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A tauberian theorem for a class of function spaces *

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Abstract

Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$, $F \subset E$, be Banach spaces. Assume that $\|.\|_F := \|.\|_E + \theta(.)$, where θ is a seminorm. It is proved that sequences in F that converge in $\|.\|_E$ and whose elements satisfy certain equicontinuous behavior, also converge in $\|.\|_F$ to the same limit points. Quantitative estimates of the degree of convergence are obtained. Examples of applications to different function spaces are presented.

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1 Introduction

Let X be either the real interval [0,1] or the multiplicative group $T = \{z \in \mathbb{C} : |z| = 1\}$. Let $Lip_\infty^\alpha X$ $(Lip_\infty^\alpha$ for short$)$, $0 < \alpha < 1$, be the Hölder space of continuous real (or complex) functions $f \in C(X)$, which satisfy the Hölder (also called Lipschitz) condition

$$\theta_\infty^\alpha(f) := \sup_{\delta>0} \theta_\infty^\alpha(f, \delta) < \infty, \tag{1}$$

where

$$\theta_\infty^\alpha(f, \delta) := \sup\{|f(x) - f(y)| / d(x,y)^\alpha : 0 < d(x,y) \le \delta\}. \tag{2}$$

Here $d(x,y) := |x-y|$ if $X = [0,1]$ or equal to the length of the shortest arc which joins x and y if $X = T$. In the last case, if functions on T are identified with 2π-periodic functions on $\mathbb{R}$, d should be the semidistance between elements of $\mathbb{R}$, given by

$$d(x + 2j\pi, y + 2k\pi) := \min\{|x-y|, 2\pi - |x-y| : x, y \in [0, 2\pi[;\ j, k \in \mathbb{Z}\}. \tag{3}$$

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Setting

$$\|f\|_{\alpha,\infty} := \|f\|_{\infty} + \theta_{\infty}^{\alpha}(f) \tag{4}$$

or another equivalent norm, the linear space Lip_{∞}^{α} becomes a Banach space. Further, denote by $lip_{\infty}^{\alpha}X$ (lip_{∞}^{α} for short), $0 < \alpha < 1$, the Banach subspace of those functions $f \in Lip_{\infty}^{\alpha}$, for which

$$\theta_{\infty}^{\alpha}(f,\delta) \longrightarrow 0 \text{ as } \delta \longrightarrow 0. \tag{5}$$

Basic results on Hölder spaces can be found in [4] and [5]. A recent survey of approximation in these spaces is given in [3].

From (4), a sequence that converges in Lip_{∞}^{α} also converges in the sup-norm $\|.\|_{\infty}$, with the same limit. The converse is false, of course. However, there is a certain tauberian condition (*) which lets us to prove the following assertion:

$$(f_n) \subset lip_p^{\alpha}\ ,\quad \|f_n - f\|_p \longrightarrow 0 \text{ and } (*) \Longrightarrow \|f_n - f\|_{\alpha,p} \longrightarrow 0. \tag{6}$$

In fact, in 1985, Leindler, Meir and Totik proved a first result of type (6) for X being the group T and (f_n) defined by a convolution process $K_n * f$, $f \in lip_{\infty}^{\alpha}$ (see [8]). They also estimated the degree of convergence. Later, Bustamante-Jiménez [2] introduced the following tauberian condition: A sequence $(f_n) \subset lip_{\infty}^{\alpha}X$, $0 < \alpha < 1$, is called *equilipschitzian* if (5) holds uniformly in n, i.e. if

$$\sup\{\ \theta_{\infty}^{\alpha}(f_n,\delta) : n \in \mathbb{N}\} \longrightarrow 0 \text{ as } \delta \longrightarrow 0. \tag{7}$$

The main theorem in [2] states that any equilipschitzian sequence (f_n) in lip_p^{α} converges in this space whenever it converges in the sup-norm, i.e. (6). Since sequences defined by convolution processes $(K_n * f)$, $f \in lip_p^{\alpha}(T)$ and (K_n) bounded in $L^1(T)$, are equilipschitzian, we get another view of the qualitative part of paper [8].

