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# Classification and reduction of polynomial Hamiltonians with three degrees of freedom

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**Abstract**

We classify all possible normal forms associated with the quadratic part of a polynomial Hamiltonian function in three degrees of freedom (3 DOF). Then we make an analysis based on normalizations which allow us to reduce the number of degrees of freedom at least one unit. We consider an arbitrary polynomial Hamiltonian whose principal part is quadratic in positions and momenta. The procedure is based on the extension of an integral of the unperturbed part to the whole system, up to a certain order. Finally we present some features of an algorithm developed so as to simplify an arbitrary $n$ DOF polynomial Hamiltonian.

**Key words and expressions:** Formal integral, normal form, reduction. **AMS (MOS) Subject Classification:** 15A21, 34C20, 70F15.

## 1 Introduction

This paper applies the theory developed in [11, 12] to 3 DOF polynomial Hamiltonian systems. We consider Hamiltonian functions of the form:

$$\mathcal{H}(\mathbf{x}) = \mathcal{H}_0(\mathbf{x}) + \mathcal{H}_1(\mathbf{x}) + \dots, \tag{1}$$

where $\mathbf{x}$ is a six-dimensional vector in the coordinates $x, y, z$ and respective momenta $X, Y, Z$. Moreover, each $\mathcal{H}_i$ is a homogeneous polynomial in $\mathbf{x}$ of degree $i + 2$.

The motivation of the work was to make a qualitative analysis of these systems, i.e., classifying equilibria and bifurcations, periodic orbits and invariant manifolds. As a consequence, our purpose was to simplify system (1) by reducing its number of degrees of freedom by one, up to a certain order of approximation. In addition, the normalization up to any order becomes a cumbersome task, so the implementation of normalizing routines to simplify an arbitrary $n$ DOF polynomial Hamiltonian became another of our goals.

The principal part of (1) is a homogeneous polynomial of degree two in the coordinates and momenta. As a first step, we calculate all the possible normal forms corresponding to this polynomial. Then we apply a Lie transformation to reduce $\mathcal{H}$ in each case.

For each Hamiltonian the reduction is not unique. Indeed, it depends on the choice of the integral of the quadratic part we take to extend to the whole system.

The usual way to reduce $\mathcal{H}$ is by making use of the normal form theorem. Poincaré [13] is considered as a pioneer in that he developed a method to simplify systems of differential equations, but not necessarily of Hamiltonian nature. Birkhoff [2] considered the Hamiltonian version. The normalization of semisimple systems in equilibrium at the origin was carried out for the planar case by Whittaker [16]. Moser [10] extended the work by Birkhoff to resonant Hamiltonians in $2\,n$ dimensions. It was generalized thereafter by Meyer [6], who presented the general solution for a Hamiltonian system whose corresponding matrix related to the linear part was semisimple. Normal forms for non-semisimple matrices have been studied by Meyer and Schmidt [9], Sokol'skii [14] and van deer Meer [15], for instance. Meyer [7] established the normal form theorem for any type of equilibrium point in systems of ordinary differential equations whose main part is linear. The Hamiltonian case appears in Meyer and Hall [8]. The normal form theorem for non-Hamiltonian systems is also described in detail in the books by Arnold [1] and Wiggins [17].

This article is divided into four sections. Section 2 deals with the classification of normal forms associated with quadratic Hamiltonians in 3 DOF. In Section 3 we propose a procedure based on the concept of extended normal forms in order to normalize 3 DOF Hamiltonian systems. Finally, Section 4 contains information about an algorithm of normalization for $n$ DOF polynomial Hamiltonians.

## 2 Classification of Quadratic Hamiltonians in Three Degrees of Freedom

In this section we classify the possible normal forms corresponding to 3 DOF polynomial Hamiltonians of degree two. This classification is based on the type and number of eigenvalues of the matrix $A$ associated to the quadratic form $\mathcal{H}_0$ and the corresponding Jordan matrix.

The list for the $n$ DOF case was given by Galin [3] and was based on a work by Williamson [18]. It has been reiterated by several authors; see, for example, Arnold [1].

