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The Gravitational Parameter from the Viewpoint of Canonical Celestial Mechanics

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Abstract

We consider the introduction of the gravitational coupling parameter μ as a canonical variable. In the special case of Keplerian systems, this variable can be incorporated, as a canonical orbital element (in fact, a canonical constant), into the segment of momenta of certain canonical sets. The corresponding canonically conjugate coordinate turns out to be an eccentric–like anomaly.

Key words: Keplerian systems, gravitational coupling parameter, extended phase space, canonical transformations, canonical orbital elements, time transformations.

1 Introduction

Baumgarte (1980, §2; Appendix 1) studied canonical transformations to make a canonical variable (momentum) of the gravitational coupling parameter μ, its conjugate coordinate-type variable being –essentially– the eccentric anomaly of elliptic Keplerian motion.

In a certain sense, this way of thinking may be looked on as pertaining to a kind of *dual* version of attempts to introduce some anomaly–like parameters as canonical elements (e.g., Andoyer 1913, and references therein) within the canonical theory of Keplerian systems; in particular, this sort of *duality* between μ and an eccentric–like anomaly is found in Levi–Civita (1913, specially §2), the energy of the system being treated as an absolute constant. Certain classical results due to these two authors, as well as several proposals by some other researchers –dating back to the second decade of the 20th century–, were thoroughly examined, systematized and extended by Soudan (1953, 1955), whose analysis concentrates on canonical orbital elements for elliptic–type quasi–Keplerian motion. In this respect, see also Hagihara (1970, §5.20).

Further generalizations, not only for bound orbits, were derived by Cid & Calvo (1973/1975), starting from polar nodal variables. Palacios (1973/1977) extended Soudan's treatment of quasi–Keplerian motion to the case of hyperbolic–type orbits.

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After a review of some of these precedents for a canonical approach to the treatment and interpretation of the gravity parameter μ, we give an outline of their possible extensions within the framework of a *uniform* formulation and description of two–body motion, i.e., *regardless of the type of conic orbit* at issue (Stiefel & Scheifele 1971, §11; Floría 2000).

2 Baumgarte's First Approach: Naïve Homogeneous Formalism

In this Section we reformulate the canonical approach taken by Baumgarte (1980, §2) to deal with elliptic orbits in the Kepler problem, provided that the eccentric anomaly is chosen as the independent variable. On the basis of his results, we sketch out a proposal for their adaptation to a *universal treatment* of Keplerian orbits.

Let the canonical set of Cartesian variables be represented by the symbols $(\mathbf{p}, \mathbf{x})$, where the notation $\mathbf{p}^t \equiv (p_1, p_2, p_3)$ refers to the momenta canonically conjugate to the usual Cartesian coordinates $\mathbf{x}^t \equiv (x_1, x_2, x_3,)$; accordingly, $r^2 = || \mathbf{x} ||^2 \equiv \mathbf{x}^2 = x_1^2 + x_2^2 + x_3^2$. Here the notation $(\ )^t$ represents the usual matrix transposition, while t stands for the independent variable (physical time).

In these notations, the unperturbed two–body motion, with t as the independent variable, is governed by the standard, conservative *Keplerian Hamiltonian*

$$\mathcal{H}_K \equiv \mathcal{H}_K(\mathbf{p}; \mathbf{x}; -) = (1/2)\, \mathbf{p}^2 - (\mu/r) \ , \tag{1}$$

where μ designates the *gravitational parameter* of the two–body system (essentially attached to the attracting central mass). If $\mathcal{E}$ is the total energy of the system, the problem possesses the first integral $\mathcal{H}_K = \mathcal{E}$. In the case of bound orbits ($\mathcal{E} < 0$), one may introduce an action variable $\alpha = \mu/\sqrt{-2\,\mathcal{E}}$, which suggests the use of the *homogeneous canonical formalism* (Stiefel & Scheifele 1971, §30, §34, §37), on enlarging the ordinary phase space with two additional dimensions, that is, introducing a new pair of canonically conjugate variables (p_0, x_0), where $p_0 = -\mathcal{E}$ is minus the total energy, and $x_0 \equiv t$. In this way, the physical time t is viewed as a dependent canonical variable (more precisely, an additional coordinate occurring on equal footing with the usual spatial coordinates). In addition to this, *transformations of the independent variable* $x_0 \equiv t \longrightarrow s$, from t to a new fictitious time s, can be defined by means of differential relations of the type

