Monografías de la Academia de Ciencias de Zaragoza. **20:** 87–96, (2002).

Some Properties of Linear Positive Operators Defined in Terms of Finite Differences *

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Abstract

In this paper, we study some properties of Mastroianni operators [4] and generalized Baskakov operators [3].

1 Introduction and Notation

In [4] Mastroianni introduced and studied a generalization of the classical Bernstein operators consisting in replacing the functions $(1-x)^{n-k}$ by more general ones satisfying suitable relations. His work was motivated by the development of a general expression that cover other Bernstein-type operators.

In this paper we study some properties of these Mastroianni operators. We obtain some recursive properties of the derivatives of the operators, that allow us to give a characterization of the Szász operators. Also, we consider the linear combination of iterates $I-(I-L_n)^m$ of Mastroianni operators of fixed degree n for increasing order of iteration m and prove that these Boolean sums have a good behaviour for polynomials.

In the same manner, we consider a generalization of the Baskakov operators [3] which are related to certain functions. We study the convergence properties of the sequence of these operators and give an asymptotic expansion for them.

We will use Stirling numbers, S_j^i y σ_j^i, of first and second kind defined, respectively, by: $x^{\underline{j}} = \sum_{i=0}^{j} S_j^i x^i$ and $x^j = \sum_{i=0}^{j} \sigma_j^i x^{\underline{i}}$, with $j \in \mathbb{N}_0$. Here $x^{\underline{j}} = x(x-1)\ldots(x-j+1)$ if $j>0$ and $x^{\underline{0}} = 1$.

*This work is partially supported by Junta de Andalucía, Research Groups FQM178 and FQM268 and by Ministerio de Ciencia y Tecnología, Project BFM2000-0911.

Throughout this paper we also use the following notation:

$$E_2 = \{f \in C[0, +\infty) : \frac{f(x)}{1+x^2} \text{ is convergent as } x \to +\infty\},$$

$\mathbb{P}_r$ denotes the space of real polynomials of degree at most r and t^r is the monomial $t^r(x) = x^r$. For convenience, we define $\sum_{i=1}^{0} = \sum_{i=0}^{-1} = 1$.

Here we simply recall the Mastroianni operators. First we start with a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of real functions on $I = [0, \infty)$ which are infinitely differentiable and strictly monotone satisfying the following additional conditions:

A1) $\phi_n(0) = 1$, for every $n \in \mathbb{N} = \{1, 2, ...\}$.

A2) $(-1)^i \phi_n^{(i)}(x) \geq 0$, for every $n \in \mathbb{N}$, $x \in I$ and $i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

A3) For every $(n, i) \in \mathbb{N} \times \mathbb{N}_0$ there exists a positive integer $p(n, i) \in \mathbb{N}$ and a real function $\alpha_{n,i} : I \to \mathbb{R}$ such that

$$\phi_n^{(i+k)}(x) = (-1)^i \phi_{p(n,i)}^{(k)}(x) \alpha_{n,i}(x),$$

for every $k \in \mathbb{N}_0$ and $x \in I$ and

$$\lim_{n\to\infty} \frac{n}{p(n,i)} = \lim_{n\to\infty} \frac{\alpha_{n,i}(x)}{n^i} = 1.$$

For short, we will denote $\alpha_{n,i} = \alpha_{n,i}(0)$.

By **A3)**, $(-1)^{i+k}\phi_n^{(i+k)}(x) = (-1)^k \phi_{p(n,i)}^{(k)}(x)\alpha_{n,i}(x)$ and by **A2)** we conclude that necessarily $\alpha_{n,i}(x) \geq 0$.

Also from **A3)**, we deduce that $(-1)^i \phi_n^{(i)}(x) = \phi_{p(n,i)}(x)\alpha_{n,i}(x)$, and, in particular, **A1)** implies that $(-1)^i \phi_n^{(i)}(0) = \alpha_{n,i}$.

To the above sequence Mastroianni [4] associates a sequence of positive linear operators $\{L_n : E_2 \to C^\infty(I)\}_{n\in\mathbb{N}}$ defined by

$$L_n f(x) = \sum_{k=0}^{\infty} (-1)^k f\left(\frac{k}{n}\right) x^k \frac{\phi_n^{(k)}(x)}{k!},$$

for every $f \in E_2$ and $x \in [0, \infty)$. L_n can be represented in the form

$$L_n f(x) = \sum_{i=0}^{\infty} (-1)^i \frac{\phi_n^{(i)}(0)}{i!} \Delta_{\frac{1}{n}}^i f(0)\, x^i. \quad (1)$$

2 Derivatives.

It is straightforward to check the following relation of the derivatives.

