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Some models for the Trojan motion

Frederic Gabern* and Àngel Jorba

Departament de Matemàtica Aplicada i Anàlisi

Universitat de Barcelona. Gran Via 585, 08007 Barcelona, Spain. †

Abstract

We focus on the dynamics of a small particle near the triangular points of the Sun-Jupiter system. To try to account for the effect of Saturn (and to simulate in a more realistic way the Sun-Jupiter relative motion), we develop specific models based on the numerical computation of periodic and quasi-periodic (with two frequencies) solutions of the planar three body problem Sun-Jupiter-Saturn and write them as perturbations of the Sun-Jupiter RTBP.

1 Introduction

The dynamics around the Lagrangian L_4 and L_5 points of the Sun-Jupiter system have been studied by several authors in the Restricted Three Body Problem using semi-analytical tools such as normal forms or approximate first integrals (see [5, 10, 3, 6, 11]).

It is also known that Trojan asteroids move near the triangular points of the Sun-Jupiter system. Its dynamics has been studied by many authors (see, for example, [7, 8, 9, 12]) using the Outer Solar System model, where the Trojans are supposed to move under the attraction of the Sun and the four main outer planets (Jupiter, Saturn, Neptune and Uranus). This is a strictly numerical model, so the semi-analytical tools mentioned above cannot be used in principle.

In this paper, we briefly present two intermediate models for the motion of a Trojan asteroid. These models try to simulate in a more realistic way the relative Sun-Jupiter motion and are written as explicit perturbations of the RTBP, which allows to compute normal forms and approximated first integrals.

The first model is a natural improvement of the Sun-Jupiter RTBP that includes the effect of Saturn on the motion of Sun and Jupiter. In this model, Sun, Jupiter and Saturn move in a periodic solution of the (non-restricted) planar Three Body Problem (TBP,

*Corresponding author.

†`gabern@mat.ub.es, angel@maia.ub.es`

from now on), with the same relative period as the real one. It is possible to write the equations of motion of a fourth massless particle that moves under the attraction of these three. This is a restricted four body problem that we call Bicircular Coherent Problem (BCCP, for short). Its detailed construction and study can be found in [4].

In the second model, the periodic solution of the BCCP is used as the starting point of the computation of a 2-D invariant torus for which the osculating excentricity of Jupiter's orbit is the actual one. In this sense, the Sun-Jupiter relative motion is better simulated by this quasi-periodic solution of the planar three body problem. Afterwards, the equations of motion of a massless particle that moves under the attraction of these three main bodies (supposing that they move in the previously computed quasi-periodic solution) are easily derived. We call this restricted four body problem as the Bianular Problem (BAP, for short).

2 The Bicircular Coherent Problem

It is possible to find, in a rotating reference frame, periodic solutions of the planar three body Sun-Jupiter-Saturn problem by means of a continuation method using the masses of the planets as parameters (see [4] for details). The relative Jupiter-Saturn period can be chosen as the actual one, and its related frequency is $\omega_{sat} = 0.597039074021947$.

Assuming that these three main bodies move on this periodic orbit, it is possible to write the Hamiltonian for the motion of a fourth massless particle as:

$$\begin{aligned} H &= \frac{1}{2}\alpha_1(\theta)(p_x^2+p_y^2+p_z^2)+\alpha_2(\theta)(xp_x+yp_y+zp_z)+\alpha_3(\theta)(yp_x-xp_y) \\ &\quad +\alpha_4(\theta)x+\alpha_5(\theta)y-\alpha_6(\theta)\left[\frac{1-\mu}{q_S}+\frac{\mu}{q_J}+\frac{m_{sat}}{q_{sat}}\right], \end{aligned} \tag{1}$$

where $q_S^2 = (x-\mu)^2+y^2+z^2$, $q_J^2 = (x-\mu+1)^2+y^2+z^2$ and $q_{sat}^2 = (x-\alpha_7(\theta))^2+(y-\alpha_8(\theta))^2+z^2$. The functions $\alpha_i(\theta)$ are periodic functions in $\theta = \omega_{sat}t$ and can be explicitly computed with a Fourier analysis of the numerical periodic solution of the three body problem. The numerical values used for the mass parameters are $\mu = 0.95387536 \times 10^{-3}$ and $m_{sat} = 0.285515017438987 \times 10^{-3}$.

