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Fundamental Models of Resonance

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Abstract

Large variety of resonance problems can be often reduced to a common, simple model called a fundamental model of resonance. The paper presents four groups of symmetric fundamental models known from literature. The pendulum-like First Fundamental Model is free of parameters and very simple, but its application can only be local – sufficiently far from the region where angle variable is undetermined. The family of Second Fundamental Models is appropriate for d'Alembertian Hamiltonians, but it does not admit separatrix bifurcations. The Third Fundamental Model of Shinkin is not satisfactory, and the separatrix bifurcations can be better studied in the context of the Extended Fundamental Models proposed by the author.

1 What is a fundamental model

Let us consider a Hamiltonian system with N degrees of freedom

$$\dot{\Gamma} = -\frac{\partial \mathcal{H}}{\partial \gamma}, \qquad \dot{\gamma} = \frac{\partial \mathcal{H}}{\partial \Gamma}, \tag{1}$$

where action–angle-like variables are $\Gamma \in \mathbb{R}^N$ and $\gamma \in \mathrm{T}^N$. Our discussion will be restricted to perturbed problems with

$$\mathcal{H}(\Gamma,\, \gamma) = \mathcal{H}_0(\Gamma) + \varepsilon\, \mathcal{H}_1(\Gamma, \gamma). \tag{2}$$

A complete normalization $(\Gamma,\, \gamma,\, \mathcal{H}) \to (B,\, \beta,\, \mathcal{K})$, creating N action–angle pairs $(B_k,\, \beta_k)$ and a new Hamiltonian $\mathcal{K} = \mathcal{K}(B)$ is usually not possible. The main obstacle in the normalization procedure is the appearance of a resonance. If only one resonance is present, we can achieve a partial normalization, creating $N-1$ action–angle pairs $(B_k,\, \beta_k)$ and one "active" pair $(\Theta,\, \vartheta)$.

For the purpose of this paper, a partially normalized Hamiltonian

$$\mathcal{K} = A_0(\Theta) + \varepsilon\, A_1(\Theta)\, \cos m\, \vartheta. \quad m = 1,\, 2,\, \ldots \tag{3}$$

will be called *a resonant Hamiltonian.*[1]

Resonant Hamiltonians are various, depending on the form of $A_0(\Theta)$ and $A_1(\Theta)$, but they share many common properties. If we ask about the common features of various resonant Hamiltonians, we quite naturally arrive at the notion of *fundamental models.*

> A fundamental model is a simplified resonant Hamiltonian with a minimum possible number of parameters, but still providing a proper qualitative description of motion.

We need it to interpret observations or numerical simulations, to understand phenomena, or to create an analytical theory for a resonant motion.

2 First Fundamental Model

Let us consider a one-harmonic resonant Hamiltonian (3). The standard procedure of reducing $\mathcal{K}$ to a fundamental model will be presented here in full length. It consists of three steps:

1. Translation of momentum and folding of angle,
2. Expansion of $\mathcal{K}$ in powers of the new momentum,
3. Reduction of parameters by means of a canonical scaling.

In order to perform the first step, we establish the approximate resonance condition

$$\dot{\vartheta} \;\approx\; A_0'(\Theta) \;=\; 0, \tag{4}$$

by neglecting ε (throughout this paper, a prime stands for a partial derivative with respect to Θ). Solving Eq. (4) for Θ, we obtain some value Θ_0, that will serve as a new origin for the momentum variable. A canonical transformation $(\vartheta, \Theta) \rightarrow (\varphi, \Phi)$ that follows, consists of translation and folding:

$$\Theta = \Theta_0 + m\,\Phi, \quad \vartheta = \frac{\varphi}{m}. \tag{5}$$

Assuming a smallness of Φ, we may pass to step 2, expanding the Hamiltonian

$$\mathcal{K} = A_0(\Theta_0) + A_0'(\Theta_0)\,m\,\Phi + \frac{m^2}{2}\,A_0''(\Theta_0)\,\Phi^2 + \varepsilon\,A_1(\Theta_0)\,\cos(\varphi) + \ldots, \tag{6}$$

Rejecting a constant term $A_0(\Theta_0)$, recalling $A_0'(\Theta_0) = 0$, and neglecting Φ^3 terms, we obtain a *simplified resonant Hamiltonian*

$$\mathcal{K}^* = \frac{m^2}{2}\,A_0''(\Theta_0)\,\Phi^2 + \varepsilon\,A_1(\Theta_0)\,\cos(\varphi), \tag{7}$$

[1] Equation (3) reveals the important restriction in the subject of this paper: only "symmetric" (one harmonic) fundamental models are to be presented.

