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Ideal bases for graded polynomial rings and applications to interpolation

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Abstract

Based on a generalized algorithm for the division with remainder of polynomials in several variables, a method for the construction of standard bases for polynomial ideals with respect to arbitrary grading structures is derived. In the case of ideals with finite codimension, which can be viewed upon as a polynomial interpolation problem, an explicit representation for the interpolation space of reduced polynomials can be given.

1 Introduction

We consider polynomial rings in several variables, equipped with a graded structure induced by an arbitrary grading monoid. The goal is a construction method for ideal bases which leads, depending on the underlying grading structure, to the H-bases introduced by Macaulay [9] in 1916 as well as to the Gröbner bases which have been developed by Buchberger [4] in 1965.

For that purpose, let

$$\Pi = \mathbb{K}\left[x_1, \dots, x_d\right]$$

denote the ring of polynomials over the infinite field $\mathbb{K}$. Let Γ be a monoid (i.e., a semigroup with neutral element) and let "$<$" be a total order on Γ. The monoid Γ is called a *grading monoid* for Π if there is a direct sum decomposition

$$\Pi = \bigoplus_{\gamma \in \Gamma} \Pi_\gamma$$

such that each Π_γ is an abelian group with respect to addition and that

$$\Pi_\gamma \Pi_{\gamma'} \subset \Pi_{\gamma+\gamma'}.$$

Note that this implies that $\mathbb{K} \subset \Pi_0$. Indeed, suppose that there exist $0 \neq K \in \mathbb{K}$ and $0 < \gamma \in \Gamma$ such that $K \in \Pi_\gamma$. Then $1/K \in \Pi_{\gamma'}$ for some $\gamma' \in \Gamma$ and, consequently, we have $1 = K \cdot 1/K \in \Pi_\kappa$, where $\kappa = \gamma + \gamma' \geq \gamma > 0$. But this also yields that $1 \in \Pi_{\kappa+\kappa}$ and since $\kappa + \kappa > \kappa$ this contradicts the fact that the homogeneous spaces form a direct sum decomposition.

The canonical examples for grading monoids are $\mathbb{N}_0$ and grading by total degree, i.e.,

$$\Pi_k = \left\{ \sum_{|\alpha|=n} c_\alpha x^\alpha : c_\alpha \in \mathbb{K} \right\}$$

and $\mathbb{N}_0^d$ where

$$\Pi_\alpha = \{ c x^\alpha : c \in \mathbb{K} \}.$$

One can decompose any polynomial $p \in \Pi$ into its *homogeneous terms* p_γ, $\gamma \in \Gamma$, writing it as

$$p = \sum_{\gamma \in \Gamma} p_\gamma,$$

where only finitely many terms of the above sum are not zero. A well–ordering "$<$" on Γ naturally determines the notion of the *degree* $\delta : \Pi \to \Gamma$ which is defined for a polynomial $p \in \Pi$ as

$$\delta(p) = \max_{<} \{ \gamma : p_\gamma \neq 0 \}.$$

The *leading term* $\Lambda_\Gamma(p)$ of a polynomial p is its maximal homogeneous component; in other words,

$$\Lambda_\Gamma(p) = p_{\delta(p)}.$$

For any set of polynomials $\mathcal{P} \subset \Pi$, the ideal $\langle \mathcal{P} \rangle$ generated by $\mathcal{P}$ is

$$\langle \mathcal{P} \rangle = \left\{ \sum_{g \in \mathcal{P}} q_g g : q_g \in \Pi \right\}.$$

If $\mathcal{I} \subset \Pi$ is any polynomial ideal, then Hilbert's Basissatz tells us that there always exists a finite basis $\mathcal{B}_\mathcal{I} \subset \Pi$ such that $I = \langle \mathcal{B}_\mathcal{I} \rangle$. However, often one is interested in ideal bases which have additional desirable properties and which are also computationally effective. The probably best–known ones are the H–bases and the Gröbner bases which are, in this notation, described by the requirement that any polynomial $p \in \langle \mathcal{G} \rangle$ can be written as

$$p = \sum_{g \in \mathcal{G}} q_g g, \qquad \delta(p) \geq \delta\left(q_g g\right), \, g \in \mathcal{G}, \tag{1}$$

where Γ is either $\mathbb{N}_0$ together with ordering by total degree (in the case of H–bases) or $\mathbb{N}_0^d$ together with a term order (in the case of Gröbner bases). It is exactly the type of basis, characterized by (1), which we want to construct in this paper for arbitrary grading monoids, but without having to refine the grading to a term order. This type of basis was introduced and investigated under the name "*standard basis*" by Robbiano [11] for even more abstract and axiomatically defined graded structures on commutative rings. For our purpose here which deals with the ring of polynomials, we make the following formal definition.