Given an arbitrary 3 DOF homogeneous polynomial Hamiltonian of degree two,

$$\begin{aligned}\mathcal{H}_0(\mathbf{x}) \;=\;& a_0\,x^2 + a_1\,x\,X + a_2\,x\,y + a_3\,x\,Y + a_4\,x\,z + a_5\,x\,Z + a_6\,X^2\\ &+ a_7\,X\,y + a_8\,X\,Y + a_9\,X\,z + a_{10}\,X\,Z + a_{11}\,y^2\\ &+ a_{12}\,y\,Y + a_{13}\,y\,z + a_{14}\,y\,Z + a_{15}\,Y^2 + a_{16}\,Y\,z\end{aligned}$$

$$+\, a_{17}\, Y\, Z + a_{18}\, z^2 + a_{19} z\, Z + a_{20}\, Z^2,$$

with constant real coefficients $a_i$, $i = 0, \ldots, 20$, we associate to it the matrix $A$ such that $d\mathbf{x}/dt \;=\; A\,\mathbf{x}$. Then $\mathcal{H}_0(\mathbf{x}) = -\frac{1}{2}\,\mathbf{x}^t\,\mathcal{J}\,A\,\mathbf{x}$ and $\mathcal{J}$ denotes the skew-symmetric matrix of order 6. Now, the eigenvalues of the matrix $A$ can be of four types: (i) real pairs $(a, -a)$, (ii) pure imaginary pairs $(\imath\, b, -\imath\, b)$, (iii) $\pm a \pm \imath\, b$, and (iv) two zeros. In all situations $a$ and $b$ are taken to be nonzero.

The following list gives the Hamiltonian associated to each Jordan-block of the matrix $A$, where $x_j$ are the coordinates and $X_j$ denote the momenta of the system:

i) For a pair of $k$–order Jordan blocks (eigenvalues $\pm a$):

$$\mathcal{H}_0 \;=\; -a \sum_{j=1}^{k} X_j\, x_j \;+\; \sum_{j=1}^{k-1} X_j\, x_{j+1}.$$

ii) For a quartet of $k$–order Jordan blocks (eigenvalues $\pm a \,\pm\, \imath b$):

$$\mathcal{H}_0 \;=\; -a \sum_{j=1}^{2k} X_j\, x_j \;+\; b \sum_{j=1}^{k} (X_{2j-1}\, x_{2j} \;-\; X_{2j}\, x_{2j-1}) \;+\; \sum_{j=1}^{2k-2} X_j\, x_{j+2}.$$

iii) For a pair of $k$–order Jordan blocks (eigenvalue 0):

$$\mathcal{H}_0 \;=\; \sum_{j=1}^{k-1} X_j\, x_{j+1}.$$

iv) For a $2k$–order Jordan block (eigenvalue 0):

$$\mathcal{H}_0 \;=\; \pm\frac{1}{2} \left( \sum_{j=1}^{k-1} X_j\, X_{k-j} \;-\; \sum_{j=1}^{k} x_j\, x_{k-j+1} \right) \;-\; \sum_{j=1}^{k-1} X_j\, x_{j+1}.$$

v) For a pair of $2k+1$–order Jordan blocks (eigenvalues $\pm\imath b$):

$$\mathcal{H}_0 \;=\; \pm\frac{1}{2} \left[ \sum_{j=1}^{k} (b^2\, X_{2j}\, X_{2k-2j+2} + x_{2j}\, x_{2k-2j+2}) - \sum_{j=1}^{k+1} b^2\, X_{2j-1}\, X_{2k-2j+3} \right] - \sum_{j=1}^{2k} X_j\, x_{j+1}.$$

vi) For a pair of $2k$–order Jordan blocks (eigenvalues $\pm\imath b$):

$$\mathcal{H}_0 \;=\; \pm\frac{1}{2} \left[ \sum_{j=1}^{k} \left( \frac{1}{b^2}\, x_{2j}\, x_{2k-2j+1} \;+\; x_{2j}\, x_{2k-2j+2} \right) - \right.$$
$$\left. - \sum_{j=1}^{k-1} (b^2\, X_{2j+1}\, X_{2k-2j+1} \;+\; X_{2j+2}\, X_{2k-2j+2}) \right] \;-\; b^2 \sum_{j=1}^{k} X_{2j-1}\, x_{2j} \;+\; \sum_{j=1}^{k} X_{2j}\, x_{2j-1}.$$

For the 3 DOF case, the combinations of all possible eigenvalues yield the 41 types of Hamiltonians appearing in Table 1. Besides, we use a result by Williamson that can be stated as follows.

**Theorem 2.1 (Williamson)** *A symplectic real vector space with a given quadratic form $\mathcal{H}_0$ can be split into a direct sum of real oblique-orthogonal symplectic subspaces by pairs such that $\mathcal{H}_0$ is represented as a sum of forms of the previous types upon these subspaces.*

A work by Meyer and Laub [5] together with the previous theorem by Williamson help us to guarantee there are no cases left. The complete proof will be found in [4].

