

On the quantities estimates of the remainder in normal forms

M. Ollé, J. R. Pacha and J. Villanueva

Dept. de Matemàtica Aplicada I. Universitat Politècnica de Catalunya, Diagonal 647,
08028 Barcelona, Spain.

e-mail: merce.olle@upc.es, joanr@vilma.upc.es, jordi@vilma.upc.es

Abstract

We consider an analytic Hamiltonian system with three degrees of freedom and having a family of periodic orbits with a transition stability-complex instability. We reduce the Hamiltonian to a normal form around a transition periodic orbit and we obtain $H = Z^{(r)} + R^{(r)}$. The analysis of the (truncated) normal form, $Z^{(r)}$, allows the description of a Hopf bifurcation of 2D-tori. However, this communication will concentrate on the study of the remainder, $R^{(r)}$, and some comparison between the remainder obtained when considering the normal form around an elliptic equilibrium point and around a critical periodic orbit will be made.

Key words and expressions: Hamiltonian systems - normal forms - bounds of the remainder.

1 Introduction

We consider analytic Hamiltonian systems (with three or more degrees of freedom) in the following two situations: in a neighbourhood of an elliptic equilibrium point and of a critical (transition from stable to complex unstable) periodic solution. The behaviour of an analytic hamiltonian around a lower dimensional invariant torus has also been studied, for instance, in [5] but we shall concentrate on the first two cases in order to show that the methodology and main ideas are essentially the same but the technical details are rather different.

Our approach in both cases involves mainly five steps:

- (i) Linear normalisation of the Hamiltonian around the equilibrium point or periodic solution (invariant object).
- (ii) Complexification of the Hamiltonian.
- (iii) Nonlinear Normalisation:

- * by means of the Giorgilli-Galgani “machine”,
- * it requires to solve the homological equations,
- * we obtain the transformed Hamiltonian in normal form as $H = Z^{(r)} + R^{(r)}$.

(iv) From the truncated normal form ($Z^{(r)}$), the description of the local dynamics around the invariant object.

(v) Bounds of the remainder, $R^{(r)}$.

Giorgilli et al. ([3]) carried out the five steps when analysing the dynamics close to an elliptic equilibrium point. Our goal in this work is to study, in particular, how to obtain bounds of the remainder (fifth step) when considering the normal form around a critical periodic orbit (the first four steps for this problem were analysed in [9] and [10]).

The whole treatment of our goal is analytical, and due to the long and tedious details involved, we only describe the main ideas and results (we refer the interested reader to [10] for full details and other related references); see also a numerical approach in [7] and [11]. Actually this problem may also be considered taking 4D symplecting mappings (see [11], [1]) or the Hamiltonian itself (as we do).

On the other hand, there are plenty of examples in celestial mechanics, planetary systems and galactic dynamics where the phenomenon of transition from stability to complex instability appears (see [2], [4], [6], [7], [8], [12]) and therefore the obtained results may be directly applied.

2 Model problem

We consider an analytic n -degrees of freedom Hamiltonian H in the neighbourhood of an *elliptic equilibrium point*, which after the linear normalisation and the complexification (steps (i) and (ii)) becomes

$$H(x, y) = i \sum_{j=1}^n \omega_j x_j y_j + \dots = H_2 + \sum_{k \geq 3} H_k.$$

We assume the *nonresonant* case, that is, the characteristic exponents at the equilibrium point, $\omega_1, \dots, \omega_n$, are Q -independent, and H defined in

$$D_{R_0} = \{(x, y) \in C^{2n} : |(x, y)| \leq R_0\}$$

with $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, $l, m \in Z_+^n$, $|l|_1 = \sum_{j=1}^n |l_j|$, and

$$H_k(x, y) = \sum_{|l+m|_1=k} H_{l,m} x^l y^m$$

with $x^l = x_1^{l_1} \dots x_n^{l_n}$, $y^m = y_1^{m_1} \dots y_n^{m_n}$ and

$$\|H_k\| = \sum_{l,m} |H_{l,m}|.$$

2.1 Transformation to normal form.

In order to obtain the Hamiltonian in normal form, we use the algorithm given by Giorgilli et al. (see [3]) based in Lie transforms in a suitable space of functions E . For this problem, E will be the space of formal series in the variables $(x, y) \in C^{2n}$ and E_k the space of homogeneous polynomials of degree k .

