Monografías de la Academia de Ciencias de Zaragoza. **20:** 69–76, (2002).

An iterative scheme to approach the asymptotic behaviour of a Kolmogorov system *

M. Gámez and Mª. I. López

Department of Statistics and Applied Mathematics.

University of Almería. Spain.

Abstract

In this paper we consider a biological system consisting of several preys and several predators. We study the existence of a global attractor for such a system and obtain an approximation to the model's solution by means a numerical method.

1 Introduction.

In the last few years many papers have been devoted to the dynamics of applied populations to Biology. From a mathematical point of view it's of a great interest to determine qualitative properties on these differential systems (see [1], [2], [3]) which give information about the behaviour of the solutions, due to the impossibility, in most cases to solve these systems explicitly. For this reason, the numerical methods are reaching an important role in this subject, in order to determine some properties of the solutions of these systems.

We consider a biological Kolmogorov system consisting of several preys and several predators. The case of the usual predator-prey system has been extensively studied by many authors. For instance, see [1] for optimal results.

We show some results about the logistic equation adapted to the notation necessary for the Kolmogorov system studied. Using an iterative scheme, we find the existence of a global attractor for the positive solutions of Kolmogorov system studied, which determine an approximation of the solution of this system.

Finally, we present concrete examples determining these approximations with the help of MATHEMATICA and we compare the results with the numerical resolution of them using the Populus software. Therefore, we verify how these systems can be used to model a process of biological fight and we get in this way another tool which help us to know the development of some biological species.

*This paper was sponsored by the UAL-CAJAMAR 2001 CR-UAL-011

2 The Logistic Equation.

In this section we introduce some notations and we state some interesting properties, which will be basic in our study, of the periodic logistic equation,

$$x' = xF(t,x) \quad , \ x \geq 0. \tag{1}$$

Given $T > 0$, we denote by $\mathcal{C}_T$ the set of all continuous function $F : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ such that:
a) $F(t, x)$ is T-periodic in t and locally Lipschitz continuous in x.
b) $F(t, x)$ is decreasing in x.
c) $F(\tau, x)$ is strictly decreasing in x for some $\tau = \tau(F) \in \mathbb{R}$.
e) There exists $R = R(F) > 0$ satisfying $\int_0^T F(t, R)dt < 0$.

In "Iterative schemes for some population models"(Nonlinear Word, 3, (1996), 695-708), the following results were proved by A. Tineo.

Theorem 2.1 *If $F \in \mathcal{C}_T$ then equation (1) has a T-periodic solution U^F, which is globally asymptotically stable. That is, if u is a solution of (1) and $u(0) > 0$, then u is defined on $[0, \infty)$ and*

$$u(t) - U^F(t) \to 0 \quad as \quad t \to +\infty.$$

Moreover, $U^F > 0$ if $\int_0^T F(t,0)dt > 0$, and $U^F \equiv 0$ if $\int_0^T F(T,0)dt \leq 0$.

We say that U^F is the "*global attractor*" of (1).

Corollary 2.2 *Let $F,\ G \in \mathcal{C}_T$ and suppose $F \leq G$. Then $U^F \leq U^G$.*

Theorem 2.3 *Let $\{F_n\}$ be a sequence in $\mathcal{C}_T$ converging to $F \in \mathcal{C}_T$ uniformly on compact sets. Then $U^{F_n}(t) \to U^F(t)$, uniformly on $\mathbb{R}$.*

3 Kolmogorov System.

In this section we study a predator-prey model for a biological community consisting of $n-$prey and $m-$predators developed under a Kolmogorov system. In a more precise way we suppose the following system,

$$\begin{aligned} x_i' &= x_i f_i(t, x_1, \ldots, x_n, y_1, \ldots, y_m) \quad 1 \leq i \leq n \\ y_j' &= y_j g_j(t, x_1, \ldots, x_n, y_1, \ldots, y_m) \quad 1 \leq j \leq m. \end{aligned} \tag{2}$$

where $f_i, g_j : \mathbb{R} \times \mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}$ are continuous functions, which are T-periodic in t and locally Lipschitz continuous in (x, y).