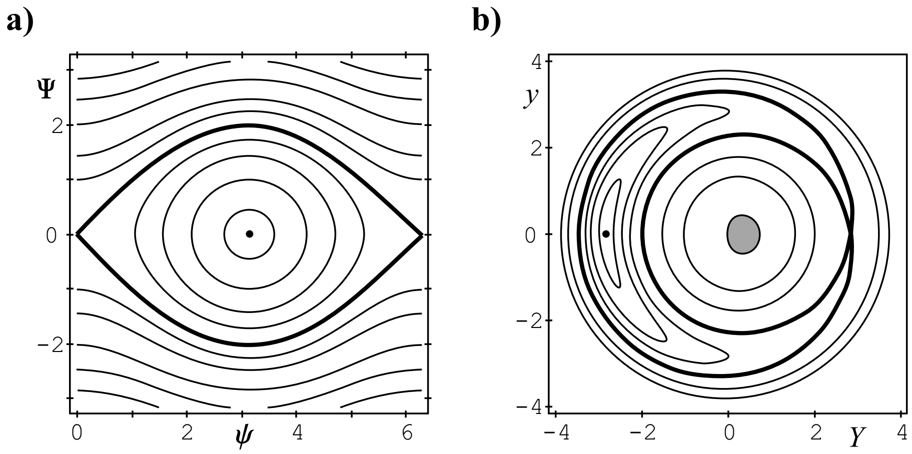

Figure 1: Phase flow of the First Fundamental Model.

The Hamiltonian $\mathcal{K}$ involves too many nonessential parameters, but we are free to modify three parameters:

i) time unit $t \to \tau = c_1\, t$,

ii) "length unit" $\Phi \to \Psi = c_2\, \Phi$,

iii) angle offset $\varphi \to \psi = \vartheta + k\,\pi$, (recommended $k = 0, 1$ to preserve the symmetry).

The above transformations permit to get rid of two parameters. If we let

$$c_2 = m\sqrt{\left|\frac{A_0''}{A_1\,\varepsilon}\right|}, \quad c_1 = \varepsilon\, A_1, \quad k = 1 \text{ if } \operatorname{sgn}(A_0''\, A_1\, \varepsilon) = -1. \tag{8}$$

we obtain a parameters-free *First Fundamental Model*

$$\mathcal{M}_{\mathrm{FFM}} = \frac{1}{2}\,\Psi^2 + \cos\psi. \tag{9}$$

We easily recognize here the Hamiltonian of a simple pendulum, which means that we know practically everything about its behaviour. The phase portrait of the pendulum model is shown in Fig. 1a. We recognize the partition of the phase space into libration and circulation zones. The "strength of resonance" is estimated by a "resonance width" Δ, measured from the line $\Psi = 0$ to the extreme point of a separatrix. Of course, the resonance width of FFM is always $\Delta = \sqrt{2}$; in order to recover the resonance width in original variables, we simply inverse the scaling:

$$\Delta_\Theta = \frac{\sqrt{2}}{c_2} = \frac{1}{m}\sqrt{\frac{2|\varepsilon\, A_1(\Theta_0)|}{|A_0''(\Theta_0)|}} = O(\sqrt{\varepsilon}). \tag{10}$$

The First Fundamental Model has been extensively used in various resonance problems at least since the first works of Laplace on Galilean satellites' resonance. Its simplicity is remarkable, but in some cases it reveals important drawbacks. Observe, that FFM implicitly assumes that Θ, ϑ are defined on an infinite cylinder ($\Theta \in \mathbb{R}$). In practice, however, the values of the momentum are bounded, say $\Theta \in [a, b]$. The question of undetermined angle ϑ when $\Theta = a$ or $\Theta = b$ becomes completely neglected in FFM. To state it briefly: FFM is not d'Alembertian.

3 Second Fundamental Model for d'Alembertian Hamiltonians

The resonant Hamiltonian (3) is d'Alembertian if its amplitudes admit a particular form of expansion in powers of Θ

$$A_0(\Theta) = a_1\,\Theta + a_2\,\Theta^2 + a_3\Theta^3 + \ldots, \tag{11}$$

$$A_1(\Theta) = b_1\Theta^{m/2} + b_2\,\Theta^{(m+1)/2} + \ldots \tag{12}$$

An equivalent definition states that $\mathcal{K}$ is d'Alembertian if it is analytic in Poincaré variables x, X

$$x = \sqrt{2\,\Theta}\,\sin\vartheta, \qquad X = \sqrt{2\,\Theta}\,\cos\vartheta. \tag{13}$$

Observing that $\Theta = \frac{1}{2}(x^2 + X^2)$, we easily explain the special form of amplitudes in (11) and (12).