Algorithm for the canonical transformation. Given the functions $G, f \in E$, $G = \sum_{k \geq 3} G_k$ (which plays the role of a generating function with $G_k \in E_k$), and $f = \sum_{k \geq 1} f_k$, $f_k \in E_k$, we define a canonical transformation $T_G : E \rightarrow E$ in the following way:

$$T_G f = \sum_{k \geq 1} F_k,$$

where,

$$F_k = \sum_{j=1}^k f_{j, k-j} \in E_k$$

with $f_{j,0} = f_j$,

$$f_{j,k} = \sum_{i=1}^k \frac{i}{k} L_{G_{2+i}} f_{j, k-i} = \sum_{i=1}^k \frac{i}{k} \{G_{2+i}, f_{j, k-i}\},$$

and we introduce the remainder

$$R^{(r)} = \sum_{k \geq r+1} F_k$$

The next result summarizes the reduction of the Hamiltonian to normal form.

Proposition A.

Given the above Hamiltonian $H = \sum_{k \geq 2} H_k$, then

(i) there exists a generating function $G^{(r)} = \sum_{k=3}^r G_k$, such that the transformed Hamiltonian $T_{G^{(r)}} H$ is in normal form up to order r , that is,

$$T_{G^{(r)}} H = Z^{(r)} + R^{(r)}$$

with

$$Z^{(r)} = \sum_{1 \leq |l|_1 \leq \frac{r}{2}} a_l (xy)^l$$

and

$$R^{(r)} = \sum_{|l+m|_1 \geq r+1} a_{lm} x^l y^m.$$

(ii) If we write the normal form as $Z^{(r)} = \sum_{k=1}^r Z_k$, then the following relations are satisfied

$$Z_2 = H_2 \tag{1}$$

$$L_{H_2} G_k + Z_k = F_k, \quad k \geq 3, \tag{2}$$

being,

$$\begin{aligned} F_3 &= H_3 \\ F_k &= \sum_{j=1}^{k-3} \frac{j}{k-2} L_{G_{2+j}} Z_{k-j} + \sum_{j=1}^{k-2} \frac{j}{k-2} H_{2+j, k-j-2}. \end{aligned}$$

for $j \geq 4$.

Remarks. 1. We denote by (x, y) both the old and new variables involved in the reduction to normal form.

2. The homological equations $L_{H_2} G_k + Z_k = F_k$ are solved in a recursive way: we obtain G_k and Z_k from F_k which is a function of the previous computed Z_3, \dots, Z_{k-1} , G_3, \dots, G_{k-1} . We also remark that, assuming $G_k, Z_k, F_k \in E_k$, the homological equations become a diagonal system of linear equations, so they can be solved (formally) very easily (see details in [3]).

2.2 Bounds of the remainder.

We are interested in getting some estimates on the radius of convergence of the transformed function and on the remainder; from the definition of the remainder and the above algorithm, we need first some estimates on $\|F_k\|$. Giorgilli et al. ([3]) proved the following result.

Proposition B.

If for $c, d > 0$, we have

$$\|H_k\| \leq c^{k-2} d, \quad k \geq 3,$$

and

$$|\nu \cdot \omega| \geq \alpha_r > 0$$

then

(i) the transformed Hamiltonian $T_{G^{(r)}} H$ is convergent in any polydisk $D_R = \{(x, y) \in C^{2n}, |(x, y)| \leq R\}$ with

$$R < R_r^* = \left[\left(9 + \frac{32r}{5} \right) \frac{d}{\alpha_r} + \left(1 + \frac{32r}{5} \right) \right]^{-1} c^{-1}.$$

(ii) One has the following estimate for the remainder $R^{(r)}$ in D_R ,

$$|R^{(r)}| \leq \frac{d}{c} R \left(\frac{R}{R_r^*} \right)^r \left(1 - \frac{R}{R_r^*} \right)^{-1},$$

the symbol $|\cdot|$ denoting the supremum norm.