We shall assume that:

P_1) $f_i(t,x,y)$ is decreasing in $(x,y) \in \mathbb{R}_+^{n+m}$ and $g_j(t,x,y)$ is increasing in $x \in \mathbb{R}_+^n$ and decreasing in $y \in \mathbb{R}_+^m$.

P_2) There exist $\tau_i;\ \theta_j \in \mathbb{R}$ such that $f_i(\tau_i, x, y);\ g_j(\theta_j, x, y)$ are strictly decreasing in x_i, y_j respectively $(i = 1, \ldots, n;\ j = 1, \ldots, m)$.

P_3) There exists $R > 0$ satisfying,

$$\int_0^T f_i(t, Re_i, 0)dt < 0 \qquad 1 \le i \le n$$

$$\int_0^T g_j(t, U^1(t), R\nu_j)dt < 0 \qquad 1 \le j \le m.$$

Here $\{e_1, \ldots, e_n\}$, $\{\nu_1, \ldots, \nu_m\}$ denote the canonical vector basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and $U^1 = (U_1^1, \ldots, U_n^1) : \mathbb{R} \to \mathbb{R}_+^n$, where $U_i^1;\ \ 1 \le i \le n$; is the global attractor of the equation,

$$z' = zf_i(t, ze_i, 0) \qquad 1 \le i \le n, \tag{3}$$

See Theorem 2.1.

4 An Iterative Scheme.

Associated to system (2), we have two sequences of nonnegative T-periodic functions $\{U^N = (U_1^N, \ldots, U_n^N)\}$ and $\{V^N = (V_1^N, \ldots, V_m^N)\}$, $N \in \mathbb{N}$, defined inductively as follows: $U^0 = V^0 \equiv 0$, and $U_i^{N+1};\ \ 1 \le i \le n$; is the global attractor of the logistic equation,

$$z' = zf_i(t, U_1^N(t), \ldots, U_{i-1}^N(t), z, U_{i+1}^N(t), \ldots, U_n^N(t), V^N(t)), \tag{4}$$

and $V_j^N\ \ 1 \le j \le m$ the global attractor of the equation

$$z' = zg_j(t, U^{N+1}(t), V_1^N(t), \ldots, V_{j-1}^N(t), z, V_{j+1}^N(t), \ldots, V_m^N(t)). \tag{5}$$

Remark. The above scheme is obtained, using some ideas in Lopez-Gomez, Ortega and Tineo in "The Periodic Predator-Prey Lotka-Volterra Model"(Avances in differential Equ. vol 1, 3, 1996, 403-423, section 3). In fact, the scheme in that paper is obtained from (4)-(5) when $m = n = 1$

These sequences are well defined and we easily get:

$$\begin{aligned} 0 &\le U^2 \le U^4 \le \ldots \le U^{2N} \le U^{2N-1} \le \ldots \le U^3 \le U^1 \\ 0 &\le V^2 \le V^4 \le \ldots \le V^{2N} \le V^{2N-1} \le \ldots \le V^3 \le V^1. \end{aligned} \tag{6}$$

By (6), $\{U^{2N-1}\}$, $\{U^{2N}\}$, $\{V^{2N-1}\}$, $\{V^{2N}\}$; $\quad N \in \mathbb{N}$ are monotone and uniformly bounded sequences. So, we have well defined functions:

$$\begin{aligned} \bar{U}(t) = \lim_{N\to\infty} U^{2N-1}(t) \quad &; \quad \underline{U}(t) = \lim_{N\to\infty} U^{2N}(t); \\ \bar{V}(t) = \lim_{N\to\infty} V^{2N-1}(t) \quad &; \quad \underline{V}(t) = \lim_{N\to\infty} V^{2N}(t). \end{aligned} \tag{7}$$

Analogously, if $(u(t), v(t))$ is a solution of (2) such that $u(0) > 0$; $v(0) > 0$.