Definition 1 *A (finite) set $\mathcal{G} \subset \Pi$ is called a* Γ–basis *(for the ideal $\langle \mathcal{G} \rangle$), if any $p \in \langle \mathcal{G} \rangle$ can be written as*

$$p = \sum_{g \in \mathcal{G}} q_g g, \qquad \delta(p) \geq \delta\left(q_g g\right), \, g \in \mathcal{G}. \tag{2}$$

Since any ideal has a finite basis and since a Γ–basis is, in particular, a basis for the ideal generated by its member polynomials, we can always assume that a Γ–basis is a finite set. Moreover, we will simply say that "$\mathcal{G}$ is a Γ–basis" instead of "$\mathcal{G}$ is a Γ–basis for $\langle \mathcal{G} \rangle$".

We also recall that an equivalent definition for a Γ–basis would be the requirement that

$$\{\Lambda_\Gamma(p) : p \in \langle \mathcal{G} \rangle\} = \langle \Lambda_\Gamma(g) : g \in \mathcal{G} \rangle. \tag{3}$$

2 A reduction algorithm

In order to formulate the reduction algorithm, we first have to introduce some more notation. Throughout this section, let $\mathcal{P}$ denote a finite set of polynomials and write $\#\mathcal{P}$ for its cardinality. When writing $(\mathcal{P})$ we want to view $\mathcal{P}$ as an *ordered* set in the sense that there exists an increasing chain of subsets

$$\emptyset = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_{\#\mathcal{P}} = \mathcal{P}, \qquad \#\mathcal{P}_j = j, \, j = 1, \dots, \#\mathcal{P},$$

where the order is arbitrary but fixed.

Also, let $(\cdot, \cdot) : \Pi \times \Pi \to \mathbb{K}$ be the scalar product (i.e., the positive definite bilinear form) given by

$$(p, q) = (p(D)q)\,(0) = \sum_{\alpha \in \mathbb{N}_0^d} \frac{p_\alpha q_\alpha}{\alpha!}, \tag{4}$$

provided that

$$p(x) = \sum_{\alpha \in \mathbb{N}_0^d} p_\alpha \frac{x^\alpha}{\alpha!} \qquad \text{and} \qquad q(x) = \sum_{\alpha \in \mathbb{N}_0^d} q_\alpha \frac{x^\alpha}{\alpha!}.$$

If $\mathbb{K} = \mathbb{C}$, one has to add complex conjugation and consider the respective sesquilinear form instead; however, we will not dwell on this explicitly. Making use of this scalar product, the Taylor expansion of a polynomial p at the origin becomes

$$p(x) = \sum_{\alpha \in \mathbb{N}_0^d} (x^\alpha, p) \, \frac{x^\alpha}{\alpha!}. \tag{5}$$

Though at various points in this paper, in particular in this section, an arbitrary scalar product may be admissible, we want to restrict ourselves to the above "standard" scalar product defined in (4).

For $\gamma \in \Gamma$ we define the homogeneous subspace

$$V_\gamma(\mathcal{P}) = \left\{ \sum_{p \in \mathcal{P}} q_p \Lambda_\Gamma(p) : q_p \in \Pi_{\gamma-\delta(p)}, \, p \in \mathcal{P} \right\} \subset \Pi_\gamma$$

with the convention that $\Pi_{\gamma-\delta(p)} = \{0\}$ if $\gamma - \delta(p) \notin \Gamma$. Using the above order on $\mathcal{P}$ and the Hilbert space structure which the scalar product $(\cdot,\cdot)$ defines on Π, we obtain an orthogonal decomposition of $V_\gamma(\mathcal{P})$ as

$$V_\gamma(\mathcal{P}) = \bigoplus_{j=1}^{\#\mathcal{P}} W_\gamma(\mathcal{P}_j),$$

where

$$W_\gamma(\mathcal{P}_j) = V_\gamma(\mathcal{P}_j) \ominus V_\gamma(\mathcal{P}_{j-1}), \qquad j = 1, \ldots, \#\mathcal{P},$$

i.e.,

$$(W_\gamma(\mathcal{P}_j), V_\gamma(\mathcal{P}_{j-1})) = 0.$$

The goal of the reduction algorithm is to decompose a given $f \in \Pi$ into

$$f = \sum_{p \in \mathcal{P}} q_p p + r, \qquad \delta(p) \geq \delta(q_p p), \, p \in \mathcal{P},$$

where the remainder $r \in \Pi$ should be in a "normalized" or "reduced" form. In the well–known context of Gröbner bases ($\Gamma = \mathbb{N}_0^d$) this means that none of the (monomial) leading terms of $\mathcal{P}$ divides any (monomial) term of r. However, when working with the grading by total degree, for example, the above requirement has to be weakened. It will turn out that orthogonality of any homogeneous term of r yields the "proper" generalization in the sense that the respective reduction process leads to an "algorithmic" characterization of Γ–bases.