3 Description of our problem

Let $H(\zeta)$, be an analytic, three degree of freedom Hamiltonian, with the associated Hamiltonian system,

$$\dot{\zeta} = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix} \nabla H(\zeta), \quad (3)$$

with I_3 the 3×3 identity matrix. Assume there is a family, $\{M_\sigma\}_{\sigma \in \mathbb{R}}$, of periodic solutions of (3), such that its nontrivial characteristic multipliers $\lambda, 1/\lambda, \mu, 1/\mu$ behave as follows: for $\sigma < 0$, they are different on the unit circle, S^1 , -i.e. a (linearly) stable periodic solution-, for $\sigma = 0$, they collide on the unit circle $\lambda = \mu = \exp(i2\pi\nu) \neq 1$ -i.e the critical periodic solution, M_0 , such that ν is irrational-, and for $\sigma > 0$, they form a quadruplet in the complex plane outside S^1 -i.e. a complex unstable periodic solution-.

Our goal is to perform the five steps mentioned in the introduction: to compute the (formal) normal form of the Hamiltonian around the critical (resonant) orbit M_0 up to an arbitrary high order r in order to describe the local dynamics around such orbit (this part involves steps (i) to (iv); see [9] and [10] for full details), and more particularly (in this communication) to bound the remainder obtained in the normal form (step (v)).

3.1 Different ingredients at the beginning

It is worth mentioning (throughout this section) the main differences between our problem and the model one.

On one hand, the linear normalization around the critical solution M_0 involves the 2π -periodic dependence of the Hamiltonian on an angular variable θ_1 and its conjugated coordinate I_1 . Once we proceed with the complexification of the Hamiltonian, it becomes

$$H(\theta_1, q, I_1, p) = H_2 + \sum_{k \geq 3} H_k,$$

with $\theta_1, I_1 \in \mathbb{C}$, $q = (q_1, q_2)$, $p = (p_1, p_2) \in \mathbb{C}^2$, defined in:

$$D(\rho_0, R_0) = \left\{ |\operatorname{Im}\theta_1| \leq \rho_0, |I_1| \leq R_0^2, |(q, p)| \leq R_0 \right\},$$

$\rho_0, R_0 > 0$, and where

$$H_2 = \omega_1 I_1 + i\omega_2(q_1 p_1 + q_2 p_2) + q_2 p_1$$

with ω_1 the frequency of the critical orbit, and $\omega_2 = \nu\omega_1$ (ν defined from the characteristic multipliers above), and, for each $k \geq 3$, H_k can be expanded as:

$$H_k = \sum_{2l+|m|_1+|n|_1=k} h_{l,m,n}(\theta_1) I_1^l q^m p^n$$

with $h_{l,m,n}(\theta_1)$ given by its Fourier series:

$$h_{l,m,n}(\theta_1) = \sum_{s \in \mathbb{Z}} h_{s,l,m,n} \exp(is\theta_1).$$

On the other hand, the natural space E considered to carry out the algorithm to transform the Hamiltonian to normal form will be the space of Poisson series; more precisely, given $f(\theta_1, I_1, q, p) \in D(\rho, R)$, $\rho > 0$, $R > 0$, 2π -periodic in θ_1 ,

$$f = \sum_{(l,m,n) \in \mathbb{N} \times \mathbb{N}^2 \times \mathbb{N}^2} f_{l,m,n}(\theta_1) I_1^l q^m p^n$$

with

$$f_{l,m,n}(\theta_1) = \sum_{s \in \mathbb{Z}} f_{s,l,m,n} \exp(is\theta_1),$$

we introduce the following norms,

$$|f_{l,m,n}|_\rho = \sum_{s \in \mathbb{Z}} |f_{s,l,m,n}| \exp(|s|\rho)$$

$$|f|_{\rho,R} = \sum_{l \in \mathbb{N}} \sum_{m \in \mathbb{N}^2} \sum_{n \in \mathbb{N}^2} |f_{l,m,n}|_\rho R^{2l+|m|_1+|n|_1}.$$

3.2 Transformation to normal form.

We describe in [9] and [10] how we compute, in a rather tricky and constructive way, the normal form. Now we just state the main result.