If the resonant Hamiltonian $\mathcal{K}$ is d'Alembertian, the reduction to a fundamental model must differ from the procedure described in the previous section at one important point: we skip the passage through (Φ, φ) and go directly to expansion around $\Theta = 0$ followed by the scaling. The two operations involved in defining Φ and φ are now prohibited. We can not translate the momentum, because this would destroy the form of expansion (12). Forbidding the folding is less evident, but equally important; it can be explained by the behaviour of an exemplary expression F

$$F = 2\,\Theta\,\cos 2\vartheta = X^2 - x^2, \tag{14}$$

All derivatives $\left.\frac{\partial^k F}{\partial X^k}\right|_0$ are regular at the origin $X = x = 0$. But if we apply a folding transformation $\varphi = 2\vartheta$, $\Phi = \Theta/2$, the expression becomes

$$F = 4\,\Phi\,\cos\varphi = 2\,X\,\sqrt{(X^2 + x^2)}, \tag{15}$$

and we see, that F is no longer analytic at the origin, because $\left.\frac{\partial^3 F}{\partial X^3}\right|_0 = \infty$.

It is quite an instructive exercise to write down the First Fundamental Model in terms of Poincaré variables. First, we notice that the variables like $x = \sqrt{2\,\Psi}\,\sin\psi$ are of no use, because they imply $\Psi \geq 0$, which is not the case. We are forced to translate the origin of momentum, introducing some new momentum $\Psi^\star = \Psi - A$; for obvious reasons, the constant A must be greater than Δ. Using Poincaré variables

$$y = \sqrt{2\,(\Psi - A)}\,\sin\psi, \qquad Y = \sqrt{2\,(\Psi - A)}\,\cos\psi, \tag{16}$$

we rewrite $\mathcal{M}_{\rm FFM}$ as

$$\mathcal{M}^{\star}_{\rm FFM} = \frac{1}{2}\left(A + \frac{1}{2}\,(y^2+Y^2)\right)^2 + \frac{Y}{\sqrt{y^2+Y^2}}. \tag{17}$$

The first derivative of the last term of Eq. (17) with respect to y is singular at the origin $y = Y = 0$, and thus the Hamiltonian $\mathcal{M}^{\star}_{\rm FFM1}$ is not analytic. Figure 1b presents the Hamiltonian (17) for $A = 4$; the exclusion of the origin has been marked by shading one of the inner contours. As a matter of fact, it is not sufficient to exclude only the point $y = Y = 0$: we have to exclude the entire contour line $\mathcal{M}^{\star}_{\rm FFM1} = {\rm const}$ passing through the origin, and a whole area bounded by this curve.

Let us return to the expansion of $\mathcal{K}$. If the translation to the resonance value of momentum is prohibited, we should directly expand $\mathcal{K}$ around $\Theta_0 = 0$, that results in the simplified resonant Hamiltonian

$$\mathcal{K}^{*}(\Theta,\vartheta) \approx \alpha_2\,\Theta^2 + \alpha_1\,\Theta + \varepsilon\beta\,\Theta^{m/2}\,\cos(m\,\vartheta). \tag{18}$$

Reducing two parameters by means of the scaling, we obtain the Second Fundamental Model (SFM), or rather a family of models SFMm

$$\mathcal{M}_{{\rm SFM}m} = \Psi^2 + \gamma\,\Psi + (2\,\Psi)^{m/2}\cos m\,\psi. \tag{19}$$

Although the Hamiltonians like (18) were studied first by Andoyer [1] and then by Jefferys [7], but it was only in 1980's when Henrard and Lemaitre reduced the number of parameters obtaining the fundamental model (19) [6, 8].

In contrast to the First Fundamental Model, the members of the family SFMm behave differently for different values of the angle multiplier m, and for each m the Hamiltonian depends on parameter γ. The dependence on γ gives rise to various bifurcations that add a special flavor to the study of apparently simple one degree of freedom systems. As an illustration of this statement, let us visit two most commonly applied cases: $m = 1$ and $m = 2$ (the first and second order resonances). A review of higher order SFMm models can be found in [9].