Inductively we can construct two sequences $\{u^N = (u_1^N, \dots, u_n^N)\}$ and $\{v^N = (v_1^N, \dots, v_m^N)\}$, defined on $[0, \infty)$ as follows: $u^0 = v^0 \equiv 0$,

$$\begin{aligned} (u_i^N)' &= u_i^N f_i(t, u_1^{N-1}(t), \dots, u_{i-1}^{N-1}(t), u_i^N, u_{i+1}^{N-1}(t), \dots, u_n^{N-1}(t), v^{N-1}(t)) \\ (v_j^N)' &= v_j^N g_j(t, u^N(t), v_1^{N-1}(t), \dots, v_{j-1}^{N-1}(t), v_i^N, v_{j+1}^{N-1}(t), \dots, v_m^{N-1}(t)) \\ u_i^N(0) &= u_i(0); \quad v_j^N(0) = v_j(0); (i = 1, \dots, n; j = 1, \dots, m; \quad N \in \mathbb{N}) \end{aligned} \tag{8}$$

It is not difficult to show that using the theorem 1.3.7 [2],

$$\begin{aligned} 0 \;&\le\; u^2 \le u^4 \le \dots \le u^{2N} \le u \le u^{2N-1} \le \dots \le u^3 \le u^1 \\ 0 \;&\le\; v^2 \le v^4 \le \dots \le v^{2N} \le v \le v^{2N-1} \le \dots \le v^3 \le v^1. \end{aligned}$$

On the other hand, using induction and Theorem 2.3 it is easy to show the following result.

Corollary 4.1 *For all $N \in \mathbb{N}$, we have*

$$u^N(t) - U^N(t) \to 0 \quad \text{as} \quad t \to +\infty,$$

$$v^N(t) - V^N(t) \to 0 \quad \text{as} \quad t \to +\infty,$$

where $u^N; v^N; U^N; V^N$ are defined in (8),(4) and (5).

Theorem 4.2 *Let $(u(t), v(t))$ be a positive solution of (2).Then, (u, v) is defined on a terminal interval of $\mathbb{R}$ and,*

$$\limsup_{t\to\infty}[u_i(t) - \bar{U}_i(t)] \le 0 \le \liminf_{t\to\infty}[u_i(t) - \underline{U}_i(t)], \quad 1 \le i \le n; \quad t \ge t_0,$$

$$\limsup_{t\to\infty}[v_j(t) - \bar{V}_j(t)] \le 0 \le \liminf_{t\to\infty}[v_j(t) - \underline{V}_j(t)], \quad 1 \le j \le m; \quad t \ge t_0.$$

That is, $[\underline{U}, \overline{U}] \times [\underline{V}, \overline{V}]$ is an approximation of the solution of the system (2).

Example 1

The following example shows the case of an autonomous system with $n = 2$ and $m = 1$ in (2):

$$x_i' = x_i \left[a_i - \sum_{j=1}^{n} b_{ij} x_j - d_i y \right] \qquad y' = y \left[-\alpha + \sum_{i=1}^{n} \beta_i x_i - \gamma y \right] \tag{9}$$

with,

$$B \equiv (b_{ij}) = \begin{pmatrix} 1.4 & -0.5 \\ -0.5 & 1.7 \end{pmatrix}$$

$\alpha = 1$, $\gamma = 14$, $\vec{\beta} = (1.2, 1.6)$, $\vec{d} = (2, 8)$ and $\vec{a} = (0.7, 0.8)$.

Then, with the help of the MATHEMATICA software, the algorithm done in the annexe I obtains the chain of global attractors or approximate solutions expressed in the next list,

	ODD				
Iteration k	1	3	...	47	49
x_1^k	0.746	0.745	...	0.6181	0.6180
x_2^k	0.690	0.687	...	0.479	0.478
y^k	0.0763	0.0758	...	0.0397	0.0396

	EVEN				
Iteration k	2	4	...	48	50
x_1^k	0.481	0.482	...	0.6088	0.6088
x_2^k	0.253	0.256	...	0.463	0.463
y^k	0.0005	0.001	...	0.0369	0.0369

In the Figure 1 we present the numerical resolution of the system done by the "Populus" software. Note that the result agrees with the approximation that we have obtained.

Example 2.