Theorem A.

There is a generating function $G^{(r)} = \sum_{k=3}^r G_k$ which transforms the Hamiltonian H into resonant normal form up to order r , that is,

$$T_{G^{(r)}} H = Z^{(r)}(I_1, q, p) + R^{(r)}(\theta_1, q, I_1, p)$$

where $Z^{(r)} = \sum_{s=2}^r Z_s$, with $Z_2 = \omega_1 I_1 + i\omega_2(q_1 p_1 + q_2 p_2) + q_2 p_1$, $Z_s = 0$ for s odd, and Z_s is an homogeneous polynomial of degree $s/2$ in $I_1, i(q_1 p_1 + q_2 p_2), q_1 p_2$, with real coefficients when s is even.

Remark. 1. This reduction to normal form for a Hamiltonian around a resonant periodic orbit corresponds to statement (i) of proposition A for the model problem (nonresonant equilibrium point).

2. Of course, statement (ii) of proposition A also applies; however the solution of the homological equations, $L_{H_2} G_k + Z_k = F_k$, is rather more involved since the coupling term $q_2 p_1$ in H_2 makes that the homological equations become a *triangular* linear system (instead of a diagonal one as in the model problem); see [10].

3. We also remark that, in this problem, an accurate control of the reduction of the domains involved (which appear in the norms $|\cdot|_{*,*}$) is required; see [10] for details.

3.3 Bound for the remainder.

Finally, we obtain an optimal r_{opt} order (as a function of the distance to the critical orbit) up to which the normal form is carried out; this allows to obtain estimates on the

remainder as well. We mention here that although the ideas are the same as in the model problem, the computations for this degenerate or resonant problem are more involved (see [10]).

Theorem B.

Let H be a complex function defined and analytic in $D_0 = D(\rho_0, R_0)$ and let us assume that the frequencies $\omega = (\omega_1, \omega_2)$ satisfy the Diophantine condition

$$|k \cdot \omega| \geq \frac{\gamma}{|k|^\tau} \quad k \in \mathbb{Z}^2 \setminus \{0\},$$

for $\tau > 1$ and for a certain $\gamma > 0$. Then, for $0 < \rho < \rho_0$ and for $0 < R < R_0$ sufficiently small:

(i) The transformed Hamiltonian $T_{G(r)}H$, is defined and analytic in the domain $D_1 = D(7\rho/8, R \exp(-\rho/8))$, with

$$T_{G(r)}H = Z^{(r)} + R^{(r)}$$

and

$$R^{(r)} = F_{r+1} + F_{r+2} + F_{r+3} + \dots$$

(ii) There exists an optimal normalizing order $\lfloor r_{opt} \rfloor$ depending on R through

$$r_{opt} \ln r_{opt} = -\frac{1}{c_1} \ln(c_2 \frac{R}{R_0}),$$

such that,

$$|R^{(\lfloor r_{opt} \rfloor)}|_{D_1} \leq c_3 \left(1 - \frac{R}{R_0}\right)^{-1} \left(c_2 \frac{R}{R_0}\right)^{\lfloor r_{opt} \rfloor / 2 - 1},$$

and c_1, c_2, c_3 are constants which depend upon ρ, τ, γ , but not on R .

(iii) The remainder $R^{(\lfloor r_{opt} \rfloor)}$ goes to zero with R/R_0 faster than any analytic order in R/R_0 . More precisely,

$$R^{(\lfloor r_{opt} \rfloor)} = o\left(\left(\frac{R}{R_0}\right)^n\right), \quad (R/R_0 \rightarrow 0)$$

for any given positive integer n .

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