Algorithm 2 **Given:** *$f \in \Pi$ and $\mathcal{P} \subset \Pi$.*

While $f \neq 0$:

1. *Set $\gamma = \delta(f)$.*
2. *For $j = 1, \ldots, \#\mathcal{P}$:*
   - (Orthogonal projections) *Find*

$$q_j^\gamma \in W_\gamma(\mathcal{P}_j), \qquad q_j^\gamma = \sum_{p \in \mathcal{P}_j} q_{j,p}^\gamma \Lambda_\Gamma(p), \tag{6}$$

such that

$$\left(\Lambda_\Gamma(f) - \sum_{k=1}^{j} q_k^\gamma, W_\gamma(\mathcal{P}_j) \right) = 0. \tag{7}$$

3. Set

$$r_\gamma := \Lambda_\Gamma(f) - \sum_{j=1}^{\#\mathcal{P}} q_j^\gamma. \tag{8}$$

4. Set

$$f := f - \sum_{p\in\mathcal{P}} \left(\sum_{\{j:p\in\mathcal{P}_j\}} q_{j,p}^\gamma \right) p - r_\gamma. \tag{9}$$

Result: *Decomposition*

$$f = \sum_{p\in\mathcal{P}} \left(\sum_{\gamma\in\Gamma} \sum_{\{j:p\in\mathcal{P}_j\}} q_{j,p}^\gamma \right) p + r, \tag{10}$$

where

$$(r_\gamma, V_\gamma(\mathcal{P})) = 0, \qquad \gamma \in \Gamma. \tag{11}$$

Motivated by equation (11) we call a polynomial $q \in \Pi$ *reduced* or *in normal form* with respect to a finite set $\mathcal{P} \subset \Pi$ if all the homogeneous terms of q are perpendicular to the respective $V_\gamma(\mathcal{P})$, i.e., if

$$(q_\gamma, V_\gamma(\mathcal{P})) = 0, \qquad \gamma \in \Gamma, \qquad q = \sum_{\gamma\in\Gamma} q_\gamma.$$

Note, however, that this notion of being reduced depends on the underlying scalar product. Nevertheless, the question whether a polynomial p is reduced with respect to $\mathcal{P}$ does *not* depend on an ordering of $\mathcal{P}$.

Proposition 3 *Algorithm (2) finishes after finitely many steps and the remainder polynomial r satisfies (11).*

Proof: We first remark that the algorithm is well–determined since the orthogonal projection $q_j^\gamma \in W_\gamma(\mathcal{P}_j)$ is unique. For the termination of the algorithm, we only have to notice that

$$r_\gamma = \Lambda_\Gamma(f) - \sum_{p\in\mathcal{P}} \sum_{\{j:p\in\mathcal{P}_j\}} q_{j,p} \Lambda_\Gamma(p),$$

hence the terms of degree γ on the right hand side of (9) are

$$\Lambda(f) - \sum_{p\in\mathcal{P}} \sum_{\{j:p\in\mathcal{P}_j\}} q_{j,p} \Lambda_\Gamma(p) - r_\gamma = 0,$$

which shows that the degree of f is strictly reduced in each step. Therefore, the algorithm terminates after a finite number of steps.

For the second claim, it is easily observed by induction that for $j = 1, \ldots, \#\mathcal{P}$ we have

$$\left(\Lambda_\Gamma(f) - \sum_{k=1}^{j} q_k, V_\gamma(\mathcal{P}_j) \right) = 0.$$

Indeed, for $j = 1$ this is exactly the requirement in the construction of q_1^γ, while for $j > 1$ the induction hypothesis and

$$q_j \in W_\gamma(\mathcal{P}_j) \perp V_\gamma(\mathcal{P}_{j-1})$$

yield

$$\left(\Lambda_\Gamma(f) - \sum_{k=1}^{j} q_k, V_\gamma(\mathcal{P}_{j-1}) \right) = 0.$$

Equation (7) and

$$V_\gamma(\mathcal{P}_j) = V_\gamma(\mathcal{P}_{j-1}) \oplus W_\gamma(\mathcal{P}_j)$$

finally advance the induction hypothesis. □

3 Reduction and Γ–bases

The first result shows that reduction with respect to a Γ–basis has a more "deterministic" outcome than reduction by a general finite set of polynomials.