Table 1: Classification of possible normal forms associated with $\mathcal{H}_0$. Third column indicates the type of the matrix (semisimple, nilpotent, or both); fourth column indicates eigenvalues and their multiplicities.

| | Normal forms | Type | Eigenvalues |
|---|---|---|---|
| Case 1 | $a\,x\,X$ | Semi. | $\pm a$, 0 (4) |
| Case 2 | $\pm\,(a^2\,x^2+X^2)/2$ | Semi. | $\pm\imath\,a$, 0 (4) |
| Case 3 | $b\,(x\,Y-X\,y)+a\,(x\,X+y\,Y)$ | Semi. | $\pm a\pm\imath\,b$, 0 (2) |
| Case 4 | $a\,x\,X\pm\,(b^2\,y^2+Y^2)/2$ | Semi. | $\pm a$, $\pm\imath\,b$, 0 (2) |
| Case 5 | $\pm\,(a^2\,x^2+X^2)/2\pm\,(b^2\,y^2+Y^2)/2$ | Semi. | $\pm\imath\,a, \pm\imath\,b, 0(2)$ |
| Case 6 | $a\,x\,X+b\,y\,Y$ | Semi. | $\pm a$, $\pm b$, 0 (2) |
| Case 7 | $a\,x\,X+b\,y\,Y+c\,z\,Z$ | Semi. | $\pm a$, $\pm b$, $\pm c$ |
| Case 8 | $a\,x\,X+b\,(y\,Y+z\,Z)\,+\,c\,(y\,Z-z\,Y)$ | Semi. | $\pm a$, $\pm b\pm\imath\,c$ |
| Case 9 | $\pm(a^2\,x^2+X^2)/2\pm(b^2\,y^2+Y^2)/2\pm(c^2\,z^2+Z^2)/2$ | Semi. | $\pm\imath\,a$, $\pm\imath\,b$, $\pm\imath\,c$ |
| Case 10 | $a\,x\,X+b\,y\,Y\pm\,(c^2\,z^2+Z^2)/2$ | Semi. | $\pm a$, $\pm b$, $\pm\imath\,c$ |
| Case 11 | $a\,x\,X\,\pm\,(b^2\,y^2+Y^2)/2\pm\,(c^2\,z^2+Z^2)/2$ | Semi. | $\pm a$, $\pm\imath\,b$, $\pm\imath\,c$ |
| Case 12 | $a\,(x\,X+y\,Y)+b\,(x\,Y-y\,X)\pm\,(c^2\,z^2+Z^2)/2$ | Semi. | $\pm a\pm\imath\,b$, $\pm\imath\,c$ |
| Case 13 | $a\,x\,X\pm\,Y^2/2$ | Semi.+Nilpo. | $\pm a$, 0 (4) |
| Case 14 | $\pm\,X^2/2\pm\,(b^2\,y^2+Y^2)/2$ | Nilpo.+Semi. | 0 (4), $\pm\imath\,b$ |
| Case 15 | $x\,Y+a\,(x\,X+y\,Y)$ | Nilpo.+Semi. | $\pm a$ (2), 0 (2) |
| Case 16 | $a\,(x\,Y-X\,y)\pm\,(x^2+y^2)/2$ | Semi.+Nilpo. | $\pm\imath\,a$ (2), 0 (2) |
| Case 17 | $a\,(x\,X+y\,Y+z\,Z)+x\,Y+y\,Z$ | Semi.+Nilpo. | $\pm a$ (3) |