3.1 SFM1

Henrard and Lemaitre [6] studied the SFM1 Hamiltonian in the form

$$\mathcal{M}_1 = \Psi^2 - 3(\delta+1)\,\Psi - 2\sqrt{2\Psi}\cos\psi = -\frac{3}{2}(\delta+1)(x^2+X^2) + \frac{1}{4}(x^2+X^2)^2 - 2\,X. \tag{20}$$

A slight difference with respect to Eq. (19), consisting in the use of δ and an extra factor 2 in the amplitude of $\cos\psi$, result from the esthetic principles only: The critical value of the parameter δ in $\mathcal{M}_1$ becomes $\delta = 0$ according to this choice. Looking at the sample phase

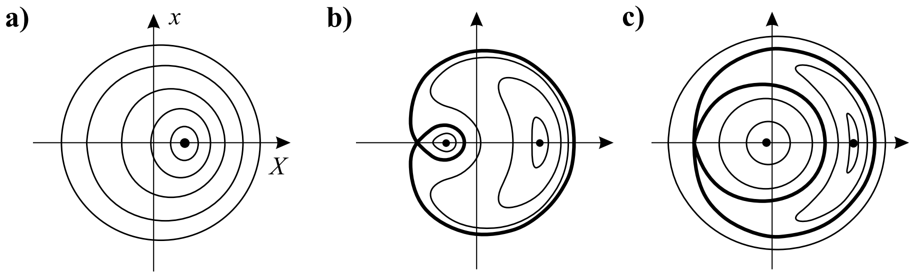

Figure 2: Two generic phase portraits of SFM1 for $\delta < 0$ (a) and $\delta > 0$ (b). For $\delta \gg 1$ the phase flow becomes similar to FFM (c).

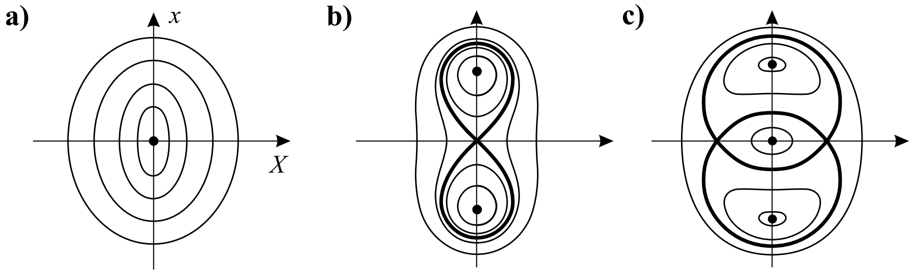

Figure 3: Generic phase portraits of SFM2 for $\delta < -1$ (a), $-1 < \delta < 0$ (b), and $\delta > 0$ (c).

portraits of SFM1 (Fig. 2), we first find a nonresonant evolution for $\delta < 0$. Although some trajectories may reveal libration, when followed in polar variables (ψ, Ψ), we should not name them resonant, because there are no unstable manifolds (separatrices) among the contour lines of $\mathcal{M}_1$. It is a tangent bifurcation at $\delta = 0$ that gives rise to the really resonant motion for $\delta > 0$. It should be observed, that in the limit $\delta \gg 1$ the libration zone is sufficiently distant from the origin, that we may locally approximate $\mathcal{M}_1$ by means of the First Fundamental Model.

3.2 SFM2

Lemaitre [8] studied the cases of $m = 2$, $m = 3$ and $m = 4$. Her SFM2 Hamiltonian is

$$\mathcal{M}_2 = 2\,\Psi^2 - (2\delta + 1)\,\Psi + \Psi \cos 2\psi = -(2\delta + 1)(x^2 + X^2) + (x^2 + X^2)^2 + X^2 - x^2. \quad (21)$$

Figure 3 provides three generic phase portraits of $\mathcal{M}_2$. The resonant cases appear only for $\delta > -1$, after a pitchfork bifurcation at $\delta = -1$. The second pitchfork bifurcation occurs at $\delta = 0$, and thus we may have up to 5 critical points in the system. If $\delta \gg 1$, we may locally perform the translation/folding transformation, that reduces SFM2 to the pendulum-like FFM.

4 Third Fundamental Model

A large number of resonances in celestial mechanics, involving asteroids, artificial satellites, planetary moons etc., can be well reduced to the SFMm. The Second Fundamental Model admits most of classical critical points bifurcations that appear in one degree of freedom Hamiltonian systems. Its application, however, is restricted to the systems that do not admit *separatrix bifurcations* (known also as *saddle connections* [2]) that appear in some problems like semi-secular resonances of artificial satellites [4] or orbital resonances of asteroids [10].