When $1 \leq p < \infty$, one defines Lip_p^{α} and lip_p^{α} in L_p, through standard procedures. Leindler, Meir and Totik announced the possibility of extending their results to $lip_p^{\alpha}(T)$. Further, in [7], Jiménez-Martínez extended most of results in [2] to these spaces.

With these antecedents at hand, one should expect a more general theorem that covers and unifies these particular results. In fact, in the next section, using a concept similar to (7), we establish and prove such a theorem. Estimates of the degree of convergence will also be obtained. The last section is devoted to applications in different function spaces.

2 Definitions and results

In order to follow the ideas of this section, let us keep our mind on the examples given by lip_p^α.

Set $\mathbb{R}_+ := \{t \in \mathbb{R}\colon t \geqslant 0\}$, $\mathbb{R}_+^* := \{t \in \mathbb{R}\colon t > 0\}$ and denote by I the real open interval $]0,b[$ (or semi-open $]0,b]$) where $I = \mathbb{R}_+^*$ is possible. Let E be a real or complex linear space and

$$\theta : E \times I \longrightarrow \mathbb{R}_+ \cup \{\infty\}, \tag{8}$$

a family $\theta(., \delta)$, $\delta \in I$, of quasi-seminorms on E, i.e. the subadditivity of usual seminorms is substituted by the most general assertion that there exists a constant $C \geq 1$ (that here we assume is independent of δ), such that for every pair of elements $f, g \in E$, one has $\theta(f+g, \delta) \leq C(\theta(f,\delta) + \theta(g,\delta))$. Without loss of generality it is also assumed that for every fixed $f \in E$, $\theta(f, .)$ is an increasing function (in the large sense) of δ. Set

$$\theta(f) := \sup\{\theta(f,\delta) : \delta \in I\}. \tag{9}$$

Consider

$$\mathbb{F} \ : \ = \{f \in E : \theta(f) < \infty\} \tag{10}$$

$$F \ : \ = \{f \in \mathbb{F} : \theta(f,\delta) \longrightarrow 0 \ \text{ as } \delta \longrightarrow 0\} \tag{11}$$

Then, $\mathbb{F}$ and F are linear subspaces of E, that are quasi-seminormed by (9) and that, eventually, could coincide .

We remark that F is a closed subspace of $(\mathbb{F}, \theta)$. In fact, let $(f_n) \subset F$ be a sequence that converges to $f \in \mathbb{F}$. Fix $\varepsilon > 0$. First, take n such that $\theta(f_n - f) \leq \varepsilon$ and then $\delta_0 > 0$ such that $\theta(f_n, \delta) \leq \varepsilon$, for every $\delta \leq \delta_0$ Thus $\theta(f,\delta) \leq C(\theta(f_n - f) + \theta(f_n, \delta)) \leq 2C\varepsilon$.

Definition 1 *A set $G \subset F$ is called 0-equicontinuous if*

$$\theta(G,\delta) := \sup\{\ \theta(g,\delta) : g \in G\} \longrightarrow 0 \ \textit{as}\ \delta \longrightarrow 0. \tag{12}$$

A sequence (f_n) is called 0-equicontinuous if the set $\{f_n : n \in \mathbb{N}\}$ is. In that case we simplify the notation by writing

$$\theta((f_n), \delta) : =\theta(\{f_n : n \in \mathbb{N}\}, \delta).$$

Of course, equilipschitzian sets in our introductory section not only are examples of 0-equicontinuous sets but also the starting point of the present definition.