At that point, we want to mention that a Bicircular Coherent problem was already developed in [1] for the Earth-Moon-Sun case to study the dynamics near the Eulerian points.

3 The Bianular Problem

In this section, we compute a quasi-periodic solution, with two basic frequencies, of the planar Sun-Jupiter-Saturn three body problem. This quasi-periodic solution lies on a 2-D torus. As the problem is Hamiltonian, this torus belongs to a family of tori. We look for a torus, on this family, for which the osculating excentricity of Jupiter's orbit is quite well adjusted to the actual one. Then, this quasi-periodic solution is used in order to obtain the Hamiltonian of the Bianular Problem.

3.1 The reduced Hamiltonian of the Three Body Problem

We take the Hamiltonian of the planar three body problem written in the Jacobi coordinates in a uniformly rotating reference frame and we make a canonical change of variables (using the angular momentum first integral) in order to reduce this Hamiltonian from 4 to 3 degrees of freedom. We obtain:

$$\begin{aligned} H(Q_1, Q_2, Q_3, P_1, P_2, P_3) &= \frac{1}{2\alpha}\left(P_1^2 + \frac{A^2}{Q_1^2}\right) + \frac{1}{2\beta}\left(P_2^2 + P_3^2\right) \\ &\quad -K - \frac{\alpha}{r} - \frac{(1-\mu)m_{sat}}{r_{13}} - \frac{\mu m_{sat}}{r_{23}} \end{aligned} \tag{2}$$

where $\alpha = \mu(1-\mu)$, $\beta = m_{sat}/(1+m_{sat})$, $A = Q_2P_3 - Q_3P_2 + K$, $r = Q_1$, $r_{13}^2 = (\mu Q_1 - Q_2)^2 + Q_3^2$, $r_{23}^2 = ((1-\mu)Q_1 + Q_2)^2 + Q_3^2$ and K is the angular momentum of the system.

3.2 A method for computing 2-D invariant tori

We are interested in finding a quasi-periodic solution (with two frequencies) of a given vector field. We reduce this problem to the one of finding an invariant curve of a suitable Poincaré map. This invariant curve is seen as a truncated Fourier series and our aim is to compute its rotation number and a representation of it. We follow roughly the method developed by [2].

3.2.1 Numerical computation of invariant curves

Let be $\dot{x} = f(x)$ $(x, f \in \mathbb{R}^n)$ an autonomous vector field of dimension n (for example, the reduced field of the three body problem given at 3.1) and $\Phi(x,t) \equiv \Phi_t(x)$ its associated flow. Let us define the Poincaré map as the time T-flow $\Phi_T(\cdot)$, where T is a prefixed value ($T = T_{sat}$, the period of Saturn in the Sun-Jupiter system, in our case).

Let ω be the rotation number of the invariant curve. Let, also, $\mathcal{C}(\mathbb{T}^1, \mathbb{R}^n)$ be the space of continuous functions from $\mathbb{T}^1$ in $\mathbb{R}^n$, and let us define the linear map $T_\omega : \mathcal{C}(\mathbb{T}^1, \mathbb{R}^n) \to \mathcal{C}(\mathbb{T}^1, \mathbb{R}^n)$ as the translation by ω, $(T_\omega \varphi)(\theta) = \varphi(\theta + \omega)$.