Theorem 4 *Let $\mathcal{G}$ be a Γ–basis and suppose that $p \in \Pi$ can be written as*

$$p = \sum_{g \in \mathcal{G}} q_g g + r,$$

where r is reduced with respect to $\mathcal{G}$. Then

$$r = p_{\underset{(\mathcal{G})}{\longrightarrow}}$$

Remark 5 *In particular, the above theorem says that for Γ–bases the remainder of the reduction algorithm does* not *depend on the ordering we impose on $\mathcal{G}$. In this case we will simply write $\underset{\mathcal{G}}{\rightarrow}$.*

Proof of Theorem (4): Let

$$p = \sum_{g \in \mathcal{G}} \tilde{q}_g g + \tilde{r}, \qquad \tilde{q}_g = \sum_{\gamma \in \Gamma} q_{\gamma,g}, \quad \tilde{r} = p_{\underset{(\mathcal{G})}{\longrightarrow}},$$

be the decomposition obtained by Algorithm (2). Then

$$r - \tilde{r} = \sum_{g \in \mathcal{G}} (\tilde{q}_g - q_g)\, g \in \langle \mathcal{G} \rangle.$$

Now suppose that $q := r - \tilde{r} \neq 0$. Since each homogeneous term of r and $\tilde{r}$ of any degree γ is orthogonal to the respective $V_\gamma(\mathcal{G})$, the same holds true for q_γ, $\gamma \in \Gamma$, and, in particular,

$$\left(\Lambda_\Gamma(q), V_{\delta(q)}(\mathcal{G}) \right) = 0. \tag{12}$$

On the other hand, since $q \in \langle \mathcal{G} \rangle$ and since $\mathcal{G}$ is a Γ–basis, we also conclude from (3) that

$$\Lambda_\Gamma(q) \in \langle \Lambda_\Gamma(\mathcal{G}) \rangle \cap \Pi_{\delta(q)} = V_{\delta(q)}(\mathcal{G}). \tag{13}$$

But (12) and (13) are contradictory if $q \neq 0$, hence we must have $q = 0$ or $r = \tilde{r} = p_{\underset{(\mathcal{G})}{\rightarrow}}$. $\square$

Next, let us recall that, for a finite set $\mathcal{P} \subset \Pi$ of polynomials, a *syzygy* for $\mathcal{P}$ is a vector of polynomials $\boldsymbol{q} \in \Pi^{\mathcal{P}}$ such that

$$\boldsymbol{q} \cdot \mathcal{P} = \sum_{p \in \mathcal{P}} q_p p = 0.$$

The set of all syzygies for $\mathcal{P}$ forms a module, denoted by $S(\mathcal{P})$. It is well–known (cf. [8]) that this module is finitely generated, i.e., there exists a finite basis $\mathcal{S} \subset S(\mathcal{P})$ such that any syzygy $\boldsymbol{q} \in S(\mathcal{P})$ can be written as

$$\boldsymbol{q} = \sum_{\boldsymbol{s} \in S(\mathcal{P})} q_{\boldsymbol{s}} \boldsymbol{s}, \qquad q_{\boldsymbol{s}} \in \Pi,\ \boldsymbol{s} \in S(\mathcal{P}).$$

We also remark that such a basis can be constructed explicitly making use of a reduced Gröbner basis of $\langle \mathcal{P} \rangle$. This has been pointed out by Buchberger in Method 6.17 of his survey paper [5].

Now, there is a Γ–bases analogue of the classical characterization of Gröbner bases via the reduction of the syzygies of leading terms. This result reads as follows.

Theorem 6 *Let $\mathcal{G} \subset \Pi$ be a finite set of polynomials and let $\mathcal{S}$ be a basis of $S(\Lambda_\Gamma(\mathcal{G}))$. Then $\mathcal{G}$ is a Γ–basis if and only if*

$$\boldsymbol{s} \cdot \mathcal{G}_{\underset{(\mathcal{G})}{\rightarrow}} = 0, \qquad \boldsymbol{s} \in S(\Lambda_\Gamma(\mathcal{G})). \tag{14}$$

Proof: Since $p_{\underset{(\mathcal{G})}{\rightarrow}}$ is unique for a Γ–basis $\mathcal{G}$ by Theorem (4) and since $\boldsymbol{s} \cdot \mathcal{G} \in \langle \mathcal{G} \rangle$, the direction "$\Rightarrow$" is clear.

The proof of "$\Leftarrow$" follows the argumentation in [10]. Pick any $p \in \langle \mathcal{G} \rangle$ which can be written as

$$p = \sum_{g \in \mathcal{G}} p_g g. \tag{15}$$

We have to show that the polynomials p_g, $g \in \mathcal{G}$, in (15) can be chosen such that $\delta(p) \geq \delta(p_g g)$, $g \in \mathcal{G}$. Assume that this is not the case in equation (15) and set

$$\gamma = \max_{<} \{\delta(p_g g) : g \in \mathcal{G}\},$$

then there is a nonempty subset $\mathcal{G}' \subset \mathcal{G}$ such that

$$\mathcal{G}' = \{g \in \mathcal{G} : \delta(p_g g) = \gamma\} \qquad \text{and} \qquad \delta(p_g g) > \delta(p),\ g \in \mathcal{G}'.$$

Consequently, the leading terms of these polynomials, which belong to Π_γ, have to cancel, i.e.