Proposition 2 *Let (f_n) be a convergent sequence in the quasi-seminormed space (F,θ). Then such a sequence is 0-equicontinuous.*

Proof: Suppose $\theta(f_n - f) \longrightarrow 0$ for some $f \in F$. Fix $\varepsilon > 0$ and choose N such that $\theta(f_n - f) \leq \varepsilon$ whenever $n > N$. Also choose $\delta_0 \in I$, such that $\theta(f, \delta_0) \leq \varepsilon$. Then, for any $0 < \tau \leq \delta_0$ and $n > N$,

$$\theta(f_n, \tau) \leq C\ \left(\theta(f_n - f, \tau) + \theta(f, \tau)\right) \leq C\ \left(\theta(f_n - f) + \theta(f, \delta_0)\right)\ \leq 2C\ \varepsilon.$$

For $i = 1, 2, ..., N$, choose δ_i such that $\theta(f_i, \delta_i) \leq \varepsilon$. Set $\delta := \min\{\delta_i : 0 \leq i \leq N\}$. Thus $\sup\{\ \theta(f_n, \delta) : n \in \mathbb{N}\} \leq 2C\ \varepsilon$. ■

In the remainder of this section we assume E to be a topological vector space whose topology is defined by a distance d_E, which is complete and translation invariant. We define another distance or quasi-distance in $\mathbb{F}$ by setting

$$d_{\mathbb{F}}(f, g) := d_E(f, g) + \theta(f - g). \tag{13}$$

Write $d_\Xi(f)$ instead of $d_\Xi(f, 0)$, where Ξ could be either E or $\mathbb{F}$. Then, $d_\Xi(f - g) = d_\Xi(f, g)$.

From (13), a sequence that converges in $(\mathbb{F}, d_{\mathbb{F}})$ also converges in (E, d_E) and to the same limit. The converse assertion is false in general. However, as we have already pointed out, we shall prove a certain converse result. In order to establish it we need a link between d_E and θ.

Definition 3 *The family of quasi-seminorms $\theta(., \delta)$, $\delta \in I$, defined above, is said to be admissible with respect to the distance d_E if the following conditions are satisfied:*

i) $(F, d_{\mathbb{F}})$ is complete

ii) There exists a constant $K > 0$ and a function $\Psi : I \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that for each $\delta \in I$,

$$\lim_{t \to 0} \Psi(\delta, t) = \Psi(\delta, 0) := 0$$

and for every $f \in F$,

$$\theta(f) \leq K\ \theta(f, \delta) + \Psi(\delta, d_E(f)). \tag{14}$$

With respect to condition i), since F is a closed subspace of $(\mathbb{F}, \theta)$, it follows from (13) that F is also a closed subspace of $(\mathbb{F}, d_{\mathbb{F}})$. Then, if $(\mathbb{F}, d_{\mathbb{F}})$ is complete, so is $(F, d_{\mathbb{F}})$.

Theorem 4 (*tauberian*) *Suppose that* $(F, d_{\mathbb{F}})$ *has been defined by a family of admissible quasi-seminorms* $\theta(., \delta)$, $\delta \in I$, *on* (E, d_E). *Let* $(f_n) \subset F$ *be a convergent sequence in* (E, d_E), *to an element* f. *If* (f_n) *is 0-equicontinuous , then* $f \in F$ *and* (f_n) *converges to* f *in* $(F, d_{\mathbb{F}})$. *Moreover, if for each* $\delta \in I$, $\Psi(\delta, .)$ *is continuous in* $\mathbb{R}_+$, *then*

$$\theta(f_n - f) \leq 2C\ K\ \ \theta((f_n), \delta) + \Psi(\delta, d_E(f_n - f)). \tag{15}$$

Proof: Assume we have already proved that $(\theta(f_n))$ is a real Cauchy sequence. Since the hypothesis of the theorem include that (f_n) is a Cauchy sequence in E, it would follow from (13) that (f_n) is a Cauchy sequence in $(F, d_{\mathbb{F}})$. But F is a complete metric space, then there exits $g \in F$ such that $d_{\mathbb{F}}(f_n - g) \longrightarrow 0$ as $n \to \infty$. Also by (13), $d_E(f_n - g) \leq d_{\mathbb{F}}(f_n - g)$, then $d_E(f_n - g) \to 0$. But $d_E(f_n - f) \longrightarrow 0$ as $n \to \infty$. That forces $f = g$. In order to prove that $(\theta(f_n))$ is a Cauchy sequence, fix $\varepsilon > 0$. For every $\delta \in I$, we use (14) to obtain,

$$\theta(f_n - f_m) \leq K\ \theta(f_n - f_m, \delta) + \Psi(\delta, d_E(f_n - f_m)). \tag{16}$$