Let us define $F : \mathcal{C}(\mathbb{T}^1, \mathbb{R}^n) \to \mathcal{C}(\mathbb{T}^1, \mathbb{R}^n)$ as

$$F(\varphi)(\theta) = \Phi_T(\varphi(\theta)) - (T_\omega \varphi)(\theta) \qquad \forall \varphi \in \mathcal{C}(\mathbb{T}^1, \mathbb{R}^n). \tag{3}$$

It is clear that the zeros of F in $\mathcal{C}(\mathbb{T}^1, \mathbb{R}^n)$ correspond to invariant curves of rotation number ω. The equation satisfied is

$$\Phi_T(\varphi(\theta)) = \varphi(\theta + \omega) \quad \forall \theta \in \mathbb{T}. \tag{4}$$

The method we want to summarize in this section boils down to looking numerically for a zero of F. Hence, let us write $\varphi(\theta)$ as a real Fourier series,

$$\varphi(\theta) = A_0 + \sum_{k>0} (A_k \cos(k\theta) + B_k \sin(k\theta)) \quad A_k, B_k \in \mathbb{R}^n \quad k \in \mathbb{N}.$$

Then, we will fix in advance a truncation value N_f for this series (the selection of the truncation value will be discussed later on), and let us try to determine (an approximation to) the $2N_f + 1$ unknown coefficients A_0, A_k and B_k, $1 \leq k \leq N_f$. To this end, we will construct a discretized version of the map F, as follows: first, we select the mesh of $2N_f+1$ points on $\mathbb{T}^1$,

$$\theta_j = \frac{2\pi j}{2N_f + 1}, \quad 0 \leq j \leq 2N_f, \tag{5}$$

and evaluate the function (3) on it. Let F_{N_f} be this discretization of F:

$$\Phi_T(\varphi(\theta_j)) - \varphi(\theta_j + \omega), \quad 0 \leq j \leq 2N_f. \tag{6}$$

So, given a (known) set of Fourier coefficients A_0, A_k and B_k ($1 \leq k \leq N_f$), we can compute the points $\varphi(\theta_j)$, then $\Phi_T(\varphi(\theta_j))$ and next the points $\Phi_T(\varphi(\theta_j)) - \varphi(\theta_j + \omega)$, $0 \leq j \leq N_f$. From these data, we can immediately obtain the Fourier coefficients of $\Phi_T(\varphi(\theta)) - \varphi(\theta + \omega)$.

To apply a Newton method to solve the equation $F_{N_f} = 0$, we also need to compute explicitly the differential of F_{N_f}. This can be done easily by applying the chain rule to the process used to compute F_{N_f}. Note that the number of equations to be solved is $(2N_f + 1)n$ and that the unknowns are $(A_0, A_1, B_1, \dots, A_{N_f}, B_{N_f})$, ω and the time T for which we fix the Poincaré map associated to the flow $(\Phi_T(\cdot))$. That is, we deal with $(2N_f + 1)n + 2$ unknowns. In each step of the Newton method, we solve a non-square linear system by means of a standard least-squares method. We want to mention that this system is degenerated unless we fix (keep constant during the computation) some of the unknowns.

Note that in the case of the reduced three body problem, an integral of motion is still left: the energy. We can easily solve the problem of the degeneracy, induced by it, fixing the time T of the Poincaré map.

3.2.2 Discretization error

Once we have solved equation (6) with a certain tolerance (error in the Newton method; we take tipically 10^{-11}), we still don't have any information on the error of the approximated invariant curve. The reason, as explained in [2], is that we have not estimated the discretization error; i.e., the error when passing from equation (3) to equation (6).

In order to do it, we compute

$$E(\varphi, \omega) = \max_{\theta \in \mathbb{T}} |\Phi_T(\varphi(\theta, z)) - \varphi(\theta + \omega, z)|$$

in a mesh of points, say, 100 times finer than the mesh (5) and consider it as an estimation of the error of the invariant curve.

If $||E||_\infty > 10^{-9}$, the solution obtained is not considered good enough and another one with the same initial approximation for the Newton method but with a greater discretization order N_f is computed. The process is repeated until the sub-infinity norm of the discretization error is smaller than 10^{-9}.

3.3 Finding the desired torus

The initial approximation to the unknowns in the Newton method is given by the linearization of the Poincaré map around a fixed point X_0 (a periodic orbit, for the flow). We use the periodic orbit computed in Section 2 for the BCCP model:

$$X_0 = \Phi_{T_{sat}}(X_0),$$

where, $\Phi_{T_{sat}}(\cdot)$ is the time T_{sat}-flow corresponding to Hamiltonian (2).