$$dt = \widetilde{f} ds \,, \quad t' = \widetilde{f} \,, \quad \text{where} \quad \widetilde{f} \equiv \widetilde{f}(\mathbf{p}, \mathbf{x}, p_0, x_0) > 0 \,, \quad (\ )' \equiv d/ds. \tag{2}$$

In *extended phase space formulation*, with t as the independent variable, the preceding Keplerian Hamiltonian reads (after adequate choice of initial conditions)

$$\mathcal{H}_h \equiv \mathcal{H}_h(\mathbf{p}, p_0; \mathbf{x}-; -) = \mathcal{H}_K + p_0 = 0. \tag{3}$$

An extended phase space Hamiltonian which vanishes initially will vanish on any solution of the system of differential equations. The corresponding homogeneous Hamiltonian, with the *fictitious time* s taking the role of the *independent variable*, is

$$\mathcal{H}_h^* \;=\; \mathcal{H}_h\, \widetilde{f} = (\mathcal{H}_K + p_0)\, \widetilde{f} = \frac{1}{2}\,\mathbf{p}^2 \widetilde{f} + p_0 \widetilde{f} - \frac{\mu}{r}\widetilde{f}\,. \tag{4}$$

This scaling function $\widetilde{f}$ for the differential time transformation (2) is now chosen so as to create and isolate an additive term with expression $-\mu/\sqrt{2p_0}$, say from

$$-\mu\widetilde{f}/r \;\;\longleftrightarrow\;\; -\mu/\sqrt{2p_0} \;\;\Longrightarrow\;\; \widetilde{f} = r/\sqrt{2p_0}. \tag{5}$$

The pseudo–time $s \equiv E$ is the eccentric anomaly, and the Keplerian Hamiltonian becomes

$$\mathcal{H}_h^* \;=\; \Big[(1/2)\,\mathbf{p}^2 + p_0\Big]\,(\,r/\sqrt{2p_0}\,) \;-\; (\,\mu/\sqrt{2p_0}\,)\,. \tag{6}$$

One may attempt to *eliminate the additive isolated term* $-\mu/\sqrt{2p_0}$ from $\mathcal{H}_h^*$: given that the coordinate x_0 is ignorable in $\mathcal{H}_h^*$, the canonically conjugate momentum p_0 is constant. Dropping the constant expression $-\mu/\sqrt{2p_0}$ allows one to omit the x_0'–equation.

Accordingly, an *alternate treatment* of this question proceeds as follows: the expression $(-\mu)$, originally a parameter, can be contemplated as a canonical momentum in $\mathcal{H}_h^*$,

$$-\mu \;=\; p_4. \tag{7}$$

The coordinate x_4, canonically conjugate to p_4, is ignorable in $\mathcal{H}_h^*$:

$$\mathcal{H}_h^* \;\equiv\; \mathcal{H}_h^*\,(p_0, p_1, p_2, p_3, p_4; -, x_1, x_2, x_3, -; -) = \left[\frac{1}{2}\,\mathbf{p}^2 + p_0\right]\frac{r}{\sqrt{2p_0}} + \frac{p_4}{\sqrt{2p_0}}\,. \tag{8}$$

Now, perform a *canonical transformation* to new variables $(\overline{p}_k, \overline{x}_k)\,, k = 0, 1, 2, 3, 4,$ such that the momenta should change according to the equations

$$p_i \;= \overline{p}_i\,, \quad i = 0, 1, 2, 3, \qquad \overline{p}_4 = p_4/\sqrt{2p_0} \;\equiv\; -\,\mu/\sqrt{2p_0}\,. \tag{9}$$