Looking for a better suited fundamental model, Shinkin [10] proposed a Hamiltonian, that he named the *Third Fundamental Model*

$$\mathcal{M}_{\rm III} = \frac{1}{2}\Psi^2 + \alpha \prod_{k=1}^{4} (\Psi + \beta_k)^{j_k/2} \cos\psi, \tag{22}$$

where $j_1 + j_2 + j_3 + j_4 \leq m$. As we see, his leading idea was to complicate the amplitude of the periodic term. Note, however, that Shinkin's $\mathcal{M}_{\rm III}$ was derived similarly to the First Fundamental Model: the resonant angle was folded and thus the Hamiltonian is not d'Alembertian in a general case. It also contains too many parameters as for a fundamental model.

5 Extended Fundamental Model

Looking for a fundamental model capable of representing separatrix bifurcations, we can adopt a simple strategy, different from the Shinkin's idea in one essential point: we follow the same way as for the SFM, but we retain a cubic term Θ^3 in the simplified resonant Hamiltonian (18)

$$\mathcal{K}^* = a_3\Theta^3 + a_2\Theta^2 + a_1\Theta + b\,(2\Theta)^{m/2}\cos m\vartheta. \tag{23}$$

The reduction of two parameters leads to the family of *Extended Fundamental Models* EFMm [4, 5]

$$\mathcal{M}_{\rm EFMm} = \Psi^3 + \frac{1}{2}\,u\,\Psi^2 + v\Psi + (2\Psi)^{m/2}\cos m\,\psi. \tag{24}$$

In order to allow a comparison between SFMm and EFMm, let us briefly inspect two cases: $m = 1$ and $m = 2$.

5.1 EFM1

The Extended Fundamental Model for first order resonance has a Hamiltonian

$$\begin{aligned} \mathcal{M}_{\rm E1} &= \Psi^3 + \frac{1}{2}\,u\,\Psi^2 + v\Psi + \sqrt{2\Psi}\cos\psi = \\ &= \frac{1}{8}\,(x^2+X^2)^3 + \frac{1}{8}\,u\,(x^2+X^2)^2 + v(x^2+X^2) + X. \end{aligned} \tag{25}$$

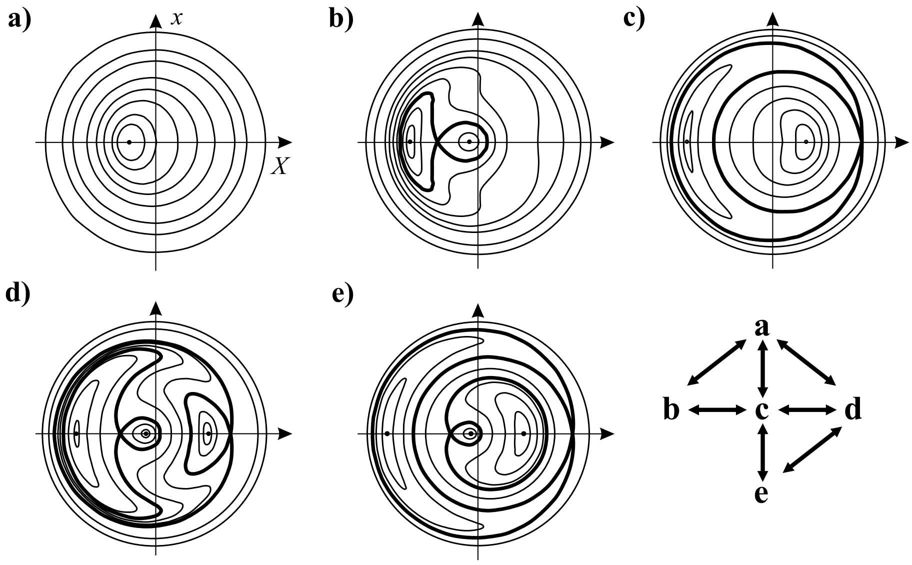

Figure 4: Generic phase portraits of EFM1 and possible transitions.

