\partial U_{eff}}{\partial z} &= z\left[-\frac{3\,\delta^2\,r^2}{(\rho^2+z^2)^4} - \frac{3\,\delta\,(-H+\beta\,r^2)}{(\rho^2+z^2)^{5/2}} + \frac{1}{(\rho^2+z^2)^{3/2}}\right] = 0.
\end{aligned}$$

In order to analyse how the presence of the perturbations distorts the Keplerian well, we study analytically the evolution of the roots located at the ρ–axis. Hence, for $z = 0$, the second equation holds, and the first one (for $H \neq 0$) gives the following third–degree polynomial equation

$$\mathcal{P}(\rho) = -2\,\delta^2 + 3\,\delta\,H\,\rho - H^2\,\rho^2 + \rho^3 - \beta\,\delta\,\rho^3 = 0.$$

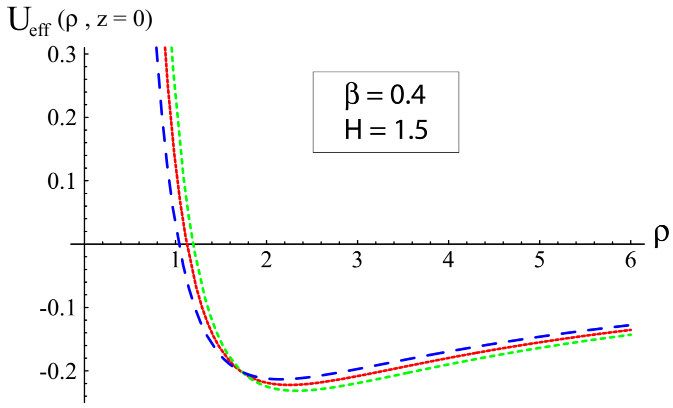

Figure 1: Effective potential U_{eff} defined in the plane $z = 0$ for for $\delta = 0.01$ (dashed line), $\delta = 0$ (solid line) and $\delta = -0.01$ (pointed line). In all curves the values for H and β have been fixed to $H = 1.5$ and $\beta = 0.4$.

The disappearance of the Keplerian well takes place when the saddle and the Keplerian minimum tend one to the other, in such a way that a double root takes place and both disappear, remaining only the inner minimum. Notice that whether $\delta \in [-0.01, 0.01]$, the Keplerian well remains far enough from the saddle point, this feature being the reason why it is only slightly deformed by the electromagnetic perturbations.

3 Delaunay Normalisation

3.1 Normalisation through first–order averaging

The aim of this section is to transform Hamilton function $\mathcal{H}$ into another Hamiltonian $\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1 + \mathcal{K}_2/2! + \mathcal{K}_3/3! + \mathcal{K}_4/4! + \cdots$, such that $\mathcal{K}_0 \equiv \mathcal{H}_0$, via a formal symplectic change of coordinates and a generating function $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3/2! + \mathcal{W}_4/3! + \cdots$.

There are certain sets of variables which are specially well-suited to deal with perturbed Keplerian systems, for example Delaunay variables (ℓ, g, h, L, G, H). This is a set of action and angle variables. If $\mathcal{H}_0$ stands for the Hamiltonian of the two-body problem, the action L is related to the two-body energy by the identity $\mathcal{H}_0 = -1/(2\,L^2)$. The action G is the norm of the angular momentum. The third component of $\mathbf{G}$ is H. The angle ℓ is named as the mean anomaly and is related to the eccentric anomaly E by means of the Kepler equation $\ell = E - e\,\sin\,E$, where e designates the eccentricity of the orbit, that in terms of Delaunay actions reads $e = \sqrt{1 - G^2/L^2}$. The angle g is the argument of the pericentre. It is reckoned from the pericentre of the orbit in the instantaneous orbital plane. The angle h is the argument of the node. Delaunay variables are not valid for either circular, or collision, or rectilinear or equatorial orbits. Thus, the domain of validity of Delaunay variables is given by the subset of $\mathbf{R}^6$

$$\Delta_D = [0, 2\,\pi) \times [0, 2\,\pi) \times [0, 2\,\pi) \times (0, +\infty) \times (0, L) \times (-G, G).$$

For more information, see the classical book by Brouwer and Clemence [1].

Specifically, if we push the computation to order bigger than one, after truncation of higher–order terms at an certain order M, the new Hamiltonian is going to be independent of the mean anomaly and subsequently, $\mathcal{K}$ will enjoy the action L as a new integral. The process to perform this transformation is called *Delaunay normalisation*, see the paper [4]; the steps to get the averaged (normalised) Hamiltonian are summarized next.

Let $n = 1/L^3$ represent the mean motion of the infinitesimal body orbiting the planet. Our interest is to perform a first–order theory, that is, to compute $\mathcal{K}_1$ and $\mathcal{W}_1$. The reason for not pushing the calculations to higher orders is that the first–order Hamiltonian retains all qualitative information we need, as all the equilibria we shall get out of the analysis of the reduced equations will be isolated, in order words, the first–order reduced system will be structurally stable.

From now on we drop the primes of the variables so that to avoid tedious notation. At first order, the transformed Hamiltonian is going to be $\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1$ where $\mathcal{K}_1$ must be independent of ℓ. Thus, $\mathcal{K}$ will define a system of two degrees of freedom in g, h, G and H. Thence, we will have that the Poisson bracket $\{\,\mathcal{K}_i\,,\,\mathcal{H}_0\,\} = 0$ for $i = 0, 1$.