Thus, $x_j = \overline{x}_j$ $(j = 1, 2, 3)$, and the transformation operates non–trivially on the variables of interest, that is, on the variables (p_0, p_4, x_0, x_4),

$$\overline{p}_4 \;=\; p_4/\sqrt{2p_0}\,, \qquad p_0 = \overline{p}_0\,, \tag{10}$$

$$x_4 \;=\; \overline{x}_4/\sqrt{2\overline{p}_0}\,, \qquad x_0 = \overline{x}_0 - (\,\overline{p}_4/2\overline{p}_0\,)\,\overline{x}_4\,, \qquad \text{with} \tag{11}$$

$$\overline{x}_4 \;=\; s = E\;\;(+\,\text{const.})\,, \tag{12}$$

after which the Hamiltonian of the Kepler problem is converted into the function

$$\overline{\mathcal{H}}_h^* \;\equiv\; \frac{\overline{r}}{\sqrt{2\overline{p}_0}}\left[\frac{1}{2}\,\overline{\mathbf{p}}^2 + \overline{p}_0\right] + \overline{p}_4 \;=\; \frac{r}{\sqrt{2p_0}}\left[\frac{1}{2}\,\mathbf{p}^2 + p_0\right] + \overline{p}_4\,, \tag{13}$$

where, for convenience in writing, one has simplified the notation by omitting the overbars according to the *convention* $\overline{r} \to r$, $\overline{\mathbf{p}} \to \mathbf{p}$, and $\overline{p}_0 \to p_0$. This function can be viewed as a *new homogeneous Hamiltonian*, in which $\overline{p}_4$ would play the part of the energy–like term of the canonical couple $(\overline{p}_4, \overline{x}_4)$. Consequently, if the isolated momentum $\overline{p}_4$ is removed,

$$\mathcal{H}^* \equiv (\, r/\sqrt{2p_0}\,) \left[(\, 1/2\,)\, \mathbf{p}^2 + p_0 \right] \tag{14}$$

can be viewed as a *non-homogeneous Hamiltonian* (in an ordinary phase–space coordinatized by the chart $(\overline{p}_i, \overline{x}_i)\,, i = 0, ..., 3$), to which an action variable $\alpha = \mathcal{H}^*$ is attached.

The new independent variable corresponding to $\mathcal{H}^*$ is the canonical variable $\overline{x}_4$, conjugate to $\overline{p}_4$, that is, $\overline{x}_4 = s = E$ (up to an additive constant).

These results, originally restricted to elliptic motion, can be generalized to the case of other types of two–body conic–section orbits. If a *unified treatment* of Keplerian orbital motion is desired, use can be made of *universal* functions, variables and parameters (Stiefel & Scheifele 1971, §11; Battin 1987, §4.5, §4.6; Floría 2000). To this end, *irrespective of the specific kind of orbit*, a *new time parameter* τ is directly introduced, instead of the elliptic eccentric anomaly, via Eq. (2): the scaling function $\widetilde{f}$ in Eq. (5) is to be taken as

$$\widetilde{f} = r \implies t \longrightarrow \tau : dt = r d\tau\,, \quad \text{(universal) Sundman's transformation,} \tag{15}$$

thanks to which the transition to variables $(\overline{p}_k, \overline{x}_k)\,, k = 0, ..., 4$, can be avoided, Hamiltonian (4) can be dealt with exactly in the same way as $\overline{\mathcal{H}}_h^*$ in Eq. (13), and the independent argument τ is, essentially, the canonical coordinate x_4 conjugate to p_4.

3 Baumgarte's Second Approach: Hamilton–Jacobi Theory

Referring to Baumgarte (1980, Appendix 1), the Kepler problem can also be treated within the extended–phase–space Hamilton–Jacobi theory formulated in (enlarged) spherical polar variables, for which the notations $(x_0, r, \theta, \varphi; p_0, p_r, p_\theta, p_\varphi)$ are adopted,

$$\begin{aligned} x_1 &= r\cos\theta\cos\varphi\,, & x_2 &= r\cos\theta\sin\varphi\,, & x_3 &= r\sin\theta\,, \\ p_r &= \dot{r}\,, & p_\theta &= r^2\dot{\theta}\,, & p_\varphi &= r^2\dot{\varphi}\cos^2\theta\,, \end{aligned}$$

with $x_0 \equiv t$. Consider also $-\mu = p_4$, and the homogeneous Keplerian Hamiltonian

$$\mathcal{H}_h \equiv \mathcal{H} + p_0 = \frac{1}{2}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\cos^2\theta} \right] + \frac{p_4}{r} + p_0 \;\; (\, = 0\,). \tag{16}$$