Take δ such that $\theta((f_n), \delta) \leq \varepsilon$. Further, take N such that for every $n > N$ and $m > N$, $\Psi(\delta, d_E(f_n - f_m)) \leq \varepsilon$. By substituting into (16),

$$\theta(f_n - f_m)\ \ \leq (2CK + 1)\,\epsilon.$$

The qualitative part of the theorem has been proved. In particular $\theta(f_n - f_m) \longrightarrow \theta(f_n - f)$ as $m \longrightarrow \infty$. Then, using (16) and the continuity of $\Psi(\delta, .)$ we deduce (15). ■

Equivalent distances to (13) are given by

$$d_{\mathbb{F}}(f) := (d_E(f)^p + \theta(f)^p)^{1/p}, \quad 1 < p < \infty, \tag{17}$$

$$d_{\mathbb{F}}(f) := \max\{d_E(f), \theta(f)\ \}, \qquad\qquad p = \infty. \tag{18}$$

In those cases, using (15), we remark that

$$d_{\mathbb{F}}(f_n - f) \leq\ (d_E(f_n - f)^p +\ [2CK\ \theta((f_n), \delta) + \Psi(\delta, d_E(f_n - f))]^p)^{1/p}, \tag{19}$$

if $1 \leq p < \infty$; or

$$d_{\mathbb{F}}(f_n - f)\ \leq \max\{\ d_E(f_n - f), 2C\ K\ \theta((f_n), \delta) + \Psi(\delta, d_E(f_n - f))\}, \tag{20}$$

if $p = \infty$.

Also we remark that formula (15) is a general one. Therefore its accuracy could be improved in particular problems. In the same way, optimal values for δ depend on the problem on hand.

Theorem 5 *Suppose that* $(F, d_{\mathbb{F}})$ *has been defined from* (E, d_E) *by a family of admissible quasi-seminorms* $\theta(., \delta)$, $\delta \in I$. *Then a set* $A \subset F$ *is compact with respect to the topology induced by* $d_{\mathbb{F}}$ *if and only if* A *is compact in* (E, d_E) *and 0-equicontinuous.*

Proof. Let $(f_n) \subset A$. If A is a compact set of (E, d_E), there exists a subsequence (f_{n_k}) that converges to an element $f \in A$ with respect to d_E. If A is a 0-equicontinuous set, then (f_{n_k}) converges to f with respect to $d_{\mathbb{F}}$. Reciprocally, if A is a compact set of $(F, d_{\mathbb{F}})$, there exists a subsequence (f_{n_k}) that converges to an element $f \in A$ with respect to $d_{\mathbb{F}}$ Then (f_{n_k}) converges to the same limit with respect to d_E. ■

3 Examples and Applications

In this section we show that well known function spaces are included in the class of spaces defined above . Of course, it is impossible to examine here the great variety of important function spaces not even to examine only a few of them in their general setting (see Triebel [9], for instance). Thus the particular examples below are conceived just to conform an illustrative sample of applications.