We first identify $\mathcal{H}_0 \equiv \mathcal{K}_0$ and we need to solve the homology equation

$$n\,\frac{\partial \mathcal{W}_1}{\partial \ell} + \mathcal{K}_1 = \mathcal{H}_1.$$

The solution of this equation is the pair $(\mathcal{K}_1, \mathcal{W}_1)$ and one must first compute the average with respect to the mean anomaly $\mathcal{K}_1 = (2\,\pi)^{-1}\int_0^{2\,\pi}\mathcal{H}_1\,d\ell$. Then, $\mathcal{W}_1$ is a periodic function of ℓ, g and h and it is calculated through the integral $\mathcal{W}_1 = n^{-1}\int(\mathcal{H}_1 - \mathcal{K}_1)\,d\ell$. In this manner the Delaunay normalisation is carried out straightforwardly and in closed form for the eccentricity and for the mean anomaly. However, in general some difficulties arise when calculating the previous integrals in closed form. For our case, this problem is circumvented using some adequate changes of variables defined through the eccentric and the true anomalies.

Thus, we arrive at the Hamiltonian $\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1$ with $\mathcal{K}_0 = -1/(2\,L^2)$ and

$$\begin{aligned}\mathcal{K}_1 \;=\; \frac{\delta}{16\,L^5\,G^7\,(L+G)}\Big[&2\,(L+G)\,\big(4\,\beta\,L^3\,G^7 + 4\,\beta\,L^3\,G^5\,H^2 - \delta\,G^4 - 8\,L^2\,G^4\,H \\ &\qquad - \delta\,G^2\,H^2 + 3\,\delta\,L^2\,G^2 + 3\,\delta\,L^2\,H^2\big) \\ &+ (L-G)\,(G^2-H^2)\,(8\,\beta\,L^3\,G^5 + \delta\,G^2 + 2\,\delta\,L\,G + \delta\,L^2)\,\cos{(2\,g)}\Big].\end{aligned}$$

We remark that the construction of $\mathcal{K}_i$ for $i \geq 2$, and therefore the construction of $\mathcal{W}_i$ for $i \geq 1$, is required if one wants to compute the expressions of the invariant manifolds related to the original Hamiltonian vector field with high accuracy. Besides if we stop at order one, the explicit formula of $\mathcal{W}_1$ is used to build the direct and inverse change of coordinates, which is essential to estimate the error made after truncating the averaged Hamiltonian. So, the explicit change of coordinates up to first order may be readily calculated using the explicit expression of $\mathcal{W}_1$. In addition, it is also required that the error made after truncation will be maintained controlled in a certain domain of the Delaunay variables.

Note that $\mathcal{K}$ and $\mathcal{W}$ are well defined if $0 < G < L$ and $r > 0$, that is, we have to exclude rectilinear, circular and collision orbits from the study. However, circular trajectories will be considered as a limit situation of our approach using appropriate variables.

4 Reductions and Reduced Phase Spaces

4.1 Passage to $S^2_L \times S^2_L$

The next step of our approach consists in expressing $\mathcal{K}$ in terms of the appropriate invariants associated to the symmetries of the problem.

The integrals associated to L are the functions that are constant along the solutions of the system defined by $\mathcal{H}_0$. All these integrals can be expressed as functions of L, the components of the angular momentum vector $\mathbf{G} = (G_1, G_2, G_3)$ and the Laplace vector $\mathbf{A}_L = (A_1, A_2, A_3)$, i.e. the vector defined as

$$\mathbf{A}_L \;=\; \mathbf{P} \times \mathbf{G} - \frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

We remark that $G_3 \equiv H$, $\|\mathbf{G}\| = G$, $\|\mathbf{A}_L\| = e$ and $\mathbf{G} \cdot \mathbf{A}_L = 0$.

Now let us consider the mapping

$$\varphi_L : \mathbf{R}^6 \setminus (\{\mathbf{0}\} \times \mathbf{R}^3) \longrightarrow \mathbf{R}^6 : (\mathbf{x}\,,\, \mathbf{P}) \mapsto (\mathbf{a}\,,\, \mathbf{b}) \equiv (\mathbf{G} + L\,\mathbf{A}_L\,,\; \mathbf{G} - L\,\mathbf{A}_L),$$

with $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. Explicitly, the functions a_i and b_i can be given in terms of the coordinates $\mathbf{x}$ and momenta $\mathbf{P}$. Henceforth, G, H, $\cos g$, $\sin g$, $\cos h$, $\sin h$, $\cos I$ and $\sin I$ can be easily expressed in terms of $\mathbf{a}$ and $\mathbf{b}$ and the positive constant L, see [2]. Now, a Hamiltonian independent of ℓ can be written as a function of the invariants $\mathbf{a}$ and $\mathbf{b}$ and the constant $L > 0$. Notice that the way in which the invariants appear in the corresponding Hamilton function depends on each specific problem.