As well known, p_φ is the z–component of the angular momentum vector, while the combination $p_\theta^2 + p_\varphi^2/\cos^2\theta$ measures the square of the magnitude of that vector. In view of the above Hamiltonian, the momenta p_0 and p_4 are already *elements*. Take

$$p_0 = P_0\,, \quad \sqrt{p_\theta^2 + (\, p_\varphi^2/\cos^2\theta\,)} = P_\psi\,, \quad p_\varphi = P_\varphi\,, \quad p_4 = P_4\,, \tag{17}$$

and the generating function $S \equiv S\,(x_0, r, \theta, \varphi, x_4; P_0, P_\psi, P_\varphi, P_4)$ as a complete solution to the Hamilton–Jacobi partial differential equation linked to Hamiltonian (16),

$$\frac{1}{2}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left\{\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{\cos^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right\}\right] + \frac{1}{r}\frac{\partial S}{\partial x_4} + \frac{\partial S}{\partial x_0} = 0. \qquad (18)$$

Subsequent calculations will take advantage of the useful abbreviations

$$Q^* \;=\; p_\theta^2 = P_\psi^2 - \frac{P_\varphi^2}{\cos^2\theta}\,, \qquad Q = p_r^2 = -2P_0 - \frac{2P_4}{r} - \frac{P_\psi^2}{r^2}\,, \qquad I_j = \int_{r_0}^{r} \frac{dr}{r^j\sqrt{Q}}\,. \quad (19)$$

The lower limit r_0 in the integral over r will be the lowest positive root of $Q = 0$. Accordingly, separation of variables leads to the expression for the generating function

$$S \;=\; \int_{r_0}^{r} \sqrt{Q}\,dr + \int_0^{\theta} \sqrt{Q^*}\,d\theta + \varphi P_\varphi + x_4 P_4 + x_0 P_0, \qquad (20)$$

which defines a canonical transformation in which all the new variables are cyclic for the problem posed by (16): the transformation generated by S brings $\mathcal{H}_h$ into the equilibrium. Four constants are obtained for the derivatives $\partial S/\partial P_0$, $\partial S/\partial P_4$, $\partial S/\partial P_\psi$ and $\partial S/\partial P_\varphi$:

$$\partial S/\partial P_0 \;=\; x_0 - I_0 = c_0\,, \qquad \partial S/\partial P_4 = x_4 - I_1 = c_4\,, \qquad (21)$$

$$\frac{\partial S}{\partial P_\psi} \;=\; -P_\psi I_2 + \int_0^{\theta} \frac{P_\psi}{\sqrt{Q^*}}\,d\theta = c_\psi, \qquad \frac{\partial S}{\partial P_\varphi} = \varphi - P_\varphi \int_0^{\theta} \frac{d\theta}{\cos^2\theta\sqrt{Q^*}} = c_\varphi\,. \quad (22)$$

With the notations (amending some typing mistakes in Baumgarte's formulae) for quantities a and e, as functions of the new momenta,

$$a \;=\; -\,P_4/(2P_0)\,, \qquad P_\psi^2 = -P_4\,a\,(1-e^2), \qquad (23)$$

after performing the required quadratures I_1 and I_0 over r, one obtains

$$I_1 \;=\; E/\sqrt{2P_0}\,, \quad E = \sqrt{2P_0}\,(x_4 - c_4)\;, \quad r = a\,(1 - e\cos E)\;, \qquad (24)$$

which gives the parametric representation of an ellipse with the eccentric anomaly E as the parameter, semi–major axis a and eccentricity e. And from relations (21),

$$dx_0/dE = r/\sqrt{2P_0} \implies x_0 - \text{ const.} = t - t_P = \left[a/\,(2P_0)^{1/2}\right](E - e\sin E\,) \;=\; I_0\,. \quad (25)$$

In these formulae r_0 is the distance of the pericentre; t_P is the time of a pericentre passage.