Proposition 1. For every $k \in \mathbb{N}_0$ and $x \in [0, \infty)$

$$L_n^{(k)} f(x) = \alpha_{n,k} \sum_{i=0}^{\infty} (-1)^i \frac{\phi_{p(n,k)}^{(i)}(0)}{i!} \ \Delta_{\frac{1}{n}}^{i+k} f(0) \ x^i. \tag{2}$$

In the particular case, $p(n, k) = n$, then

$$L_n^{(k)} f(x) = \alpha_{n,k} L_n(\Delta_{\frac{1}{n}}^k f)(x). \tag{3}$$

Proof.

$$L_n^{(k)} f(x) = \sum_{i=k}^{\infty} (-1)^i \frac{\phi_n^{(i)}(0)}{i!} \ \Delta_{\frac{1}{n}}^i f(0) \ \frac{i!}{(i-k)!} \ x^{i-k}$$
$$= \sum_{i=0}^{\infty} (-1)^{i+k} \frac{\phi_n^{(i+k)}(0)}{(i+k)!} \ \Delta_{\frac{1}{n}}^{i+k} f(0) \ \frac{(i+k)!}{i!} \ x^i.$$

From property **A3)** we get (2). Then (3) follows from (1) and (2). □

We will use this identity to compute the moments $L_n(t^s)$. Formula (3) is similar to $(B_n f)^{(k)}(x) = n^{\underline{k}} \ B_{n-k}(\Delta_{\frac{1}{n}}^k f)(x)$, which is valid for Bernstein polynomials. And it is also similar to the identity

$$S_n^{(k)} f(x) = n^k S_n(\Delta_{\frac{1}{n}}^k f)(x),$$

valid for Szász operators. In fact, the last one is a particular case of (2) and (3).

L_n operators have classical shape preserving properties. Recall the notion of higher order convexity. Given $k \in \mathbb{N}_0$, a function f is said to be convex of order k (k-convex) if for all $h > 0$ one has that $\Delta_h^k f \geq 0$. A function $f \in C^k[0, \infty)$ is convex of order k if and only if $f^{(k)} \geq 0$.

We can use (1) to deduce some properties of L_n, such as preservation of n-convexity. In this way, the following result is immediate:

Proposition 2. For any $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ if f is convex of order k, then $L_n f$ is convex of order k.

Of course, Mastroianni's operator L_n has the degree-preserving property

$$L_n \mathbb{P}_r \subset \mathbb{P}_r, \qquad (0 \leq r \leq n).$$

3 Moments and asymptotic expansion of L_n.

Now, we give explicit expressions for the moments $L_n(t^i)$ and central moments $L_n\left((t-x)^i\right)(x)$, for $i \in \mathbb{N}$ and $x \in [0, \infty)$.

Lemma 3. *For any $n \in \mathbb{N}$, $r \in \mathbb{N}_0$, one has*

$$L_n(t^r) = n^{-r} \sum_{i=1}^{r} \alpha_{n,i}\ \sigma_r^i\ t^i. \tag{4}$$

Proof. By (1) we have

$$L_n(t^r) = \sum_{i=0}^{r} (-1)^i \frac{\phi_n^{(i)}(0)}{i!}\ \Delta_{\frac{1}{n}}^i (t^r)(0)\ t^i.$$

On the other hand, it is known that $\Delta_{\frac{1}{n}}^i t^r(0) = i!\sigma_r^i n^{-r}$, (see [1, Section 24.1.4.II.C]). Now, replacing this expression in the above formula we get

$$L_n(t^r) \ = \ \sum_{i=0}^{r} (-1)^i \frac{\phi_n^{(i)}(0)}{i!}\ i!\sigma_r^i n^{-r}\ t^i = \sum_{i=0}^{r} (-1)^i \phi_n^{(i)}(0)\ \sigma_r^i n^{-r}\ t^i.$$

Finally, properties **A3)** and **A1)** are used. □

We would like to remark that in the above Lemma in the special case $r = 0$, we have $L_n(1) = 1$.