Now, fixing a value of $L > 0$, the product of the two–spheres

$$S^2_L \times S^2_L \;=\; \{(\mathbf{a}, \mathbf{b}) \in \mathbf{R}^6 \;\mid\; a_1^2 + a_2^2 + a_3^2 = L^2, \quad b_1^2 + b_2^2 + b_3^2 = L^2\} \tag{8}$$

is the phase space associated to Hamiltonian systems of Keplerian type independent of ℓ, that is, perturbed Keplerian Hamiltonians for which L is an integral. This result was first reported by Moser [9] using a regularisation technique based on stereographic projections. Later on it has been described and used by Cushman [2]. We stress that the reduction is regular, that is to say, $S^2_L \times S^2_L$ is a smooth manifold.

The introduction of the invariants extends the use of Delaunay and polar–nodal variables, as we can include equatorial, circular and rectilinear orbits, see for instance [11].

4.2 Reduction of the axial symmetry

Now we briefly analyse what happens for systems invariant under the axial symmetry, that is, for Hamilton functions independent of the argument of the node. We start by fixing a value of H (with $|H| \leq G$), this integral H can be understood as an S^1–action, or the action of the one–dimensional unitary group $U(1)$ over the space of coordinates and momenta such that

$$\begin{array}{rccc} \varrho: & S^1 \times (\mathbf{R}^6 \setminus (\{\mathbf{0}\} \times \mathbf{R}^3)) & \longrightarrow & \mathbf{R}^3 \times \mathbf{R}^3 \\ & (R_h, (\mathbf{x}, \mathbf{P})) & \mapsto & (R_h \mathbf{x}, R_h \mathbf{P}), \end{array} \tag{9}$$

where

$$R_h = \begin{pmatrix} \cos h & \sin h & 0 \\ -\sin h & \cos h & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $0 \leq h < 2\pi$. This is a singular (non–free) action because there are non–trivial isotropy groups. The subspace defined by $\{(0,0,z) \mid z \in \mathbf{R}\}$ is invariant under all rotations around the axis z. This is in contrast to the regular reduction obtained by doing L an integral, where all the isotropy groups were trivial.

If we denote $\tau = (\tau_1, \tau_2, \tau_3)$, we can define the mapping

$$\pi_H : S^2_L \times S^2_L \longrightarrow \{H\} \times \mathbf{R}^3 : (\mathbf{a}, \mathbf{b}) \mapsto (H, \tau_1, \tau_2, \tau_3) \equiv (H, \tau),$$

where we can express the invariants τ_1, τ_2 and τ_3 in terms of $\mathbf{a}$ and $\mathbf{b}$ as

$$\tau_1 = \frac{1}{2}(a_3 - b_3), \quad \tau_2 = a_1 b_2 - a_2 b_1, \quad \tau_3 = a_1 b_1 + a_2 b_2.$$

Next the corresponding phase space, named $\mathcal{T}_{L,H}$, is defined as the image of the product $S^2_L \times S^2_L$ by π_H, that is,

$$\mathcal{T}_{L,H} = \pi_H(S^2_L \times S^2_L) = \{\tau \in \mathbf{R}^3 \mid \tau_2^2 + \tau_3^2 = [(L+\tau_1)^2 - H^2]\,[(L-\tau_1)^2 - H^2]\}, \tag{10}$$

for $0 \leq |H| \leq L$ and $L > 0$. Note that τ_2 and τ_3 always belong to the interval $[H^2 - L^2, L^2 - H^2]$, whereas τ_1 belongs to $[|H| - L, L - |H|]$.

Rectilinear motions must satisfy $G = H = 0$. Using the constraint appearing in (10), we know that they are defined on the one–dimensional set

$$\mathcal{RE}_{L,0} = \{\tau \in \mathbf{R}^3 \mid \tau_2 = 0, \quad \tau_3 = \tau_1^2 - L^2\}.$$

Thus, excepting orbits with $\|\mathbf{x}\| = 0$ we could analyse rectilinear trajectories. Circular type of orbits are concentrated on a unique point of $\mathcal{T}_{L,H}$ with coordinates $(0,0,L^2 - H^2)$ — or on a unique point of $\mathcal{T}_{L,0}$ with coordinates $(0,0,L^2)$ — whereas equatorial trajectories in this twice–reduced phase space are represented in the negative extreme point of $\mathcal{T}_{L,H}$ with coordinates $(0,0,H^2 - L^2)$, respectively in the point $(0,0,-L^2)$ of $\mathcal{T}_{L,0}$.

This time the twice–reduced system is represented by a Hamiltonian expressed in terms of the τ's. It defines a one–degree–of–freedom system, with L and H as two independent integrals. After dropping constant terms, $\mathcal{K}$ is written in terms of τ, the integrals L and H and the constant term δ. Thus, we get a the rational function in the τ's.

4.3 Reduction of the finite symmetries

In the next paragraphs we follow similar steps to those of Cushman and Sadovskií [3] in their treatment of the Zeeman–Stark effect using singular reduction.

First we notice that the original Hamilton function $\mathcal{H}$ enjoys the following discrete symmetries:

$$\begin{aligned}
\mathcal{R}_1: &\ (x, y, z, P_x, P_y, P_z) \longrightarrow (x, -y, -z, -P_x, P_y, P_z),\\
\mathcal{R}_2: &\ (x, y, z, P_x, P_y, P_z) \longrightarrow (x, -y, z, -P_x, P_y, -P_z),\\
\mathcal{R}_3: &\ (x, y, z, P_x, P_y, P_z) \longrightarrow (x, y, -z, P_x, P_y, -P_z).
\end{aligned} \tag{11}$$

It is clear that $\mathcal{R}_3$ can be expressed as the combination of $\mathcal{R}_1$ and $\mathcal{R}_2$.