| Case 18 | $-x\,Y-y\,Z\pm\,[a^2\,y^2+Y^2-2(a^2\,x\,z+X\,Z)]/2$ | Semi.+Nilpo. | $\pm\imath\,a$ (3) |
| Case 19 | $a\,x\,X+b\,(y\,Y+z\,Z)+y\,Z$ | Semi.+Nilpo. | $\pm a$, $\pm b$ (2) |
| Case 20 | $a\,x\,X+y\,Z$ | Semi.+Nilpo. | $\pm a$, 0 (4) |
| Case 21 | $a\,x\,X+b\,y\,Y\pm\,Z^2/2$ | Semi.+Nilpo. | $\pm a$, $\pm b$, 0 (2) |
| Case 22 | $a\,(x\,X+y\,Y)+x\,Y\pm\,Z^2/2$ | Semi.+Nilpo. | $\pm a$ (2), 0 (2) |
| Case 23 | $a\,(x\,X+y\,Y)+b\,(x\,Y-y\,X)\pm\,Z^2/2$ | Semi.+Nilpo. | $\pm a\pm\imath\,b$, 0 (2) |
| Case 24 | $a\,x\,X\pm\,z^2/2+y\,Z$ | Semi.+Nilpo. | $\pm a$, 0 (4) |
| Case 25 | $a\,(x\,X+y\,Y)\pm\,(b^2\,z^2+Z^2)/2+x\,Y$ | Semi.+Nilpo. | $\pm a$ (2), $\pm\imath\,b$ |
| Case 26 | $x\,Y\pm\,(b^2\,z^2+Z^2)/2$ | Nilpo.+Semi. | 0 (4), $\pm\imath\,b$ |
| Case 27 | $\pm\,X^2/2\pm\,(a^2\,y^2+Y^2)/2\pm\,(b^2\,z^2+Z^2)/2$ | Nilpo.+Semi. | $0(2), \pm\imath\,a, \pm\imath\,b$ |
| Case 28 | $\pm\,y^2/2+x\,Y\pm\,(b^2\,z^2+Z^2)/2$ | Nilpo.+Semi. | 0 (4), $\pm\imath\,b$ |
| Case 29 | $a\,x\,X\pm\,(b^2\,y^2+Y^2)/2\pm\,Z^2/2$ | Semi.+Nilpo. | $\pm a$, $\pm\imath\,b$, 0 (2) |
| Case 30 | $a\,x\,X+b\,(y\,Z-Y\,z)\pm\,(y^2+z^2)/2$ | Semi.+Nilpo. | $\pm a$, $\pm\imath\,b$ (2) |
| Case 31 | $\pm X^2/2\pm(y^2+z^2)/2+b(yZ-Yz)$ | Nilpo.+Semi. | 0 (2), $\pm\imath\,b$ (2) |
| Case 32 | $\pm\,(a^2\,x^2+X^2)/2+b\,(y\,Z-Y\,z)\pm\,(y^2+z^2)/2$ | Semi.+Nilpo. | $\pm\imath\,a$, $\pm\imath\,b$ (2) |
| Case 33 | $\pm\,X^2/2$ | Nilpo. | 0 (6) |
| Case 34 | $x\,Y$ | Nilpo. | 0 (6) |
| Case 35 | $\pm\,y^2/2+x\,Y$ | Nilpo. | 0 (6) |
| Case 36 | $\pm\,(x^2+y^2)/2$ | Nilpo. | 0 (6) |
| Case 37 | $X\,y+Y\,z$ | Nilpo. | 0 (6) |
| Case 38 | $-Xy-Yz\pm(-y^2+2XY-2xz)/2$ | Nilpo. | 0 (6) |
| Case 39 | $x\,Y\pm\,Z^2/2$ | Nilpo. | 0 (6) |
| Case 40 | $\pm\,(x^2+y^2+z^2)/2$ | Nilpo. | 0 (6) |
| Case 41 | $\pm\,X^2/2+Y\,Z\pm\,z^2/2$ | Nilpo. | 0 (6) |