Lemma 4. *For any $n \in \mathbb{N}$, $p \in \mathbb{N}_0$, $x \in [0, \infty)$, one has*

$$L_n\left((t-x)^p\right)(x) = n^{-p} \sum_{j=0}^{p} (-1)^j (j - nx)^p G(j, n, x),$$

where $G(j,n,x) = \sum_{i=j}^{p} \frac{\phi_n^{(i)}(0)}{i!} \binom{i}{j}\ x^i = \sum_{i=j}^{p} (-1)^i \frac{\alpha_{n,i}}{i!} \binom{i}{j}\ x^i$.

Proof. First by (1)

$$L_n\left((t-x)^p\right)(x) = \sum_{i=0}^{\infty} (-1)^i \frac{\phi_n^{(i)}(0)}{i!} \Delta_{\frac{1}{n}}^i ((t-x)^p)(0)\ x^i.$$

From the definition of Δ

$$\begin{aligned}\Delta_{\frac{1}{n}}^i ((t-x)^p)(0) &= \sum_{j=0}^{i} \binom{i}{j} (-1)^{i+j} ((t-x)^p) \left(\frac{j}{n}\right) \\ &= \sum_{j=0}^{i} \binom{i}{j} (-1)^{i+j} \left(\frac{j}{n} - x\right)^p = n^{-p} \sum_{j=0}^{i} \binom{i}{j} (-1)^{i+j} (j - nx)^p.\end{aligned}$$

It is evident that $\Delta_{\frac{1}{n}}^i ((t-x)^p)(0) = 0$ for $i \geq p$. Replacing it in the above expression we get

$$\begin{aligned}L_n\left((t-x)^p\right)(x) &= \sum_{i=0}^{p} (-1)^i \frac{\phi_n^{(i)}(0)}{i!} n^{-p} \sum_{j=0}^{i} \binom{i}{j} (-1)^{i+j} (j-nx)^p\ x^i \\ &= n^{-p} \sum_{j=0}^{p} (-1)^j (j-nx)^p \sum_{i=j}^{p} \frac{\phi_n^{(i)}(0)}{i!} \binom{i}{j}\ x^i.\end{aligned}$$

□

Now, we give the asymptotic expansion of the sequence of L_n operators.

Theorem 5. *Let $f \in E_2$ r times differentiable at $x \in [0, \infty)$, then*

$$L_n f(x) = \sum_{j=0}^{r} (-1)^j G(j, n, x) \sum_{p=j}^{r} \frac{f^{(p)}(x)}{p!} (\frac{j}{n} - x)^p + o(L_n((t - x)^r)),$$

where $G(j, n, x)$ is given as in Lemma 4.

Proof. The proof of Sikkema Theorem in [6] is valid to check that

$$L_n f(x) = \sum_{p=0}^{r} \frac{f^{(p)}(x)}{p!} L_n \left((t-x)^p\right)(x) + o(L_n((t-x)^r)), \tag{5}$$

for every $x \in [0, \infty)$. We finish the proof using Lemma 4. $\square$

4 Some Limiting Properties.

Mastroianni [4] proves that the sequence of iterations $\{I - (I - L_n)^m\}_{n \in \mathbb{N}}$ converges as m tends to infinity under certain assumptions. More precisely, he showed that

$$\lim_{m \to \infty} \left(I - (I - L_n)^m\right)(f)(x) = f(0) + \sum_{i=1}^{\infty} \frac{(nx)^{\underline{i}}}{n^i} \Delta_{\frac{1}{n}}^{i-1} f(0) \tag{6}$$

holds for all $f \in C[0, b]$ and $x \in [0, b]$ if and only if $\left|\frac{\phi_n^{(i)}(0)}{n^2}\right| < 2$ for every $i \geq 2$. In the general case, such a convergence does not always hold but we are going to see that at least we can obtain a good behavior for polynomials. For this purpose we employ a modification of the technique used by Sevy [5] for Bernstein operators.

Theorem 6. *Given $n, r \in \mathbb{N}$, let us suppose that $0 < \frac{\alpha_{n,i}}{n^i} < 2$ for every $i \in \{0, \ldots, r\}$. Then, for any $p \in \mathbb{P}_r$ we have*

$$\lim_{m \to \infty} \left(I - (I - L_n)^m\right)(p) = p$$

uniformly on compact sets.