These $\mathbf{Z}_2$–symmetries (reflections) are conserved through the two previous reductions. In particular, in terms of the τ's, these discrete symmetries (reflections) are given by:

$$\begin{aligned}
\mathcal{R}_1: &\ (\tau_1, \tau_2, \tau_3) \longrightarrow (-\tau_1, \tau_2, \tau_3),\\
\mathcal{R}_2: &\ (\tau_1, \tau_2, \tau_3) \longrightarrow (\tau_1, -\tau_2, \tau_3),\\
\mathcal{R}_3: &\ (\tau_1, \tau_2, \tau_3) \longrightarrow (-\tau_1, -\tau_2, \tau_3).
\end{aligned}$$

Hamiltonian $\bar{\mathcal{K}}$ can be written as a function of the type $\bar{\mathcal{K}}(\tau_1^2, -, \tau_3; \delta, \beta, L, H)$, and therefore it enjoys the symmetries $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$, as it should be expected. Hence, it is possible to further reduce Hamiltonian $\bar{\mathcal{K}}$.

In this way, we introduce new functions with the aim of taking advantage of these discrete symmetries. Indeed we define the functions σ_1 and σ_2 as:

$$\sigma_1 = (L - |H|)^2 - \tau_1^2, \qquad \sigma_2 = \frac{\sqrt{L^2 + H^2 - \tau_1^2 + \tau_3}}{\sqrt{2}}. \tag{12}$$

At this point we stress some remarks:

- The functions σ_1 and σ_2 are indeed invariants under the action of the three finite symmetries $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$, but we could have defined other invariants, say $\bar{\sigma}_1$ and $\bar{\sigma}_2$, as two functions depending on τ_1^2 and τ_3, i.e. $\bar{\sigma}_1 \equiv \bar{\sigma}_1(\tau_1^2, \tau_3; L, H)$ and $\bar{\sigma}_2 \equiv \bar{\sigma}_2(\tau_1^2, \tau_3; L, H)$.

- With our choice σ_1 depends on both G and g whereas σ_2 is exactly the norm of the angular momentum vector. The relationship between the Delaunay variables g and

G with the σ's is as follows:

$$
\begin{aligned}
G &= \sigma_2, \\
\cos g &= \pm\sqrt{\frac{L^2\,H^2 - 4\,\sigma_1\,\sigma_2^2 + 4\,\sigma_2^4 - 2\,L\,|H|\,(\sigma_1 + 2\,\sigma_2^2)}{5\,L^2\,H^2 - 4\,(L^2+H^2)\,\sigma_2^2 + 4\,\sigma_2^4 - 2\,L\,|H|\,(L^2+H^2-2\,\sigma_2^2)}}, \qquad (13)\\
\sin g &= \pm\sqrt{\frac{4\,L^2\,H^2 - 4\,(L^2+H^2-\sigma_1)\,\sigma_2^2 - 2\,L\,|H|\,(L^2+H^2-\sigma_1-4\,\sigma_2^2)}{5\,L^2\,H^2 - 4\,(L^2+H^2)\,\sigma_2^2 + 4\,\sigma_2^4 - 2\,L\,|H|\,(L^2+H^2-2\,\sigma_2^2)}}.
\end{aligned}
$$

- The quantities sin I and cos I are also functions of σ_2.

The reduction process is now achieved by using a suitable map:

$$\sigma_{L,H} : \mathcal{T}_{L,H} \longrightarrow \mathcal{U}_{L,H} : (\tau_1, \tau_2, \tau_3) \mapsto (\sigma_1, \sigma_2) \quad \text{for} \quad 0 \le |H| \le L,$$

such that σ_1 and σ_2 are given through (12). The resulting space is the most reduced phase space and it is denoted by $\mathcal{U}_{L,H}$ and, for $H = 0$, by $\mathcal{U}_{L,0}$. For $|H| > 0$, the space $\mathcal{U}_{L,H}$ is given by:

$$\mathcal{U}_{L,H} = \Big\{(\sigma_1, \sigma_2) \in \mathbf{R}^2 \;\mid\; \frac{(\sigma_2^2 - L\,|H|)^2}{\sigma_2^2} \le \sigma_1 \le (L - |H|)^2, \quad |H| \le \sigma_2 \le L\Big\}, \qquad (14)$$

whereas for $H = 0$, the space $\mathcal{U}_{L,0}$ is given through:

$$\mathcal{U}_{L,0} = \{(\sigma_1, \sigma_2) \in \mathbf{R}^2 \;\mid\; \sigma_2^2 \le \sigma_1 \le L^2, \quad 0 \le \sigma_2 \le L\}. \qquad (15)$$

Indeed the constraints between the new invariants are deduced from (10):

(i) $|H| > 0$: the fully–reduced phase space $\mathcal{U}_{L,H}$ is bounded by the curves $\sigma_1\,\sigma_2^2 = (\sigma_2^2 - L\,|H|)^2$ and $\sigma_1 = (L - |H|)^2$,

(ii) $H = 0$: the fully–reduced phase space $\mathcal{U}_{L,0}$ is bounded by the lines $\sigma_1 = \sigma_2^2, \quad \sigma_2 = 0$ and $\sigma_1 = L^2$.