It is easy to see, by looking at the eigenvalues of $D\Phi_{T_{sat}}(X_0)$, that there are two different non-neutral normal directions to the periodic orbit X_0. Thus, two families of tori arise from it. We call them Family1 and Family2.

We compute a first torus for each family (they are called Torus1 and Torus2) with the method described in 3.2. In Figure 1, the projection for Torus1 and Torus2 into the (q_1, q_2), $(\dot{q}_1, \dot{q}_2)$ and (q_3, q_4) planes are shown (where $(q_i)_{i=1\div 4}$ are the Jacobi coordinates for the three body problem in a rotating system). We can see (left and center plots) that the relative Sun-Jupiter motion is a libration around the point $(-1, 0, 0, -1)$, and that Saturn's orbit has fattened a little bit (right plots).

The two families of tori can be parameterized by the angular momentum K. It is straightforward from the computations that there is a strong relationship between the angular momentum, K, and the osculating orbital elements of Jupiter's and Saturn's orbits. As we want to simulate in a more reallistic way the Sun-Jupiter relative motion, we are more interested in adjusting Jupiter's orbital elements than Saturn's ones. As there is one degree of freedom (we are allowed to set K), we select the osculating excentricity of Jupiter's orbit as the targeting value. Thus, by means of a continuation method, we try to find another torus inside Family1 or Family2 for which the osculating excentricity of Jupiter is approximately 0.0484.

In order to continue the families, we add to the invariant curve equations the following one:

$$\mathrm{excen}(Q_1, Q_2, Q_3, P_1, P_2, P_3, K) = e,$$

where excen$(\cdot)$ is a function that gives us Jupiter's osculating excentricity at a given moment (we evaluate it when Sun, Jupiter and Saturn are in a particular collinear config-