$$\sum_{g\in\mathcal{G}'} \Lambda_\Gamma(p_g g) = \sum_{g\in\mathcal{G}'} \Lambda_\Gamma(p_g)\,\Lambda_\Gamma(g) = 0$$

and therefore

$$\boldsymbol{q} = (q_g : g \in \mathcal{G})\,, \qquad q_g = \begin{cases} \Lambda_\Gamma(p_g) & g \in \mathcal{G}', \\ 0 & g \in \mathcal{G}\setminus\mathcal{G}', \end{cases}$$

belongs to $S\left(\Lambda_\Gamma(\mathcal{G})\right)$. By assumption,

$$\boldsymbol{q}\cdot\mathcal{G}\underset{(\mathcal{G})}{\rightarrow} = \sum_{\boldsymbol{s}\in\mathcal{S}} q_{\boldsymbol{s}}\,(\boldsymbol{s}\cdot\mathcal{G})\underset{(\mathcal{G})}{\rightarrow} = 0,$$

hence, there exist polynomials $\tilde{p}_g \in \Pi$, $g \in \mathcal{G}$, such that

$$\sum_{g\in\mathcal{G}'} \Lambda_\Gamma(p_g)\,g = \boldsymbol{q}\cdot\mathcal{G} = \sum_{g\in\mathcal{G}} \tilde{p}_g g,$$

where $\delta(p_g g) < \gamma$, since $\delta(\boldsymbol{q}\cdot\mathcal{G}) < \gamma$. This yields that

$$p = \sum_{g\in\mathcal{G}'} \left(p_g - \Lambda_\Gamma(p_g) + \tilde{p}_g\right) g + \sum_{g\in\mathcal{G}\setminus\mathcal{G}'} \left(p_g + \tilde{p}_g\right) g = \sum_{g\in\mathcal{G}} \widehat{p}_g\, g,$$

which is again a representation of the form (15), but now with the property that $\delta(\widehat{p}_g\, g) < \gamma$, $g \in \mathcal{G}$. Repeating this process, we arrive, after finitely many steps, at a "minimal" representation of the form (15), where $\delta(p_g g) \le \delta(p)$, $g \in \mathcal{G}$, which shows that $\mathcal{G}$ is a Γ–basis. □

This allows us to finally formulate a crude version of Buchbergers algorithm for the computation of a Γ–basis for the ideal $\langle\mathcal{P}\rangle$, where $\mathcal{P}$ is any finite set of polynomials.

Algorithm 7 **Given:** *Finite set* $\mathcal{P} \subset \Pi$.

1. *Set* $\mathcal{G} = \mathcal{P}$ *and* $\mathcal{G}' = \emptyset$.
2. *While* $\mathcal{G}' \neq \mathcal{G}$:
   (a) *Set* $\mathcal{G}' = \mathcal{G}$.
   (b) *Compute a basis* $\mathcal{S}$ *of* $S\left(\Lambda_\Gamma(\mathcal{G})\right)$.
   (c) *For* $\boldsymbol{s} \in \mathcal{S}$:
      i. *Compute* $h = \boldsymbol{s}\cdot\mathcal{G}'\underset{(\mathcal{G}')}{\rightarrow}$.
      ii. *If* $h \neq 0$ *then set* $\mathcal{G} = \mathcal{G} \cup \{h\}$.

Result: Γ–*basis* $\mathcal{G}$.

Theorem 8 *Algorithm (7) generates a* Γ–*basis after finitely many steps.*

Proof: The argument is identical with the one for termination of Buchberger's algorithm for Gröbner bases, cf. [6]. Let $\mathcal{G}_k$, $k \in \mathbb{N}_0$, denote the set $\mathcal{G}$ after the kth step of the algorithm. Then $\mathcal{G}_k \subset \langle \mathcal{P} \rangle$, $k \in \mathbb{N}_0$, and, by construction, as long as $\mathcal{G}_k \subset \mathcal{G}_{k+1}$ is a strict inclusion, then the inclusion of homogeneous ideals $\langle \Lambda_\Gamma(\mathcal{G}_k) \rangle \subset \langle \Lambda_\Gamma(\mathcal{G}_k) \rangle$ is also a strict one. However, after finitely many steps the sequence of homogeneous ideal $\langle \Lambda_\Gamma(\mathcal{G}_k) \rangle$ must stabilise which means that there exists $k_0 \in \mathbb{N}$ such that $\mathcal{G}_k = \mathcal{G}_{k+1}$ for all $k \geq k_0$. But then Γ_{k_0} is a Γ–basis since all syzygies reduce to zero. $\square$

4 Least interpolation

In this section we will connect the technique of Γ–bases to multivariate polynomial interpolation of (Γ–) minimal degree and of Γ–least interpolation which extends and generalizes the approach from [12, 13]. To clarify these notions, we have to introduce some more terminology.