In Fig. 2 we have plotted the fully–reduced phase space for $H = 0$ and for $H \neq 0$.

Note that the definition of σ_1 and σ_2 together with the constraints inherited from the τ's, are used to define the spaces $\mathcal{U}_{L,H}$ and $\mathcal{U}_{L,0}$ from the reduction of the twice–reduced phase spaces $\mathcal{T}_{L,H}$ and $\mathcal{T}_{L,0}$.

We also observe that $\mathcal{U}_{L,H}$ has two singular points: $((L-|H|)^2, |H|)$ and $((L-|H|)^2, L)$ while $\mathcal{U}_{L,0}$ has three singular points: $(L^2, 0)$, (L^2, L) and $(0, 0)$. The singularities of $\mathcal{U}_{L,H}$ have been introduced through the mapping $\sigma_{L,H}$ and are indeed spurious. As well the singularities $(L^2, 0)$, (L^2, L) have been introduced by reducing out the discrete symmetries and they are spurious too. The point $(0, 0)$ is the singularity coming from the singular points $(\pm L, 0, 0)$ of $\mathcal{T}_{L,0}$.

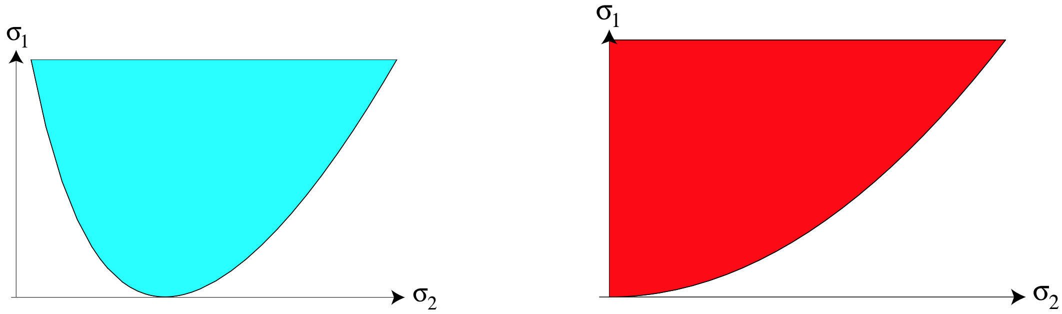

Figure 2: On the right, fully–reduced phase space for $|H| > 0$. The coordinates of the extreme points of $\mathcal{U}_{L,H}$ are $((L-|H|)^2, |H|)$ and $((L-|H|)^2, L)$ whereas the space reaches its lowest point at $(0, \sqrt{L\,|H|})$. On the left fully–reduced phase space for $H = 0$. The coordinates of the extreme points of $\mathcal{U}_{L,0}$ are $(L^2, 0)$, (L^2, L) and $(0, 0)$.

A single point in the interior of $\mathcal{U}_{L,0}$ or in the interior of $\mathcal{U}_{L,H}$ is in correspondence with four points in the space $\mathcal{T}_{L,0}$ or in $\mathcal{T}_{L,H}$, respectively. Besides, a single point in the regular part of the boundaries of either $\mathcal{U}_{L,0}$ or $\mathcal{U}_{L,H}$ is respectively related to two points of $\mathcal{T}_{L,0}$ or of $\mathcal{T}_{L,H}$. In addition to this, to each of the two singular points of the boundary of $\mathcal{U}_{L,H}$, it corresponds one point of $\mathcal{T}_{L,H}$. Finally, the points of $\mathcal{U}_{L,0}$ with coordinates $(L^2, 0)$ and (L^2, L) are related respectively to the points $(0, 0, -L^2)$ and $(0, 0, L^2)$ on $\mathcal{T}_{L,0}$ whereas the point whose coordinate is $(0, 0)$ in $\mathcal{U}_{L,0}$ corresponds to the singular points $(\pm L, 0, 0)$ of $\mathcal{T}_{L,0}$.

Next we stress that equatorial, rectilinear and circular type of motions are easily characterised in the fully–reduced phase space. More specifically, for locating circular "trajectories" we need that $\sigma_2 = L$, and so the set of circular "orbits" is zero–dimensional and is defined by the point $((L-|H|)^2, L)$ in $\mathcal{U}_{L,H}$ and by (L^2, L) in $\mathcal{U}_{L,0}$ respectively. For equatorial "orbits" we make $\sigma_2 = |H|$ henceforth the set of equatorial "orbits" is defined by the points $((L-|H|)^2, |H|)$ in $\mathcal{U}_{L,H}$ and by $(L^2, 0)$ in $\mathcal{U}_{L,0}$. Finally, rectilinear "orbits" define a one–dimensional set in $\mathcal{U}_{L,0}$ that, since $\sigma_2 = 0$, is simply characterised by the segment of $\mathcal{U}_{L,0}$ given by $(\sigma_1, 0)$ with $0 \le \sigma_1 \le L^2$). We have depicted this type of special motions in Fig. 3.