## 3 Extension of an Integral of $\mathcal{H}_0$

This section deals with the reduction of Hamiltonian systems through the construction of formal integrals. To get that target we use the well-known normal form theorem given by Meyer and Hall [8] and a generalization of normal forms due to Palacián and Yanguas that can be found in [11].

### *3.1 Normalizing theorems*

Let $\mathcal{H}$ be an $n$ degrees of freedom Hamilton function such that

$$\mathcal{H}(\mathbf{x};\varepsilon) \;=\; \sum_{i\geq 0} \frac{\varepsilon^i}{i!}\,\mathcal{H}_i(\mathbf{x}), \tag{2}$$

where $\mathbf{x} = (x_1, \ldots, x_n, X_1, \ldots, X_n)$ and each $\mathcal{H}_i(\mathbf{x})$ is a homogeneous polynomial of degree $i+2$ in $\mathbf{x} \in \mathbf{R}^{2n}$.

Through the application of the normal form theorem [8], we reduce by one the number of degrees of freedom of Hamiltonian systems whose matrix $A$ is not nilpotent. As a consequence, $\mathcal{H}_0$ becomes a formal integral of the reduced system. Applying a more general theorem by Palacián and Yanguas we reduce by one the number of degrees of freedom of any Hamiltonian system up to a certain order and the new formal integral is not necessarily $\mathcal{H}_0$, but any integral $\mathcal{G}$ of $\mathcal{H}_0$ we choose previously. In all the cases, we get a symplectic change of variables $\mathbf{x} \rightarrow \mathbf{y}$ that transforms $\mathcal{H}$ into the normalized Hamiltonian $\mathcal{K}$, with

$$\mathcal{K}(\mathbf{y};\varepsilon) = \sum_{i=0}^{L} \frac{\varepsilon^i}{i!}\,\mathcal{K}_i(\mathbf{y}) \;+\; \mathcal{O}(\varepsilon^{L+1}), \tag{3}$$

where $\mathcal{K}_0(\mathbf{y}) \equiv \mathcal{H}_0(\mathbf{x})$ and each $\mathcal{K}_i(\mathbf{y})$ is a homogeneous polynomial of degree $i+2$ in $\mathbf{y} \in \mathbf{R}^{2n}$.

The construction of $\mathcal{K}$ must be done order by order; i.e., one has to proceed in an ascendent way from $i = 1$ to $i = L$ to determine each $\mathcal{K}_i$. For that, the homology equation

$$\{\,\mathcal{W}_i\,,\,\mathcal{H}_0\,\} + \mathcal{K}_i \;=\; \widetilde{\mathcal{H}}_i, \tag{4}$$

has to be solved with the extra condition $\{\,\mathcal{K}_i\,,\,\mathcal{G}\,\} = 0$ for $i = 1, \ldots, L$. From now on, the operator $\{\;,\;\}$ denotes usual the Poisson bracket. Note that the terms $\widetilde{\mathcal{H}}_i$ are the ones known from the previous orders and the solution of (4) is the pair $(\mathcal{W}_i, \mathcal{K}_i)$, where $\mathcal{W}_i$ denotes the generating function computed at order $i$.

The drawback of the generalized method is that $\mathcal{W}_i$ is not necessarily a polynomial function of degree $i+2$, as it occurs in the normal form theorem, but rational, logarithmic or arctangent functions, so we have to exclude the singularities from the domain.

### *3.2 Types of reduction*

Taking into account the so-called Jordan decomposition of the matrix $A$ into its semisimple and nilpotent components, that is $A = S + N$, we classify the types of reduction into three remarkable cases:

**(a)** Semisimple case: $A = S$. We apply the normal form theorem with $\mathcal{G}(\mathbf{x}) = \mathcal{H}_0(\mathbf{x})$. If the reduced system has zero degrees of freedom we choose another $\mathcal{G}$.

**(b)** Semisimple plus nilpotent case: $A = S + N$ with $S, N \neq 0$. We proceed with the normal form theorem taking $\mathcal{G}(\mathbf{x}) = -\frac{1}{2}\, \mathbf{x}^t\, \mathcal{J}\, S\, \mathbf{x} \neq \mathcal{H}_0(\mathbf{x})$.

**(c)** Nilpotent case: $A = N$. We choose $\mathcal{G}(\mathbf{x}) = \alpha\, \mathcal{H}_0$ with $\alpha \in \mathbf{R}$.

Thus, once we have computed $\mathcal{G}$, integral of the truncated system independent of it, the function

$$\mathcal{I}(\mathbf{x};\varepsilon) \;=\; \mathcal{G}(\mathbf{x}) \;+\; \sum_{i=1}^{L} \frac{\varepsilon^i}{i!}\, \mathcal{L}^i_{-\mathcal{W}}\, [\mathcal{G}(\mathbf{x})] \tag{5}$$

becomes an integral of $\mathcal{H}$ functionally independent of it with an order of approximation of $\mathcal{O}(\varepsilon^{L+1})$. $\mathcal{L}_{-\mathcal{W}}$ refers to the Lie operator $\mathcal{L}_{-\mathcal{W}} : F \longrightarrow \{\mathcal{W}, F\}$. In addition to that, the Lie operator $\mathcal{L}^i_{-\mathcal{W}}(\mathbf{y}) = \mathcal{L}_{-\mathcal{W}}(\mathcal{L}^{i-1}_{-\mathcal{W}}(\mathbf{y}))$ for $i \geq 2$. With this, we will be able to reduce any polynomial Hamiltonian with three DOF by at least one degree of freedom.

## 4 Algorithm

This section is devoted to show some features of routines we developed in order to apply the normalization process to an arbitrary $n$ DOF polynomial Hamiltonian up to any order.