Example 6 *Set* $E := C(X)$. *Taking* $\theta(f, \delta) := \theta_\infty^\alpha(f, \delta)$, *defined in* (2), *we obtain* $\mathbb{F} = Lip_\infty^\alpha$ *and* $F = lip_\infty^\alpha$. *Set* $K := 1$. *Thus , with* $\Psi(\delta, t) := 2t/\delta^\alpha$, *the family of seminorms is admissible. An application of* (15) *leads to*

$$\|f_n - f\|_{\alpha,\infty} \leq (1{+}2/\delta^\alpha) \|f_n - f\|_\infty + 2\theta((f_n), \delta) \tag{21}$$

The qualitative part of this application is the main theorem in Bustamante-Jiménez [2]. In particular, the sequence of Bernstein polynomials $(B_n f)$, $f \in lip_\infty^\alpha([0,1])$ is *0*-equicontinuous . In fact, Bustamante-Jiménez proved that $(B_n f)$ converges to f in $lip_p^\alpha[0,1]$, i.e. in the norm (4) which implies convergence in the seminorm (1). Then Proposition 2 asserts that $(B_n f)$ is *0*-equicontinuous. On the other hand, theorem 5 characterizes the compact sets in lip_∞^α in the same way that it was done in [2].

Example 7 *In the last example, take* $X := T$ *and change* (2) *by*

$$\begin{aligned} \theta(f, \delta) \ &: \ = \sup\{\zeta(f, t) : 0 < t \leq \delta\}, \\ \zeta(f, t) \ &: \ = \sup\{|f(x+t) - f(x)| / \varphi(t) : x \in T\}, \end{aligned}$$

where $\varphi : \mathbb{R}_+^* \to \mathbb{R}_+^*$, *is an increasing function. For* $f \in F$, *define the sequence* $f_n := K_n * f$, *where* $K_n \in L^1(T)$ *and* $M := \sup\{\|K_n\|_1 : n \in \mathbb{N}\} < \infty$. *Then* (f_n) *is 0-equicontinuous with* $\theta((f_n), \delta) \leq M\ \theta(f, \delta)$. *Assume that* $f_n \longrightarrow f$ *in uniform norm. Set* $K := 1$ *and* $\Psi(\delta, t) := 2t/\varphi(\delta)$. *In this situation* (21) *is transformed into*

$$\|f_n - f\|_{\mathbb{F}} \leq (1{+}2/\varphi(\delta)) \ \ \|f_n - f\|_\infty + 2\ M\ \theta(f, \delta).$$

This is the estimate given by Leindler, Meir and Totik, from which several implications to Fourier series follow ([8])

Example 8 *We can use modulus of smoothness of higher order r. For instance, set $X := T$ and $E := L^p(T)$, $1 \leq p < \infty$. Define*

$$\begin{aligned} \theta(f,\delta) &:= \sup\{\zeta(f,t) : 0 < t \leq \delta\}, \\ \zeta(f,t) &:= \left\{ \left(\frac{1}{2\pi} \int_0^{2\pi} |\Delta_t^r f(x)|^p \, d(x) \right)^{1/p} / t^\alpha \right\}. \end{aligned}$$

Here $\Delta_t f := \Delta_t^1 f := f(.+t) - f$, $\Delta_t^r f := \Delta_t\left(\Delta_t^{r-1} f\right)$. Set $K := 1$. Then, with the function $\Psi(\delta,t) := 2^r t/\delta^\alpha$, the family of semi-norms is admissible in definition 3.

Example 9 *Set $E := L_p(T)$, $1 \leq p < \infty$. In [6], the author has defined homogeneous Hölder spaces B_p^α, $\alpha > 0$, which are equivalent in norm to certain Besov spaces. A function $f \in L_p(T)$ is in B_p^α, if $F_\alpha(x,y) := (f(x) - f(y))/d(x,y)^\alpha \in L_p(T^2)$. A crucial point here is that d is given by (3) and then F_α has period 2π in each variable. Set*

$$\theta(f) := \left(\frac{1}{4\pi^2} \int_0^{2\pi} \left(\int_0^{2\pi} |F_\alpha(x,y)|^p \; dx \right) dy \right)^{1/p}.$$

Then B_p^α becomes a homogeneous Banach (Hilbert if $p = 2$) space under the norm

$$\|f\|_{\alpha,p} := \left(\|f\|_{L^p(T)}^p + \|F_\alpha\|_{L^p(T^2)}^p \right)^{1/p}.$$

Taking

$$\theta(f,\delta) := \left(\frac{1}{2\pi^2} \int_0^{\delta} \left(\int_0^{2\pi} |\Delta_t f(x)/t^\alpha|^p \; dx \right) dt \right)^{1/p},$$

we can show that $\theta(f) = \theta(f,\pi)$.