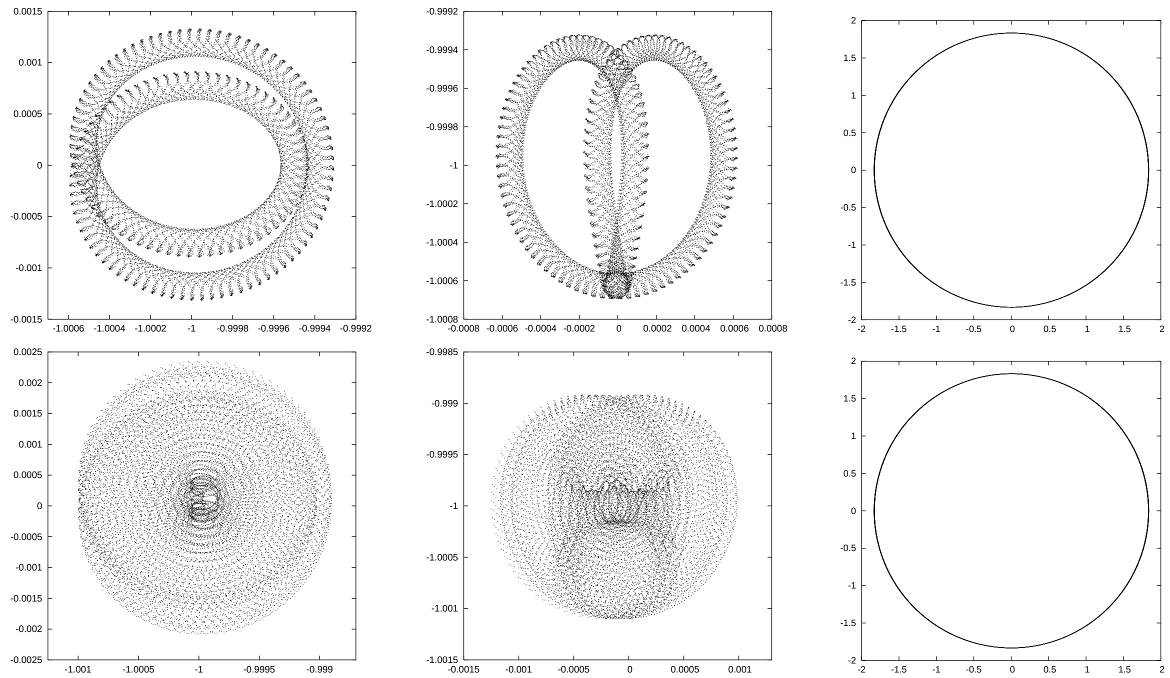

Figure 1: Left: Sun-Jupiter relative motion for Torus1 (top) and Torus2 (bottom). Center: Momenta for the Sun-Jupiter relative motion for Torus1 (top) and Torus2 (bottom). Right: Saturn's orbit around the Sun-Jupiter barycenter for Torus1 (top) and Torus2 (bottom).

uration), and e is a fixed constant that is used as a control parameter. We try to continue each family increasing the parameter e to its actual value.

In Family1, we start increasing little by little the parameter e in order to have a good enough initial point for the Newton method in each step of the continuation process. What is observed is that when e increases, the number of harmonics (N_f) has also to be increased if we want the discretization error of the invariant curve to be smaller than a certain tolerance (tipically we take 10^{-9}). We stop the continuation when the number of harmonics is about 180. At this moment, if we look at the orbital elements of Jupiter's and Saturn's orbits, we see that they do not evolve in the desired direction, but they are getting farther from the real ones. In Figure 2, the projection of this solution into the configuration space is shown. This solution is far from a planetary one because, for example, the big variation of the two semi-major axes. Thus, increasing Jupiter's excentricity inside Family1 forces us to move away from the desired solution.

In Family2, we are able to increase e up to its actual value ($e = 0.0484$), the number of harmonics doesn't grow up very much (actually, if we ask the invariant curve to have an error smaller than 10^{-9}, N_f increases from 6 to 9) and the solution obtained is of the planetary type. In Figure 3, we plot the variation of the angular momentum K of the planar SJS Three Body Problem when the parameter e is increased in the continuation

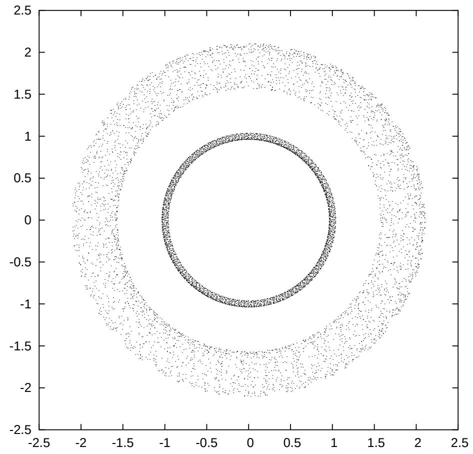

Figure 2: Projection into the configuration space of the torus belonging to Family1 when the continuation procedure is stopped. See the text for more details.

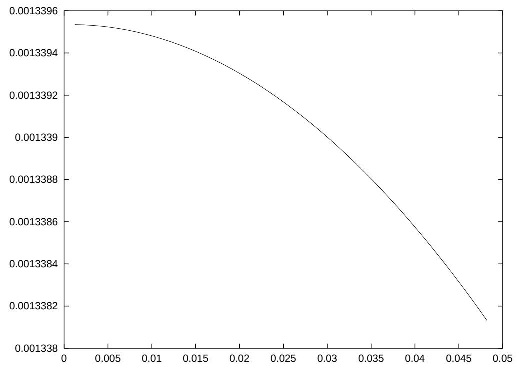

Figure 3: Plot of the evolution of the angular momentum K when the parameter e is increased from 0.00121 (corresponding to Torus2) to 0.0484 (the desired value) in the continuation of Family2.

process.