Also we can pose the autonomous case supposing that the predator breed, for that we use the model analogy to (9), changing the second equation:

$$x_i' = x_i \left[a_i - \sum_{j=1}^{n} b_{ij} x_j - d_i y \right], \qquad y' = y \left[\alpha + \sum_{i=1}^{n} \beta_i x_i - \gamma y \right] \tag{10}$$

Like in the Example 1 we are going to see a concrete case for the model (10), using the next coefficients,

$$B \equiv \begin{pmatrix} 1.3 & -0.2 \\ -0.1 & 1 \end{pmatrix},$$

and that, $\alpha = 0.2$, $\gamma = 8.5$, $\vec{\beta} = (0.2, 0.7)$, $\vec{d} = (13, 5)$, and $\vec{a} = (1.3, 0.5)$. Again by the algorithm of the Annexe I we obtain,

	ODD				
Iteration k	1	3	...	49	50
x_1^k	1.094	0.832	...	0.475445	0.475444
x_2^k	0.609	0.463	...	0.264891	0.26489
y^k	0.009	0.081	...	0.0565309	0.0565308

	EVEN				
Iteration k	2	4	...	48	50
x_1^k	0.006	0.205	...	0.475443	0.475444
x_2^k	0.003	0.114	...	0.26489	0.26489
y^k	0.0239	0.0378	...	0.0565307	0.0565308

Again the graphic expression of the numerical resolution, using the same software, end up as the next form.

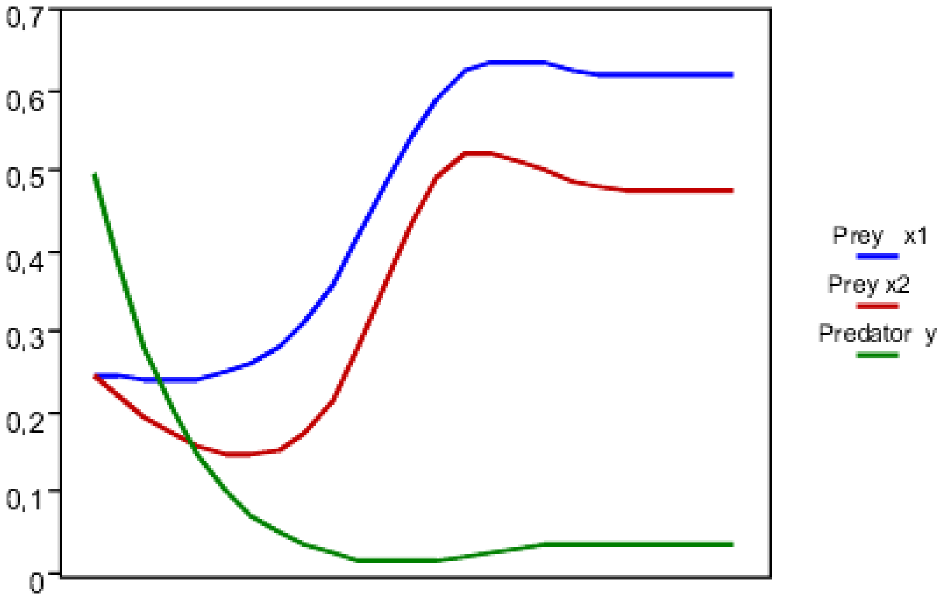

Figure 1: Numerical resolution to example 1

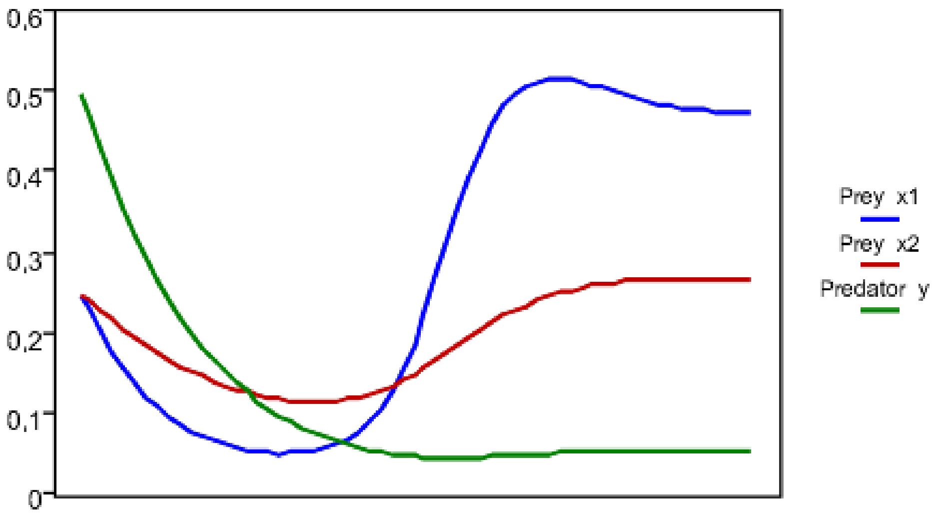

Figure 2: Numerical resolution to example 2

ANNEX I

Using Mathematica, we have developed the next program which has been used for the iterative schemes of the examples 1 and 2 posed in the beginning of this chapter.