Let $\Theta \subset \Pi'$ be a finite set of linearly independent linear functionals defined on Π. Following a terminology of G. Birkhoff [1], we say that Θ *admits an ideal interpolation scheme* if

$$\ker \Theta = \{p : \Theta(p) = 0\} \subset \Pi$$

is an ideal in Π.

It is well–known that $\theta \in \Pi'$ can be identified with a formal power series $f_\theta \in \mathbb{K}[\![x_1, \dots, x_d]\!]$ by the assignment

$$\theta(p) = (p, f_\theta)\,;$$

clearly, the scalar product $(\cdot, \cdot)$ can be extended to $\Pi \times \mathbb{K}[\![x_1, \dots, x_d]\!]$ since the sum of coefficients still runs over a finite index set only. For example, to the point evaluation functional $\theta = \delta_x$ the power series of $f_\theta(y) = e^{x \cdot y}$ is associated. The following characterization of ideal interpolation schemes has been given by de Boor and Ron [3].

Theorem 9 *A finite set Θ of linear functionals admits an ideal interpolation scheme if and only if the subspace*

$$f_\Theta = span_{\mathbb{K}}\{f_\theta : \theta \in \Theta\} \subset \mathbb{K}[\![x_1, \dots, x_d]\!]$$

is closed under formal differentiation.

Assume that Θ admits an ideal interpolation scheme. We say that a linear subspace $\mathcal{P} \subset \Pi$ is an *interpolation space* with respect to Θ if for any $y \in \mathbb{K}^\Theta$ there exists a *unique* polynomial $p \in \mathcal{P}$ such that

$$\Theta(p) = y.$$

If $\mathcal{P}$ is an interpolation space with respect to Θ, then we denote by

$$L(\mathcal{P};\cdot):\mathbb{K}^{\Theta}\to\mathcal{P}$$

the *interpolation operator* (which is a clearly linear operator). Moreover, $L(\mathcal{P};\Theta(\cdot)):\Pi\to\mathcal{P}$ is a linear projection from $\Pi\to\mathcal{P}$. We say that $\mathcal{P}$ is a Γ*–minimal degree interpolation space* with respect to Θ if $\mathcal{P}$ is an interpolation space and the projection $L(\mathcal{P};\Theta(\cdot))$ is *degree reducing*, i.e.,

$$\delta\left(L\left(\mathcal{P};\Theta(p)\right)\right)\leq\delta(p),\qquad p\in\Pi,$$

which is a desirable behaviour of polynomial projections.

The following result tells us that the normal forms (or, reduced polynomials) with respect to a Γ–basis $\mathcal{G}$ for $\ker\Theta$ are always a canonical minimal degree interpolation space with respect to Θ.

Theorem 10 *Suppose that $\Theta\subset\Pi'$ admits an ideal interpolation scheme and let $\mathcal{G}$ be a Γ–basis of* $\ker\Theta$. *Then $\mathcal{P}_\Theta=\Pi_{\underset{\mathcal{G}}{\rightarrow}}$ is a Γ–minimal degree interpolation space with respect to Θ and*

$$L(\mathcal{P}_\Theta;\Theta(p))=p_{\underset{\mathcal{G}}{\rightarrow}}.$$

Proof: Since the functionals in Θ are linearly independent, there exist *dual polynomials* $p_\Theta\in\Pi^\Theta$ such that

$$\theta\left(p_{\theta'}\right)=\delta_{\theta,\theta'},\qquad\theta,\theta'\in\Theta.$$

Since $\Theta\left(p-p_{\underset{\mathcal{G}}{\rightarrow}}\right)=0$ for any $p\in\Pi$, the polynomials $p_{\Theta\underset{\mathcal{G}}{\rightarrow}}$ are also dual to Θ and therefore also linearly independent. Consequently, for any data $y\in\mathbb{K}^\Theta$, the polynomial

$$p_y=\sum_{\theta\in\Theta}y_\theta p_{\theta\underset{\mathcal{G}}{\rightarrow}}\in\Pi_{\underset{\mathcal{G}}{\rightarrow}}$$

satisfies $\Theta(p_y)=y$. In addition, all polynomials $p\in\Pi$ which have the property that $\Theta(p)=y$, differ by an element of $\langle\mathcal{G}\rangle$ and therefore the p_y above is the *unique* (because $\mathcal{G}$ is a Γ–basis) reduced interpolant which proves that $\Pi_{\underset{\mathcal{G}}{\rightarrow}}$ is an interpolation space where the interpolation operator is given by reduction. Since the reduction process is also degree–reducing, we finally find that $\Pi_{\underset{\mathcal{G}}{\rightarrow}}$ is a Γ–minimal degree interpolation space. $\square$

For a power series $f\in\mathbb{K}[\![x_1,\dots,x_d]\!]$ we denote its Γ*–least term* by

$$\lambda_\Gamma(f)=\min_\gamma\left\{f_\gamma:f_\gamma\neq0\right\},$$

in other words, its homogeneous term of Γ–minimal degree. The vector space of the least terms of a subspace F of $\mathbb{K}[\![x_1,\dots,x_d]\!]$ will be denoted by $\lambda_\Gamma(F)$. The *least space* related to Θ, $\lambda_\Gamma\left(f_\Theta\right)$, has been discovered as very useful in interpolation by de Boor and Ron [2] for the case of grading by total degree.