Proof. From Lemma 3 it is straightforward that L_n is a linear map, $L_n : \mathbb{P}_r \to \mathbb{P}_r$, whose eigenvalues are $\lambda_i^{(n)} = \frac{\alpha_{n,i}}{n^i}$, $i = 0 \ldots, r$. It is also clear that for every eigenvalue, $\lambda_i^{(n)}$, we can find an eigenvector $p_i^{(n)}$ which is a polynomial with exact degree i and also that $p_0^{(n)} = 1$.

Take the operator $V = I - L_n$. It is not difficult to check that for all $m, N \in \mathbb{N}$,

$$V^{m+N} - V^m = -(I - L_n)^m \left(\sum_{i=0}^{N-1} (I - L_n)^i\right) L_n. \tag{7}$$

If $\lambda_i^{(n)} = 1$ then $V^m(p_i^{(n)}) = 0$ for every $m \in \mathbb{N}$. If $\lambda_i^{(n)} \neq 1$ from (7) we have

$$V^{m+N}(p_i^{(n)}) - V^m(p_i^{(n)}) = -(1-\lambda_i^{(n)})^m \lambda_i^{(n)} \frac{1-(\lambda_i^{(n)})^N}{1-\lambda_i^{(n)}} p_i^{(n)}.$$

Since $0 < \lambda_i^{(n)} < 2$, one has $|1-\lambda_i^{(n)}| < 1$. In both cases, as $p_i^{(n)}$ is continuous, taking into account the preceding identity we can conclude that the sequence $\{V_m(p_i^{(n)})\}_{n\in\mathbb{N}}$ satisfies the Cauchy condition on compact subsets from which we deduce that such a sequence converges towards $g \in C[0,\infty)$. Furthermore, it is immediate that $V^m(p_i^{(n)}) \in \mathbb{P}_r$ for any $m \in \mathbb{N}$ so that $g \in \mathbb{P}_r$ because $C[0,a]$ is closed and the convergence is uniform in $[0,a]$.

We know that the linear operator $L_n : \mathbb{P}_r \to \mathbb{P}_r$ is always continuous because $\mathbb{P}_r$ is a finite dimensional space. Then, $V(g) = \lim_{m\to\infty} V(V^m(p_i^{(n)})) = \lim_{m\to\infty} V^{m+1}(p_i^{(n)}) = g$ from which $g = V(g) = g - L_n(g)$ and then $L_n(g) = 0$, that is possible only when $g(\frac{i}{n}) = 0$ for all $i \in \{0,\dots,r\}$ (see Remark 8). Hence, g is a polynomial of degree at most r that vanishes at $r+1$ points which implies that $g = 0$. Therefore

$$\lim_{m\to\infty} \left(I - (I-L_n)^m\right)(p_i^{(n)}) = \lim_{m\to\infty} \left(p_i^{(i)} - V^m(p_i^{(n)})\right) = p_i^{(n)}.$$

If $p \in \mathbb{P}_r$ then we can write $p = \sum_{i=0}^r A_i p_i^{(n)}$. Since $I-(I-L_n)^m$ is a linear and continuous operators the results obtained for $p_i^{(n)}$ also hold for p. □

Corollary 7. *Given $r \in \mathbb{N}$ and $p \in \mathbb{P}_r$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$\lim_{m\to\infty} \left(I - (I-L_n)^m\right)(p) = p.$$

Proof. From the definition of the Mastroianni operators, for every $i \in \mathbb{N}$,

$$\lim_{n\to\infty} \lambda_i^{(n)} = \lim_{n\to\infty} \frac{\alpha_{n,i}}{n^i} = 1.$$

Therefore, we can find n_0 large enough such that $|1-\lambda_i^{(n)}| < 1$ for $i \in \{0,\dots,r\}$ and $n \geq n_0$. □

Remark 8. From the hypotheses of Theorem 6 we know that $0 < \alpha_{n,i}$ so that $\phi_n^{(i)}(0) \neq 0$, $i \in \{0,\dots,r\}$. By means of (1), if $L_n(g) = 0$ we easily deduce that

$$(-1)^i \frac{\phi_n^{(i)}(0)}{i!} \, \Delta_{\frac{1}{n}}^i g(0) = 0, \ \forall i \in \mathbb{N}_0$$

and hence $\Delta_{\frac{1}{n}}^i g(0) = 0$ for $i \in \{0,\dots,r\}$ (because the corresponding $\phi_n^{(i)}(0)$ does not vanish) from which it is straightforward that $g\left(\frac{i}{n}\right) = 0$ for $i \in \{0,\dots,r\}$.