### *4.1 Homology equation resolution*

The main effort we make consists in solving the homology equation (4). For achieving that, as a first step we split each $\widetilde{\mathcal{H}}_i$ into the sum $\widetilde{\mathcal{H}}_i = \widetilde{\mathcal{H}}_i^{\,*} + \widetilde{\mathcal{H}}_i^{\,\#}$ where $\mathcal{K}_i = \widetilde{\mathcal{H}}_i^{\,*}$, with $\widetilde{\mathcal{H}}_i^{\,*} \in \ker(\mathcal{L}_{\mathcal{G}})$. Along the process we express $\mathcal{K}_i$ in Cartesian coordinates[1].

After achieving that we balance the coefficients of $\mathcal{K}_i$ with the ones coming from $\widetilde{\mathcal{H}}_i$ using all possible monomials of $\mathbf{x}$ of degree $i+2$. The function `Solve` of Mathematica yields the expected result. Then, we discuss the solution of the equation $\mathcal{L}_{\mathcal{H}_0}(\mathcal{W}_i) = \widetilde{\mathcal{H}}_i^{\,\#}$. At this step we emphasize that the code used for *semisimple* and *semisimple plus nilpotent* cases differs with the one of the *nilpotent* cases. For the last one the function `Solve` does not work properly as the generating functions can be of rational or logarithmic character. Still we can make use of the Mathematica function `DSolve` to determine $\mathcal{W}_i$.

[1]From a computational point of view, as a future purpose, it will be convenient to use an adequate set of symplectic variables so that the homology equation (4) presents a "nice" aspect, for example polar-nodal, spherical or complex variables. Sometimes it is best to use a combination of them.

### *4.2 Algorithm code*

Finally, we offer the most remarkable shares of the algorithm. We split the procedure into four steps.

**STEP 1:** Previously we have developed an auxiliary subalgorithm which provides the Jordan decomposition of the Hamiltonian matrix associated to the quadratic terms of the original Hamiltonian. Thereafter we proceed with the computation of $arb1$, an arbitrary polynomial of degree $i \geq 3$ in $\mathbf{x} = (x_1, \ldots, x_n, X_1, \ldots, X_n)$, which is the general solution of $\{\mathcal{G}, \texttt{arb1}\} = 0$. From now on, $\mathrm{pb}[F_1,\ F_2]$ stands for the Poisson bracket $\{F_1, F_2\}$. Besides, `lista[i]` provides the list of all possible monomials of degree $i + 2$ formed by the components of $\mathbf{x}$.

```
mon = lista[i];
arb1 = Sum[coe1[j]*mon[[j]], {j, Length[mon]}];
zero = pb[G, arb1];
coef = Coefficient[zero, mon];
sol1 = Solve[Thread[coef == 0], Table[coe1[j], {j, Length[mon]}]];
arb11 = Expand[arb1 /.  sol1[[1]]];
```

**STEP 2:** The following share of code is applied in the semisimple and semisimple plus nilpotent cases. It is used to solve the homology equation at order $i \geq 1$, that is, it represents the piece of code where $\mathcal{W}_i$ and $\mathcal{K}_i$ are determined according to the requirements of the Lie transformation one performs.

```
arb2 = Sum[coe2[j]*mon[[j]], {j, Length[mon]}];
zero = pb[arb2, H0] - (H̃i - arb11);
coef = Coefficient[zero, mon];
sol2 = Solve[Thread[coef == 0], Join[Table[coe1[j], {j, Length[mon]}],
                                     Table[coe2[j], {j, Length[mon]}]]];
{Ki,Wi} = Expand[{arb11,arb2} /.  sol1[[1]] /.  sol2[[1]]];
```

**STEP 3:** At this point we deal with the nilpotent case. We also apply STEP 3 whenever for Hamiltonians with non–null semisimple part we obtain a 0 DOF normalized Hamiltonian at STEP 2. Now `x` denotes the $2n$–dimensional array $\mathbf{x} = (x_1, \ldots, x_n, X_1, \ldots, X_n)$.

```
Ki = arb11;
der = H̃i - Ki;
pbn = pb[W[x], H0];
dsol = DSolve[{pbn == der}, W[x], {x}];
Wi = dsol[[1,3]];
```

**STEP 4:** At last we can easily include all this in the well–known Lie–Deprit recursion process so as to obtain the normalized Hamiltonian and the generating function with an order of approximation $\mathcal{O}(\varepsilon^{L+1})$.


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