We can see the projection of the final torus into the configuration space in Figure 4. This solution of the planar Sun-Jupiter-Saturn TBP is what we call the Bianular solution of the TBP. This torus is parameterized with the angles $(\theta_1, \theta_2) = (\omega_1 t + \theta_1^0, \omega_2 t + \theta_2^0)$, where the frequencies are $\omega_1 = \omega_{sat} = 0.597039074021947$ and $\omega_2 = \frac{\omega_1 \bar{\omega}}{2\pi} = 0.194113943490717$ ($\bar{\omega}$ is the rotation number of the invariant curve), and $\theta_{1,2}^0$ are the initial phases.

Let us comment this result: Recall that our main goal was to simulate in the most reallistic way possible the relative Sun-Jupiter motion. We have obtained (numerically) a quasi-periodic solution of the TBP where the osculating excentricity, the semimajor axis, the period and the mean motion of Jupiter's orbit are quite well adjusted (see Figure 5, where we have plot the evolution of the osculating orbital elements of Jupiter and Saturn in a time span of 52.6195485402068 adimensional units). The argument of the perihelion still oscilates too much. We have tried to adjust it better by moving the rotation number

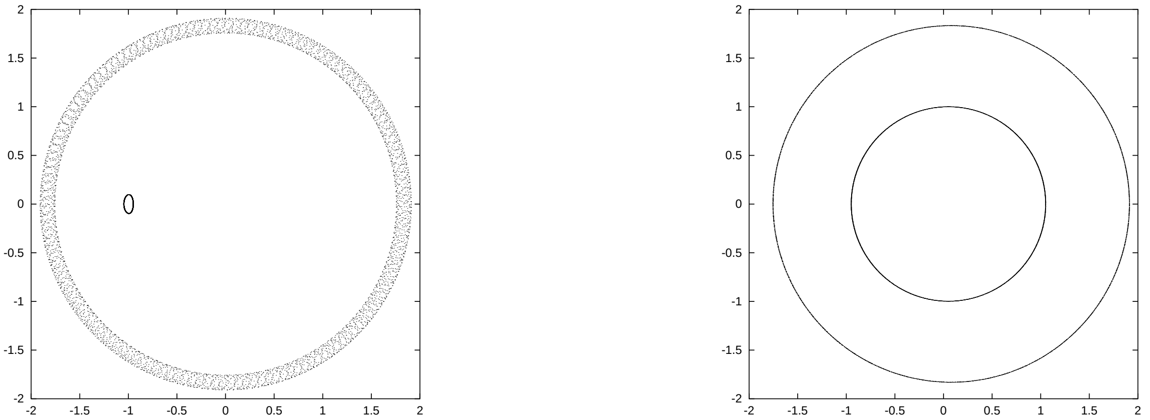

Figure 4: Projection into the configuration space of the Bianular solution of the planar three body problem Sun-Jupiter-Saturn in the rotating reference frame (left plot) and in an inertial system (right plot).

ω, but what happens is that we loose the accuracy in the Jupiter's mean motion (we desire that it librates quasi-periodically around 1).

Concerning Saturn's motion on the torus, we have quite well fitted the semimajor axis and the period of its orbit; the obtained excentricity is about the 80% of the actual one and the argument of the perihelion still oscilates too much. We have tried to adjust better the excentricity of Saturn's orbit by changing the rotation number ω, but if we do so, we loose again accuracy on the Jupiter's mean motion.

3.4 The Hamiltonian of the BAP Model

Finally, it is possible to obtain the equations of a massless particle that moves under the attraction of the three primaries. The corresponding Hamiltonian is:

$$\begin{aligned} H_{BAP} &= \frac{1}{2}\alpha_1(\theta_1,\theta_2)(p_x^2+p_y^2+p_z^2) \\ &\quad +\alpha_2(\theta_1,\theta_2)(xp_x+yp_y+zp_z) \\ &\quad +\alpha_3(\theta_1,\theta_2)(yp_x-xp_y)+\alpha_4(\theta_1,\theta_2)x \\ &\quad +\alpha_5(\theta_1,\theta_2)y-\alpha_6(\theta_1,\theta_2)\left[\frac{1-\mu}{q_S}+\frac{\mu}{q_J}+\frac{m_{sat}}{q_{sat}}\right], \end{aligned}$$

where $q_S^2=(x-\mu)^2+y^2+z^2$, $q_J^2=(x-\mu+1)^2+y^2+z^2$, $q_{sat}^2=(x-\alpha_7(\theta_1,\theta_2))^2+(y-\alpha_8(\theta_1,\theta_2))^2+z^2$, $\theta_1=\omega_1 t+\theta_1^0$ and $\theta_2=\omega_2 t+\theta_2^0$.