Finally, we assume that the grading induced by Γ is a *monomial grading* which requires that for $\gamma \in \Gamma$

$$\Pi_\gamma = \operatorname{span}_{\mathbb{K}} \{x^\alpha : \alpha \in I_\gamma\}, \qquad I_\gamma \subset \mathbb{N}_0^d,\ \gamma \in \Gamma.$$

In particular, each monomial x^α, $\alpha \in \mathbb{N}_0^d$, belongs to some homogeneous space Π_γ; we denote the respective index γ by $\gamma(\alpha)$.
Then we have the following result.

Theorem 11 *Suppose that $\Theta \subset \Pi'$ admits an ideal interpolation scheme and let $\mathcal{G}$ be a Γ–basis of* $\ker \Theta$. *Then*

$$\Pi_{\overrightarrow{\mathcal{G}}} = \lambda_\Gamma (f_\Theta).$$

Corollary 12 *If $\Theta \subset \Pi'$ admits an ideal interpolation scheme and Γ is a monomial grading, then $\lambda_\Gamma(f_\Theta)$ is a Γ–minimal degree interpolation space.*

For the prove of the theorem we begin with collecting some auxiliary results. First we remark that it follows directly from (4) that for $f, g, p \in \Pi$ one has

$$(f, p(D)g) = (fp, g). \tag{16}$$

Next, we give a simple observation on monomial gradings.

Lemma 13 *Suppose that Γ induces a monomial grading. If $p, q \in \Pi$ satisfy $\delta(p) < \delta(q)$, then $\Lambda_\Gamma(q)(D)p = 0$.*

Proof: Pick any $\beta \in \mathbb{N}_0^d$ such that $\gamma(\beta) > \delta(p)$. By (5) and (16) we obtain that

$$D^\beta p = \sum_{\alpha \in \mathbb{N}_0^d} \left(x^\alpha, D^\beta p\right) \frac{x^\alpha}{\alpha!} = \sum_{\alpha \in \mathbb{N}_0^d} \left(x^{\alpha+\beta}, p\right) \frac{x^\alpha}{\alpha!}.$$

On the other hand, equation (5) yields that $\left(x^{\alpha+\beta}, p\right)$ is the coefficient of p with respect to the monomial

$$\frac{x^{\alpha+\beta}}{(\alpha+\beta)!} \in \Pi_{\gamma(\alpha)+\gamma(\beta)}.$$

Since $\gamma(\alpha) + \gamma(\beta) \geq \gamma(\beta) > \delta(p)$ and since the grading yields a direct sum decomposition, this coefficient has to be zero. Since

$$\Lambda(q) \in \operatorname{span}_{\mathbb{K}} \left\{x^\alpha : \alpha \in I_{\delta(q)}\right\},$$

the result follows. $\square$

To prove the theorem, we make use of the following additional characterization of reduced polynomials with respect to a Γ–basis $\mathcal{G}$.

Proposition 14 *Let $\mathcal{G}$ be a Γ–basis for $\langle \mathcal{G} \rangle$ and let Γ induce a monomial grading. Then a polynomial $p \in \Pi$ is reduced with respect to $\mathcal{G}$ if and only if*

$$p \in \bigcap_{g \in \mathcal{G}} \ker \Lambda_\Gamma(g)(D). \tag{17}$$

Proof: We first remark that

$$\bigcap_{g \in \mathcal{G}} \ker \Lambda_\Gamma(g)(D) = \bigcap_{g \in \langle \mathcal{G} \rangle} \ker \Lambda_\Gamma(g)(D). \tag{18}$$

Indeed, the inclusion "$\supset$" is trivial since $\mathcal{G} \subset \langle \mathcal{G} \rangle$. For "$\subset$" we pick any $q \in \langle \mathcal{G} \rangle$, and write it as $q = \sum_g q_g g$. Since $\mathcal{G}$ is a Γ–basis for $\langle \mathcal{G} \rangle$, we know that $\delta(q_g g) \leq \delta(q)$ and defining the subset $\mathcal{G}' \subset \mathcal{G}$ as

$$\mathcal{G}' = \{ g \in \mathcal{G} : \delta(q_g g) = \delta(q) \}$$

we have that

$$\Lambda_\Gamma(q) = \Lambda_\Gamma \left(\sum_{g \in \mathcal{G}'} q_g g \right) = \sum_{g \in \mathcal{G}'} \Lambda_\Gamma(q_g) \Lambda_\Gamma(g),$$

hence,

$$\Lambda_\Gamma(q)(D)p = \sum_{g \in \mathcal{G}'} \Lambda_\Gamma(q_g)(D) \underbrace{(\Lambda_\Gamma(g)(D)p)}_{=0} = 0.$$