So, we can conclude that under the conditions of Theorem 6, given $g \in C[0,\infty)$ such that $L_n(g) = 0$ we have that $g\left(\frac{i}{n}\right) = 0$, $i \in \{0,\dots,r\}$.

5 Generalized Baskakov Operators.

For every $f \in E_2$ and $x \in [0, \infty)$, Baskakov operators are defined by

$$M_n f(x) = \sum_{k=0}^{\infty} (-1)^k f\left(\frac{k}{n}\right) x^k \frac{\phi_n^{(k)}(x)}{k!},$$

(cf. Martini [3]) and they are generated by a sequence of analytic functions on $[0, \infty)$, $\phi_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, satisfying **A1)**, **A2)** and

$$\phi_n^{(k)}(x) = -(n + l_1)\phi_{n+l_2}^{(k-1)}(x)$$

for all $k \in \mathbb{N}$ and $x \in [0, \infty)$, where $l_1, l_2 \in \mathbb{N}_0$ are independent of n, k and x.

By induction **A3)** is also verified for $\alpha_{n,i} = \alpha_{n,i}(x) = (n+l_1)^{(i,-l_2)}$ and $p(n,i) = n+il_2$, where $x^{(0,l)} = 1$ and $x^{(i,-l)} = x(x + l)...(x + (i - 1)l)$, for $i > 0$.

Furthermore, from suitable choices of the sequences $\{\phi_n\}_{n\in\mathbb{N}_0}$ we obtain some well known operators. For example, choosing $l_2 = 0$, we have the Schurer-Szász-Mirakjan operators [2] $S_{l_1,n}$ with

$$S_{l_1,n} f(x) = e^{-(n+l_1)x} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(n + l_1)^k}{k!} x^k.$$

From (2) and (3) we get

Proposition 9. For every $k \in \mathbb{N}_0$ and $x \in [0, \infty)$

$$M_n^{(k)} f(x) = (n + l_1)^{(k,-l_2)} \sum_{i=0}^{\infty} (-1)^i \frac{\phi_{n+kl_2}^{(i)}(0)}{i!} \, \Delta_{\frac{1}{n}}^{i+k} f(0) \, x^i. \tag{8}$$

In particular,

$$S_{l_1,n}^{(k)} f(x) = (n + l_1)^k S_{l_1,n}(\Delta_{\frac{1}{n}}^k f)(x). \tag{9}$$

Lemma 3 allows us to state the moments of the operators.

Proposition 10. For any $n \in \mathbb{N}$, $r \in \mathbb{N}_0$, one has

$$M_n(t^r) = \sum_{\beta=0}^{r} n^{-\beta} \sum_{s=0}^{min\{r-1,\beta\}} A(r - s, r - \beta) \; \sigma_{\frac{1}{r}}^{r-s} \; t^{r-s}, \tag{10}$$

where $A(i, \alpha) = \sum_{j=\alpha}^{i} \binom{j}{\alpha} l_2^{i-j} S_i^j \, (l_1 + l_2(i - 1))^{j-\alpha}$.

Proof. Observe that

$$x^{(i,-l)} = l^i \left(\frac{x}{l} + i - 1\right)^{\underline{i}} = l^i \sum_{j=0}^{i} S_i^j \left(\frac{x}{l} + i - 1\right)^j$$

and in particular

$$(n+l_1)^{(i,-l_2)} = l_2^i \sum_{j=0}^{i} S_i^j \left(\frac{n+l_1}{l_2}+i-1\right)^j = \sum_{j=0}^{i} l_2^{i-j} S_i^j \left(n+l_1+l_2(i-1)\right)^j$$

$$= \sum_{j=0}^{i} l_2^{i-j} S_i^j \sum_{\alpha=0}^{j} \binom{j}{\alpha} n^\alpha \left(l_1+l_2(i-1)\right)^{j-\alpha}$$

$$= \sum_{\alpha=0}^{i} n^\alpha \sum_{j=\alpha}^{i} \binom{j}{\alpha} l_2^{i-j} S_i^j \left(l_1+l_2(i-1)\right)^{j-\alpha}.$$