Its behaviour was studied in [5]. The parametric plane (u, v) is divided by bifurcation lines $f(v, u) = 0$ into 5 sectors with qualitatively distinct phase portraits (the latter are shown in Fig. 4). Similarly to SFM1, we find tangent bifurcations, but they generate more critical points: up to 5 in EFM1, compared to 3 in SFM1. What is more important, if two unstable points have the same value of cM_{E1}, the separatrices asymptotic to these points merge. This separatrix bifurcation changes qualitatively the phase portrait, but it could not be detected by means of usual considerations based on the variational equations in the vicinity of critical points.

The reduction of EFM1 to previous models is locally possible, provided the two libration zones are sufficiently distant one from another (reduction to SFM1) and from the axes origin (reduction to FFM).

5.2 EFM2

The results for the second order EFM2 have not been published yet, but the lunisolar resonance Hamiltonian discussed in [3] can be easily reduced to the general form

$$\begin{aligned} \mathcal{M}_{\mathrm{E2}} &= \Psi^3 + \frac{1}{2}\,u\,\Psi^2 + v\Psi + 2\Psi\cos 2\psi = \\ &= \frac{1}{8}\,(x^2+X^2)^3 + \frac{1}{8}\,u\,(x^2+X^2)^2 + v(x^2+X^2) + X^2 - x^2. \end{aligned} \tag{26}$$

The bifurcation sequence presented in [3] is actually the one of EFM2 (Fig. 5). Compared to SFM2, the model not only admits more critical points (up to 9) but also it involves both tangent and pitchfork bifurcations, not to mention the separatrix bifurcations. The phase portaits are surprisingly rich as for an apparently simple Hamiltonian (26).

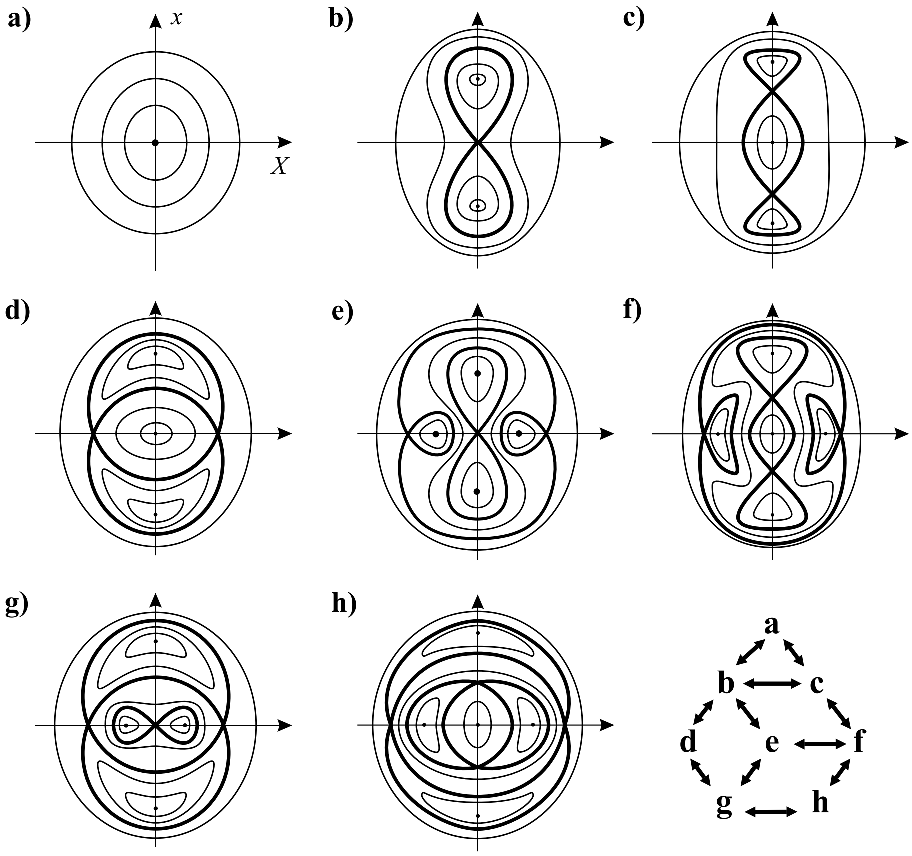

Figure 5: Generic phase portraits of EFM2 and possible transitions.

6 Conclusions

Fundamental models play a crucial role in understanding the dynamics of resonant systems. They help us to understand the results of numerical simulations. They can be used to predict the onset of chaos due to various phenomena like the resonance overlap or the appearance of saddle connections. Recognizing the fundamental model appropriate for a given problem saves quite a lot of work that otherwise would be spent on studying well known and widespread patterns.

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