4.4 The fully–reduced Hamilton function

From (12) we easily deduce that $\tau_1^2 = (L-|H|)^2 - \sigma_1$ and $\tau_4 = \sqrt{2}\,\sigma_2$. Hence, Hamilton function $\bar{\mathcal{K}}$ can be expressed in terms of the new invariants σ_1 and σ_2 quite straightfor-

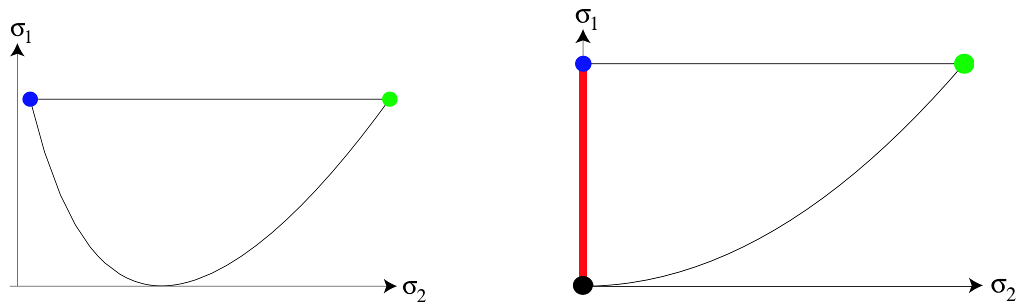

Figure 3: In the two fully–reduced phase spaces the leftmost points correspond to equatorial motions whereas the rightmost points represent circular motions. For the phase space $\mathcal{U}_{L,0}$, the vertical segment in the axis σ_1 corresponds to rectilinear motions whereas the point of coordinates $(0,0)$ represents the non–spurious singularity.

wardly. We arrive at:

$$\begin{aligned}\bar{\bar{\mathcal{K}}} \;=\; \frac{\delta}{16\,L^5\,\sigma_2^7\,(L+\sigma_2)^2} \Big\{ & \delta\,(L+\sigma_2)^2 \Big[5\,L^2\,H^2 \\ & - (3\,H^2 - 4\,L\,|H| - 5\,L^2 - 2\,\sigma_1)\,\sigma_2^2 - 3\,\sigma_2^4\Big] \\ & + 16\,L^2\,\sigma_2^4 \Big[- L^2\,H - 2\,L\,H\,\sigma_2 + H\,\left(-1+\beta\,L^2\,H\right)\,\sigma_2^2 \\ & + \beta\,L\,\left(2\,L\,|H| + \sigma_1\right)\,\sigma_2^3 + \beta\,L^2\,\sigma_2^4\Big]\Big\}\end{aligned} \tag{16}$$

Notice that $\bar{\bar{\mathcal{K}}}$ is singular for $\sigma_2 = 0$. It is not a surprise as σ_2 is equivalent to G and the original Hamilton function $\mathcal{H}$ is not well defined for rectilinear trajectories. A way to circumvent this trouble is based on regularisation techniques, but this is outside the purpose of the present paper. Nevertheless we need to be very careful when analysing orbits with σ_2 small, since the perturbation $\bar{\bar{\mathcal{K}}}$ could be bigger than the unperturbed part and therefore our study could have no sense for almost rectilinear trajectories. The reader can consult [8] to see how we have avoided this problem, controlling the size of $|\bar{\bar{\mathcal{K}}}|$.

In the next section we shall analyse the equilibrium points of the equation related to $\bar{\bar{\mathcal{K}}}$ through the study of the absolute and relative extremes of it.

5 Relative Equilibria and Bifurcations

5.1 Simplification: the case $\delta^2 = 0$

Assuming first that $\delta^2 = 0$ in $\bar{\bar{\mathcal{K}}}$, the analysis of the resulting system becomes much easier, as it is seen with detail in [8].

There always exist two equilibrium points, those points where the two curves delimiting the boundary of $\mathcal{U}_{L,H}$ meet. Their coordinates are:

$$E_1 \equiv ((L-|H|)^2, |H|), \qquad E_2 \equiv ((L-|H|)^2, L),$$

and they correspond to the class of equatorial and circular "orbits" respectively.

To determine the rest of the equilibria, two cases must be considered:

(a) those equilibria located on the rectilinear part of the boundary given by the curve $\sigma_1 = (L - |H|)^2$, under the restriction $|H| \le \sigma_2 \le L$,

(b) those equilibria located on the curved part of the boundary defined by $\sigma_1\,\sigma_2^2 = (\sigma_2^2 - L\,|H|)^2$ and $|H| \le \sigma_2 \le L$.

Concerning *(a)*, a new equilibrium appears whose coordinate σ_2 corresponds to as a root of a quartic polynomial equation. We do not give the details here, but the point has coordinates

$$E_3 \equiv ((L - H)^2, \sigma_2^*),$$

for some $\sigma_2^* \in [|H|, L]$. It is worth to note that this point appears or disappears whenever one crosses the hypersurfaces:

$$\begin{aligned} \Gamma_1 &\equiv 3\,\beta\,L^2\,H^2 + \beta\,L^4 - 12\,H = 0,\\ \Gamma_2 &\equiv 2\,\beta\,L\,H^3 - 3\,(L + H) = 0. \end{aligned} \tag{17}$$