Lemma 3 yields

$$M_n(t^r) = n^{-r} \sum_{i=1}^{r} (n+l_1)^{(i,-l_2)}\ \sigma_r^i\ t^i; \tag{11}$$

i.e.,

$$M_n(t^r) \quad = \quad n^{-r} \sum_{i=1}^{r} \sum_{\alpha=0}^{i} n^\alpha A(i,\alpha) \quad \sigma_r^i \quad t^i \quad = \quad \sum_{\alpha=0}^{r} n^{\alpha-r} \sum_{i=max\{1,\alpha\}}^{r} A(i,\alpha) \quad \sigma_r^i \quad t^i.$$

Replacing $\alpha = r-\beta$ and $i = r-s$, the proof is concluded. □

A characterization of $S_{l_1,n}$ operators is obtained using Proposition 10. (In fact we use (11)). We show the operator M_n satisfies (9) if and only if M_n is the Schurer-Szász-Mirakjan operator.

Corollary 11.

$$M_n^{(k)} f = (n+l_1)^k M_n(\Delta_{\frac{1}{n}}^k f),\ \textit{for every } k \in \mathbb{N}_0 \Leftrightarrow l_2 = 0.$$

Proof. If we suppose that M_n satisfies (9), then in particular (9) holds for $f = t^k$. Now, (11) implies that $(n+l_1)^{(k,-l_2)} = (n+l_1)^k$ and so $l_2 = 0$. □

In order to give the asymptotic expansion for the M_n operators, we study their central moments.

Proposition 12. Given $p \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $x \in [0,\infty)$ the following identity holds:

$$M_n\left((t-x)^p\right)(x) = \sum_{\beta=0}^{p} n^{-\beta} \sum_{r=\beta}^{p} (-1)^{p-r} \binom{p}{r} \sum_{s=0}^{min\{r-1,\beta\}} A(r-s, r-\beta)\ \sigma_r^{r-s}\ x^{p-s}, \tag{12}$$

where $A(\ ,\ )$ is given in Proposition 10.

Proof. From Proposition 10 we obtain

$$M_n\left((t-x)^p\right)(x) = \sum_{r=0}^{p} \binom{p}{r} (-x)^{p-r} M_n\left(t^r\right)(x)$$
$$= \sum_{r=0}^{p} \binom{p}{r} (-1)^{p-r} \sum_{\beta=0}^{r} n^{-\beta} \sum_{s=0}^{min\{r-1,\beta\}} A(r-s, r-\beta)\ \sigma_r^{r-s}\ x^{p-s}$$
$$= \sum_{\beta=0}^{p} n^{-\beta} \sum_{r=\beta}^{p} (-1)^{p-r} \binom{p}{r} \sum_{s=0}^{min\{r-1,\beta\}} A(r-s, r-\beta)\ \sigma_r^{r-s}\ x^{p-s}.$$

□

Remark 13. Using the same arguments as Sikkema [6] for Szász operators, we can deduce that, in fact, in (12) β runs from $[\frac{p+1}{2}]$, the greatest integer less than or equal to $\frac{p+1}{2}$, to p.

The main result of this section is:

Theorem 14. *Let $f \in E_2$ r times differentiable at $x \in [0, \infty)$. Then*

$$M_n f(x) = f(x) + \sum_{\beta=1}^{r} n^{-\beta} a(\beta, r, f, x) + o(n^{-r}), \tag{13}$$

where

$$a(\beta, r, f, x) = \sum_{p=\beta}^{r} \frac{f^{(p)}(x)}{p!} \sum_{r=\beta}^{p} (-1)^{p-r} \binom{p}{r} \sum_{s=0}^{min\{r-1,\beta\}} A(r-s, r-\beta)\ \sigma_r^{r-s}\ x^{p-s}.$$

Proof. By (5), the proof follows from Proposition 12. □

For the convenience of the reader we list the initial summands of the expansion (13), for $r = 3$;

$$M_n f(x) = f(x) + \frac{l_1 x}{n} f'(x) + \frac{nx(1 + l_2 x) + x(l_1 + l_1(l_1 + l_2)x)}{2n^2} f''(x)$$
$$+ \left[\frac{nx\left(1 + 3(l_1 + l_2)x + l_2(3l_1 + 2l_2)x^2\right)}{6n^3}\right.$$
$$\left. + \frac{x(l_1 + 3l_1(l_1 + l_2)x + l_1(l_1 + l_2)(l_1 + 2l_2)x^2)}{6n^3}\right] f'''(x) + o(n^{-3}),$$

as $n \to \infty$.

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