Now, pick any *homogeneous* polynomial $p \in \Pi_\gamma$ for some $\gamma \in \Gamma$ and $q \in \langle \mathcal{G} \rangle$. If $\delta(p) < \delta(q)$, then $\Lambda_\Gamma(q)(D)p = 0$ by Lemma (13). If, on the other hand, $\delta(p) \geq \delta(q)$, then

$$\Lambda_\Gamma(q)(D)p = \sum_{\alpha) \in \mathbb{N}_0^d} (x^\alpha, \Lambda_\Gamma(q)(D)p) \frac{x^\alpha}{\alpha!} = \sum_{\{\alpha : \delta(q) + \gamma(\alpha) = \gamma\}} (x^\alpha \Lambda_\Gamma(q), p) \frac{x^\alpha}{\alpha!}.$$

Hence, $\Lambda_\Gamma(q)(D)p = 0$ holds if and only if

$$(x^\alpha \Lambda_\Gamma(q), p), \qquad \gamma(\alpha) + \delta(q) = \gamma.$$

However,

$$V_\gamma(\mathcal{G}) = \bigoplus_{g \in \mathcal{G}} \operatorname{span}_{\mathbb{K}} \{ x^\alpha g : \gamma(\alpha) + \delta(g) = \gamma \}$$

and therefore $\Lambda_\Gamma(q)(D)p = 0$, $q \in \langle \mathcal{G} \rangle$ is equivalent to

$$(p, V_\gamma(\mathcal{G})) = 0.$$

This immediately yields the statement of the proposition. $\square$

Hence, the proof of Proposition (14) also yields the following description of the joint kernels of homogeneous differential operators, cf. [7].

Corollary 15 *Suppose that Θ admits an ideal interpolation scheme and let $\mathcal{G}$ be a Γ–basis of* $\ker \Theta$. *Then*

$$\Pi_{\overrightarrow{\mathcal{G}}} = \bigcap_{g \in \mathcal{G}} \ker \Lambda_\Gamma(g)(D)$$

Another immediate consequence is the following "algorithmic" description of the joint kernels of homogeneous differential operators with constant coefficients.

Corollary 16 *Let $\mathcal{P} \subset \Pi$ be a finite set of homogeneous polynomials and let $\mathcal{G}$ be a Γ–basis for $\langle \mathcal{P} \rangle$. Then*

$$\bigcap_{p\in\mathcal{P}} \ker p(D) = \Pi_{\overrightarrow{\mathcal{G}}}.$$

Proof of Theorem (11): By Corollary (15) it suffices to prove that

$$\bigcap_{g\in\langle\mathcal{G}\rangle} \ker \Lambda_\Gamma(g)(D) = \lambda_\Gamma\left(f_\Theta\right). \tag{19}$$

To prove the inclusion "$\supset$", we assume that there exists some $f \in \mathbb{K}[\![x_1,\dots,x_d]\!]$, $f = \sum_\theta c_\theta f_\theta$, and $q \in \langle\mathcal{G}\rangle = \ker\, Theta$ such that

$$\Lambda_\Gamma(q)(D)\lambda_\Gamma(f) \neq 0.$$

Hence, $\delta(q) = \delta\left(\lambda_\Gamma(f)\right)$ and therefore, by Lemma (13) we have that

$$\Lambda_\Gamma(q)(D)\lambda_\Gamma(f) = \left(q(D)f\right)(0) = (f,q) = \sum_{\theta\in\Theta} c_\theta\left(f_\theta,q\right) = \sum_{\theta\in\Theta} c_\theta\theta(q) = 0$$

since $q \in \ker\,\Theta$, which is a contradiction.

Conversely, since the functionals in Θ were assumed to be linearly independent and therefore

$$\lambda_\Gamma\left(\sum_{\theta\in\Theta} c_\theta f_\theta\right) = 0 \quad\Leftrightarrow\quad \sum_{\theta\in\Theta} c_\theta f_\theta = 0 \quad\Leftrightarrow\quad c_\theta = 0,\ \theta\in\Theta,$$

we conclude that

$$\dim \lambda_\Gamma(f_\Theta) = \dim f_\Theta = \dim \operatorname{span}_{\mathbb{K}} \Theta = \#\Theta = \dim \mathcal{P}_\Theta.$$

Hence, $\lambda_\Gamma\left(f_\Theta\right)$ is a linear subspace of $\mathcal{P}_\Theta$ which has the same dimension. Consequently, the spaces are the same. $\square$

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