As a conclusion, taking $-\mu = p_4$ leads to the introduction of the eccentric anomaly E as a variable related to the canonical coordinate x_4, conjugate to p_4.

The above considerations can be made *universal*, that is, uniformly valid for any kind of conic–section orbit, if principles of the nature of those presented, e.g., in Stiefel & Scheifele (1971, §11), Battin (1987, §4.5, §4.6), or Floría (2000), are applied. With this aim in view, the expressions in Eq. (23) are replaced by

$$2P_0 \;=\; -\,P_4(1-e)/q\,, \qquad P_\psi^2 = -P_4\,q\,(1+e) = -P_4\,p\,, \qquad (26)$$

the distance of the pericentre q playing a role similar to (although more general than) the one taken by the semi–major axis in derivations concerning elliptic or hyperbolic motion.

4 On Canonical Sets of Orbital Elements for Quasi–Keplerian Systems

Starting from polar nodal variables, previous results due to Soudan (1953, 1955), concerning *quasi–Keplerian systems*, were generalized by Cid & Calvo (1973/1975). Proceeding along the same basic guide–lines as Soudan, Palacios (1973/1977) derived the corresponding sets of canonical variables in the case of hyperbolic–type orbital motion.

Examining and amending some developments due to Andoyer (1913), Soudan (1953, 1955) devised a systematic procedure to obtain six canonical sets of orbital variables that are *elements* for elliptic–type motion acted upon by a certain central potential of the form $V(r) = -\mu/r - \mu'/r^2$, which is linked to the so–called quasi–Keplerian systems. Five of these canonical sets generalize the classical ones usually known after Delaunay, Levi–Civita, Hill, de Sitter and Jekhovsky in the case of pure, unperturbed Keplerian motion, while a new set was uncovered by Soudan. As in the Andoyer (1913) proposal, this unified procedure was based on a thorough analysis of the Hamilton–Jacobi equation in spherical polar variables and certain parameters; different special choices of parameters (playing the role of dynamical variables –canonical momenta– or absolute constants) in a generating function lead to the said sets of canonical variables.

In terms of *polar nodal variables*, $(r, \theta, \nu; p_r, p_\theta, p_\nu)$, the study of orbital motion in a central force field with potential $V(r)$ hinges on the analysis of the Hamiltonian

$$\mathcal{H} \;=\; (1/2)\left[p_r^2 + (p_\theta^2/r^2)\right] + V(r)\,. \tag{27}$$

Subsequent developments will be related to the specific type of quasi–Keplerian potential

$$V(r) \;=\; -\mu/r - \mu'/r^2\,, \quad \text{where } \mu \text{ and } \mu' \text{ are parameters} \tag{28}$$

(Cid & Calvo 1973/1975). Here the meaning of (θ, p_θ) is not the same as in our Section 3. Let $\mathcal{E}$ be the constant energy of the solution. The Hamilton–Jacobi equation is

$$(1/2)\left[(\partial S/\partial r)^2 + (1/r^2)(\partial S/\partial \theta)^2\right] - (\mu/r) - (\mu'/r^2) \;=\; \mathcal{E}\,, \tag{29}$$

to which a solution by separation of variables is found under the form

$$S \;=\; S(r, \theta, \nu; H, G, \mathcal{E}, \mu, \mu') = \theta G + \nu H + \int_{r_0}^{r} \sqrt{2\mathcal{E} + \frac{2\mu}{r} + \frac{2\mu' - G^2}{r^2}}\, dr, \tag{30}$$

that is, a function depending on the variables (r, θ, ν) and five parameters, where G and H are separation constants. As for r_0, it is usually taken as the lowest zero of the function of r under the radical sign. This solution contains the quantities $(G, H, \mathcal{E}, \mu, \mu')$; in the case of pure Keplerian systems, G is the (constant) norm of the angular momentum vector, H represents the (constant) vertical component of the angular momentum, $\mathcal{E}$ is the energy constant, $\mu' = 0$, and the gravity parameter μ is an absolute constant. Nevertheless, in what follows these quantities $(G, H, \mathcal{E}, \mu, \mu')$ are contemplated as parameters, and three of them can be taken as the new momenta. Replacing μ' with a new parameter η ,