In case *(b)*, we need to distinguish between prograde ($H \ge 0$) and retrograde ($H < 0$) motions. Whenever $H < 0$, the critical point:

$$E_0 \equiv \left(\frac{(\sigma_2^{-2} - L\,H)^2}{\sigma_2^{-2}}, \sigma_2^- \right)$$

for some $\sigma_2^- \in [|H|, L]$, a root of a quartic polynomial equation, appears or disappears as one crosses the hypersurface

$$\Gamma_0 \equiv 5\,\beta\,L^2\,H^2 - \beta\,L^4 - 12\,H = 0. \tag{18}$$

For the case of prograde "orbits" two equilibrium points can be obtained

$$E_4 \equiv \left(\frac{(\sigma_{20}^2 - L\,H)^2}{\sigma_{20}^2}, \sigma_{20} \right), \qquad E_5 \equiv \left(\frac{(\sigma_{21}^2 - L\,H)^2}{\sigma_{21}^2}, \sigma_{21} \right),$$

where σ_{20} and σ_{21} are the two positive real roots of a quartic polynomial. Moreover, these points appear or disappear if one crosses the hypersurfaces:

$$\begin{aligned} \Gamma_3 &\equiv 2\,\beta\,L^2\,H^2 - 3\,(L + H) = 0,\\ \Gamma_4 &\equiv \beta\,L^2\,(L^2 - 5\,H^2) + 12\,H = 0,\\ \Gamma_5 &\equiv -3\,H + 8\,\beta\,L^2\,H^2 - 16\,\beta\,L^4 - 6\,\beta^2\,L^4\,H^3 + \beta^4\,L^8\,H^5 = 0. \end{aligned} \tag{19}$$

All lines Γ_k correspond to parametric bifurcations of pitchfork type except for Γ_5 that corresponds to a saddle–centre bifurcation. This conclusion follows from the number of equilibrium points involved in the bifurcation together with the Index Theorem and a theorem on the multiplicity of a root for a vanishing resultant. From the discussion above it follows that for a fixed value of β, the plane (H, L) is divided into different regions where the number of equilibria changes. These regions are determined by the curves defined by (17), (18) and (19) together with the constraint $|H| \leq L$ as is depicted in Fig. 4.

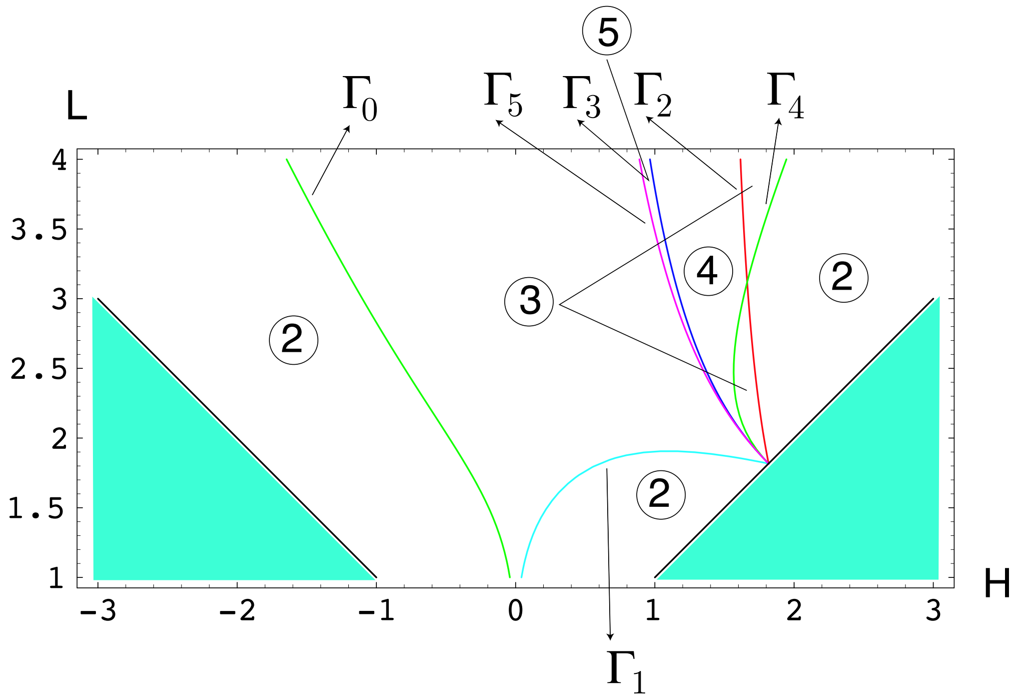

Figure 4: The case $\delta^2 = 0$: plane of parameters in terms of the number of equilibria for $\delta = 0.01$ and $\beta = 0.4$. The number of equilibria in each region delimited by the curves Γ_k is encircled.

We stress the presence of a saddle–connection bifurcation. Note that E_3 appears after a pitchfork bifurcation involving E_1 and it disappears through a pitchfork bifurcation involving E_2. Then, the homoclinic loops attached to equilibria E_1 and E_2 eventually merge and then interchange the stable points they encircle. This happens when the energy for E_1 and E_2 "orbits" is the same, that is, the saddle–connection bifurcation takes place in the hypersurface:

$$\Gamma_6 \equiv \beta \, L^2 \, H^2 \, (L + H) - 2 \, (L^2 + 2 \, L \, H + H^2) = 0.$$

5.2 The case $\delta^2 \neq 0$

In the general situation, the number of critical points and bifurcation lines change. We do not give details here and refer to the paper [8].

For $\delta > 0$ the negative half part of the parameter plane does not experience noticeable changes and its aspect is the same as the one observed in Fig. 4. However, some changes must be remarked in the right half part of the parameter plane, see Fig. 5.