- **Main Procedure.**

```
Clear ["Global"];
Presa[mb_,a_,d_,aalfa_,bbeta_,ggamma_,n_,error_,flag_]:=
Module[{sx,sy=0,y,iter=0,listax={},listay={},er=109̂},
   While [iter<n &&er>error, iter++; sx=LinearSolve[mb,a-sy*d];
   If[EvaluacionSinAlfa, sy=Solve[ggamma*y==-aalfa+({bbeta}.
   Transpose[{sx}])[[1,1]],y][[1,1,2]], sy=Solve[ggamma*y==aalfa+({bbeta}.
   Transpose[{sx}])[[1,1]],y][[1,1,2]] ];
   If[iter>1,
      er=Max[Abs[{sx-Last[listax], sy-Last[Listay]}]] ];
   AppendTo[listax,sx];AppendTo[listay,sy]; ];
  Return[If[flag==1,{listax,listay,er},{sx,sy,er}]]]
```

- **Receipts of Data.**

```
file=Input["Archivo de datos:"];
datos=ReadList[file,Number,NullRecord->True,RecordList->True];
n=Lenght[datos[[1]]]; mb=Table[datos[[i]],{i,1,n}];a=datos[[n+1]];
d=datos[[n+2]]; bbeta=datos[[n+3]];niter=datos[[n+4,1]];
error=datos[[n+5,1]];flag=datos[[n+6]];
If[datos[[n+7]]=={},EvaluacionSinAlfa=True, aalfa=datos[[n+7]];flag=0;
  (*Fin entrada*)

  If[EvaluacionSinAlfa,
  (*Case in which the variable "alpha" haven't taken of the card index of the data*)
  ss=Solve[({bbeta}.Inverse[mb].Transpose[{a}])[[1,1]]-x==0,x];
  mensaje1="Alfa(<"<>ToString[ss[[1,1,2]]]<>"):"; aalfa=Input[mensaje1];
  var2=Max[{Max[Table[(({bbeta}.Inverse[mb].Transpose[{a}])[[1,1]]-aalfa*
     (Inverse[mb].Transpose[{d}])[[i,1]]/(Inverse[mb].Transpose[{a}])[[i,1]],
     {i,1,Length[a]}]], ({bbeta}.Inverse[mb].Transpose[{d}])[[1,1]]}],

  (*Case in which the variable "alpha" can take only positive values*)
  var2=Max[Table[(({bbeta}.Inverse[mb].Transpose[{a}])[[1,1]]+aalfa*
  (Inverse[mb].Transpose[{d}])[[i,1]]/(Inverse[mb].Transpose[{a}])[[i,1]],
  {i,1,Length[a]}]]];
  mensaje2="Gamma(>"<>ToString[var2]<>"):";
  ggamma=Input[mensaje2];
```

- **Numerical Resolution. Presentation of Results.**

*(*Resolution by Iterative Methods of Differential System*)*

```
sol=Presa[mb,a,d,aalfa,bbeta,ggamma,niter,error,flag];
```

*(*Presentation of Results*)*

```
If[flag==1,
    (*Visualization of the sequence of the iterations obtained*)
    Print["Lista x:",MatrixForm[sol[[1]]]]; Print[""]; Ip={};li={};
    For[i=1,i<=Length[sol[[2]]],i++, If[Mod[i,2]==0,
    AppendTo[Ip,sol[[2,i]]],AppendTo[li,sol[[2,i]]]]]; Print[li,lp],
    (*Final solution adjusted to the level of error stablished.*)
    Print["x=",sol[[1]]]; Print["y=",sol[[2]]]; Print["Errores=",sol[[3]]];]
```

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