$$\mu' \longrightarrow \eta : \quad 2\mu' - G^2 = -\, G^2/(1+\eta)\,, \quad 2\mu' = [\,\eta/(1+\eta)\,]G^2\,, \tag{31}$$

$$S = S\,(r, \theta, \nu; H, G, \mathcal{E}, \mu, \eta) = \theta G + \nu H + \int_{r_0}^{r} \sqrt{2\mathcal{E} + \frac{2\mu}{r} - \frac{G^2}{(1+\eta)r^2}}\, dr\,, \tag{32}$$

$$p_r \equiv \dot r = \frac{\partial S}{\partial r} = \sqrt{2\mathcal{E} + \frac{2\mu}{r} - \frac{G^2}{(1+\eta)r^2}}\,, \quad p_\theta = \frac{\partial S}{\partial \theta} = G\,, \tag{33}$$

while $\nu = Q_H = h$ and $p_\nu = H$ remain unchanged. To complete the system of transformation equations implicitly defined by S, according to the standard procedure in dealing with the Kepler problem, introduce a true–like anomaly f by means of a differential relation $r^2 df = G dt$, such that $f_0 = 0 \rightarrow r(f_0) = r_0$. Notice also that $\eta = 0$ when $\mu' = 0$.

The five parameters $(G, H, \mathcal{E}, \mu, \eta)$ are still available, and different choices of new canonical momenta created by the transformation generated by S are possible, while the remaining quantities would act as parameters or absolute constants in the transformation. Some illustrative cases are presented, in which μ is introduced as a canonical variable.

• Take $\mathcal{E}$ and η as constants, and consider the new canonical set $(\mu, G, H; Q_\mu, Q_G, h)$. The remaining, non–trivial transformation equations are

$$Q_G = \frac{\partial S}{\partial G} = \theta - \int_{r_0}^{r} \frac{G\, dr}{(1+\eta) r^2 \dot r} = \theta - \frac{f}{1+\eta}\,, \quad Q_\mu = \frac{\partial S}{\partial \mu} = \int_{r_0}^{r} \frac{dr}{r\, \dot r} = \int_{T}^{t} \frac{dt}{r} = s \tag{34}$$

where s is a regularizing variable, an eccentric–like anomaly, as introduced by means of this last integral, and T is a value of t corresponding to r_0: $r(T) = r_0 = r\mid_{(s_0=0)}$. The preceding expression for Q_G would become somewhat simpler in terms of another true–like anomaly φ given as $r^2(1+\eta)d\varphi = G dt$, with $r_0 = r\mid_{(\varphi_0=0)}$.

• Take G and η as constants; the new set is $(\mu, \mathcal{E}, H; Q_\mu, Q_\mathcal{E}, h)$. The significant transformation equations are now

$$Q_\mathcal{E} = \frac{\partial S}{\partial \mathcal{E}} = \int_{r_0}^{r} \frac{dr}{\dot r} = \int_{T}^{t} dt = t - T\,, \qquad Q_\mu = \frac{\partial S}{\partial \mu} = \int_{r_0}^{r} \frac{dr}{r\, \dot r} = \int_{T}^{t} \frac{dt}{r} = s\,. \tag{35}$$

The preceding variable s in (34) and (35), coinciding with Q_μ (the coordinate conjugate to μ), satisfies formally the same differential relation (15) as the fictitious time τ in Section 2, and can be expressed as a *universal variable* –in terms of universal functions– thanks to an adequate *modification of Sundman's transformation* (Stiefel & Scheifele 1971, §11, Eqs. [54] and [60]), suitably adapted to the case of *perturbed* Kepler problems.

More involved considerations are to be taken into account when η is not an absolute constant, in which case lengthy reckoning work is required (see Cid & Calvo 1973/1975).

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