Thus $\mathbb{F} = F = B_p^\alpha$. Set $K := 1$. Therefore, with the function

$$\Psi(\delta,t) := \left(\frac{2}{\pi} \int_\delta^\pi \frac{dx}{x^{\alpha p}} \right)^{1/p} t,$$

the family of seminorms is admissible.

With the following two examples, we show the connection of section 2 with the theory of Measure and Integration and also the convenience of considering the general scope in which the tauberian theorem above has been established.

Example 10 *Let E be the complex linear space of all bounded complex functions f on $\mathbb{R}$ that are continuous to the right and such that $f(x) \longrightarrow 0$ as $x \to -\infty$. For all $\delta > 0$, set*

$$(22) \quad \theta(f,\delta) := \sup \left\{ \begin{array}{c} \sum_{1\leq i\leq m} |f(y_i) - f(x_i)| : x_1 < y_1 \leq x_2 < ... < y_m; \\ m = 1,2,..; \quad \sum_{1\leq i\leq m} y_i - x_i \leq \delta \end{array} \right\}.$$

Thus $\theta(f)$ stands for the total variation of f in $\mathbb{R}$; $(\mathbb{F},\theta)$ is defined to be the Banach space of functions of bounded variation and F is its closed subspace of absolutely continuous functions.

We remark that for a given function f, it could happen that $\theta(f,\delta) \longrightarrow 0$ as $\delta \to 0$, but $\theta(f) = \infty$. For instance, $f(x) := \sin(x)/x$. However such a function is not in F by [10] and [11].

On the other hand, since (22) is equal to

$$\sup\left\{\int_A |f\,'(x)|\,d(x) : \ \text{meas}(A) = \delta\right\}, \ f \in F,$$

this example is connected with the next one, for which the theoretical background can be found in chapter 4 of [1]. However, to avoid technical difficulties that are not any objective at present, we restrict ourself to a set of finite measure.

Example 11 *Let E be the complex linear space of all measurable complex functions f on $[0,1]$. We identify functions that are equal Lebesgue almost everywhere and consider any complete and translation invariant distance d_E which characterize the convergence in measure. .*

For any $f \in E$, $0 < p < \infty$ and $0 < \delta \leq 1$, define

$$\theta(f,\delta) := \sup\left\{\left(\int_A |f|^p\,d(x)\right)^{1/p} : \ meas(A) = \delta\right\}.$$

Then $\mathbb{F} = F = L^p[0,1]$. A sequence (f_n) is 0-equicontinuous if and only if it is equi-integrable and it is known that convergence of (f_n) in $L^p[0,1]$, occurs if and only if (f_n) is a Cauchy sequence in measure and equi-integrable. In this example, the function Ψ depends on the particular distance d_E at hands. In fact, for a given function $f \in F$ and $0 < \delta \leq 1$, fix a measurable set A, with meas$(A) = \delta$, such that for any pair $x \in A$ and $y \in A^c$, $f(y) \leq f(x)$. Using typical procedures, we obtain

$$\theta(f) = \left[\int_0^1 |f|^p\,d(x)\right]^{1/p} \leq K\left[\theta(f,\delta) + \left[\int_{A^c} |f|^p\,d(x)\right]^{1/p}\right],$$

with $K := C = 1$ if $1 \leq p < \infty$ or $K := C = 2^{1/p}$ if $0 < p < 1$. Then, in terms of the sequence (f_n) and its limit in measure f,

$$\theta(f_n - f) \leq 2K^2\,\theta((f_n),\delta) + K\ \beta_n\ (1-\delta)^{1/p},$$

where the sequence β_n, that converges to 0 when $n \to \infty$, can be expressed in terms of $d_E(f_n - f)$.

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