The auxiliar functions $\alpha_i(\theta_1,\theta_2)_{i=1\div 8}$ are quasi-periodic functions that can be computed by a Fourier analysis of the solution found in 3.3.

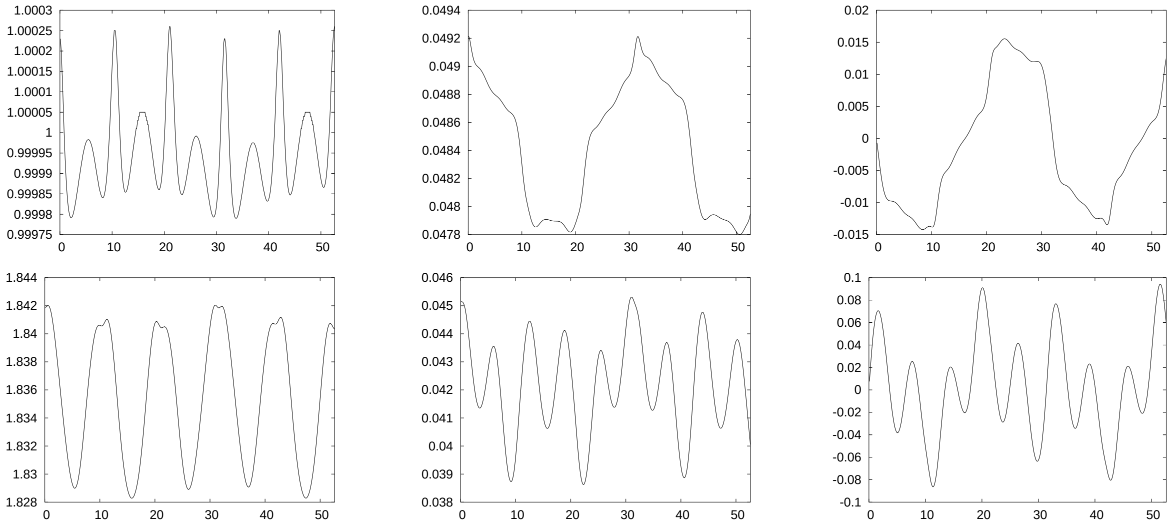

Figure 5: From left to right: Evolution of the osculating semimajor axis, excentricity and argument of the perihelion of Jupiter's orbit (top) and Saturn's orbit (bottom) for the Bianular solution of the planar three body problem.

4 Conclusions

We have seen two particular examples of a much more general methodology for constructing semi-analytic models of the Solar System and writing them as "perturbations" of the Sun-Jupiter RTBP. For instance, if a quasi-periodic solution of the N-Body Problem with m frequencies is known, it is then possible to write the Hamiltonian of the Restricted Problem of $(N+1)$ bodies as:

$$\begin{aligned} H &= \frac{1}{2}\alpha_1(\theta)(p_x^2+p_y^2+p_z^2)+\alpha_2(\theta)(xp_x+yp_y+zp_z) \\ &\quad +\alpha_3(\theta)(yp_x-xp_y)+\alpha_4(\theta)x+\alpha_5(\theta)y-\alpha_6(\theta)\sum_{i=0}^{N} G\frac{m_i}{\rho_i}, \end{aligned}$$

where the functions $\alpha_i(\theta)$ are also quasi-periodic with the same m frequencies ($\theta \in \mathbb{T}^m$) and ρ_i is the distance between the particle and the i-th body written in a "rotating-pulsating" reference system.

All these models (as BCCP and BAP) are specially written in order that semi-analytical tools (such as Normal Forms or numerical First Integrals techniques) can be applied.

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