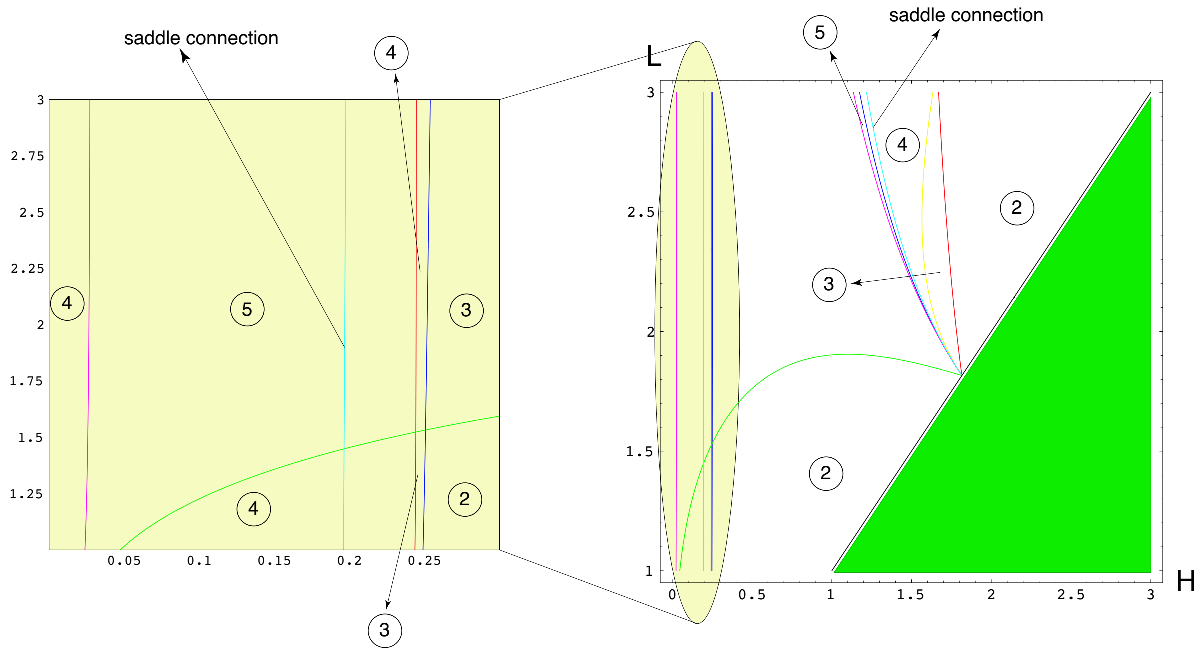

Figure 5: The case $\delta^2 \neq 0$: new bifurcation lines for $\delta > 0$ in the positive half part of the parameter plane. We have fixed $\delta = 0.01$ and $\beta = 0.4$. The number of equilibria in each region is encircled.

When $\delta < 0$, besides the presence of new critical points, the bifurcation lines associated to them in the curved part of the boundary of $\mathcal{U}_{L,H}$, namely Γ_0 and $H = 0$, are slightly modified. In this way, the first one does not experience noticeable changes but the second ($H < 0$) is displaced to the left in such a way that it intersects the leftmost point of the region where the interior equilibrium exists, see Fig. 6. These changes imply a richer scenario of bifurcations, but it involves equilibria that correspond to collision "trajectories". Moreover, the new bifurcation lines are outside the region of meaningful orbits and all the new bifurcations we can report involve meaningless equilibria from a physical standpoint.

In conclusion, if we do not consider those points corresponding to collision "orbits", which are not relevant from a physical point of view, the influence of the terms in δ^2 does not modify the description of bifurcations reported within this section and neither the stability character of the equilibria.

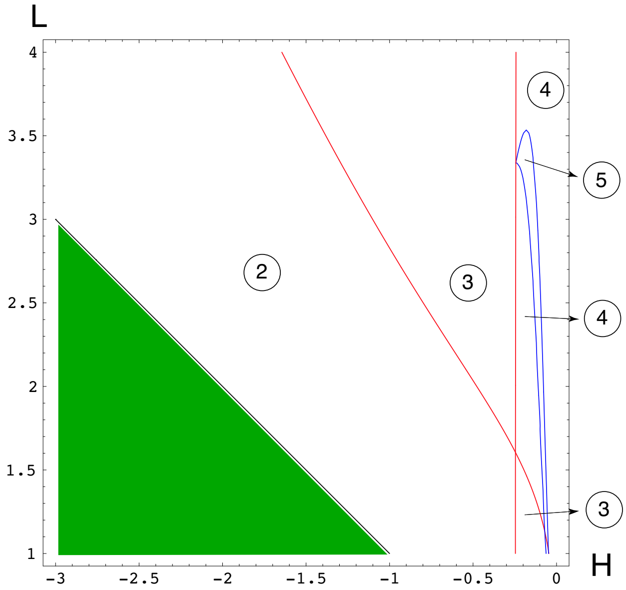

Figure 6: The case $\delta^2 \neq 0$: new bifurcation lines for $\delta < 0$ in the negative half part of the parameter plane. For this picture we have taken $\delta = -0.01$ and $\beta = 0.4$. The number of equilibrium points in each region is encircled. When there are five equilibria, one of them is located at the interior of the fully–reduced phase space.

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