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On saturation in conservative approximation*

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Abstract

In this paper we state a pointwise saturation result for sequences of linear operators that preserve the sign of the k-th derivative of the functions. We apply it to some well known sequences of operators.

1. Notation and introduction

Let $A \subset \mathbb{R}$, $i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. As usual, we denote by $C^i(A)$ the space of all real-valued i-times continuously differentiable functions defined on A and by D^i the i-th differential operator. $C^i_B(A)$ denotes the subspace formed by the functions of $C^i(A)$ which are bounded on A, and we write e_i for the polynomial $e_i(t) = t^i$. A function $f \in C^i(A)$ is said to be i-convex if $D^i f \geq 0$ on A and a linear operator is said to be i-convex if it maps i-convex functions onto i-convex functions.

Now let I be a closed real interval, let $k \in \mathbb{N}_0$ and let $L_n : C^k(I) \longrightarrow C^k(I)$ be a sequence of linear operators satisfying the following asymptotic condition:

A) there exist a sequence λ_n of real positive numbers, and a function $p \in C^k(I)$ strictly positive on $\mathrm{Int}(I)$ such that for all $g \in C^k_B(I)$, $k+2$-times differentiable in some neighborhood of a point $x \in \mathrm{Int}(I)$,

$$\lim_{n\to+\infty} \lambda_n \left(D^k L_n g(x) - D^k g(x) \right) = D^k \left(p D^2 g \right)(x). \tag{1}$$

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Some recent papers have proved that this formula is satisfied for many known operators (see [1], [4], [7]). It informs that the speed of convergence of $D^k L_n g(x) - D^k g(x)$ to 0 cannot overcome in general the one of λ_n^{-1}. Those functions f that satisfy $D^k L_n f(x) - D^k f(x) = o(\lambda_n^{-1})$ for $x \in (a, b)$ with $a, b \in Int(I)$ form the trivial class for the local saturation problem of $D^k L_n$, while the functions that satisfy $D^k L_n f(x) - D^k f(x) = O(\lambda_n^{-1})$ for $x \in (a, b)$ form the so-called saturation class. Recently, in [5] the authors have found these classes assuming also the following shape preserving property:

B) for all $n \in \mathbb{N}, L_n$ is k-convex.

They have extended the results obtained by Mühlbach [11] for the case $k = 0$, taking into account the outstanding work by Lorent and Schumaker [9] in 1972. Simultaneously, Berens [3] dealt with this matter from a more general point of view but also just for $k = 0$. Here the basic tools were convexity arguments through the use of the theory of extended complete Tchebycheff systems (ECT-Systems) and a generalization of the parabola technique introduced by Bajsanski and Bojanić [2].

In the present paper we assume A) and B) and prove a pointwise saturation result for $D^k L_n$ that extends this last one in the sense of considering $k > 0$. We will also apply it in the last section to the well-known operators of Bernstein and Szász-Mirakjan. Firstly, we prove the general formulation of the parabola technique we shall use here.

Lemma 1 *Let $0 = Ly := D^2 y + a_1(t) D^1 y + a_2(t) y = 0$ be a second-order linear differential equation with $a_1, a_2 \in C(I)$ and assume that it has a unique solution taking any two given real values at any two given points within $Int(I)$. Let $g \in C(I)$ and $t_1, t_2 \in Int(I)$. If $f \in C(I)$ verifies that $f(t_1) = f(t_2) = 0$ and $f(t_0) > 0$ for some $t_0 \in (t_1, t_2)$, then there exist a real constant $\alpha < 0$ and a solution of the previous differential equation, $\tilde{y}$, such that for all $t \in [t_1, t_2]$, $\alpha g(t) + \tilde{y}(t) > f(t)$ and for some $s \in (a, b)$, $\alpha g(s) + \tilde{y}(s) = f(s)$.*

Proof Let L_g be the unique solution of $Ly = 0$ satisfying $L_g(t_i) = g(t_i),\ i = 1, 2$, let y_0 be a solution of $Ly = 0$ satisfying $y_0(t) > 0\ \forall t \in [t_1, t_2]$ (whose existence is garanteed taking into account that $t_1, t_2 \in Int(I)$), and let $\epsilon > 0$ be sufficiently small so that $f(t_0) - \epsilon (L_g(t_0) - g(t_0)) > 0$. Then the function

$$\frac{f - \epsilon (L_g - g)}{y_0}$$

is continuous in $[t_1, t_2]$, it vanishes at the end points of this interval and it is strictly positive at the point t_0, so it reaches a maximum value, say M, at a point $s \in (t_1, t_2)$. Consequently, for all $t \in [t_1, t_2]$

$$\epsilon (L_g(t) - g(t)) + M y_0(t) \geq f(t)$$

and

$$\epsilon\left(L_g(s) - g(s)\right) + My_0(s) = f(s).$$

Now the proof is over taking $\alpha = -\epsilon$ and $\tilde{y} = \epsilon L_g + My_0$. $\square$

2. The results

Along this section we assume that the operators L_n defined in Section 1. satisfy A) and B). In [5] we proved the following result

Lemma 2 *a) For $k \in \mathbb{N}$ the ordinary linear differential equation*

$$D^k(pD^2y) \equiv 0, \tag{2}$$

has a fundamental system of solutions of the form $\{e_0, \ldots, e_{k-1}, y_0, y_1\}$, and using the change of variable $z = D^k y$ it can be reduced to

$$D^2z + \frac{kD^1p}{p}D^1z + \frac{k(k-1)D^2p}{2p}z \equiv 0 \tag{3}$$

(p is necessarily a polynomial of degree less than or equal to 2).

b) If $f \in C^k_B(I)$ is a solution of (2) on some neighborhood of a point $x \in Int(I)$, then

$$D^kL_nf(x) - D^kf(x) = o(\lambda_n^{-1}).$$

c) Let $f, g \in C^k_B(I)$. If $D^kf \leq D^kg$ on some neighborhood of a point $x \in Int(I)$, then

$$D^kL_nf(x) \leq D^kL_ng(x) + o(\lambda_n^{-1}).$$

In the sequel, if not specified in other sense, solutions of equation (2) and (3) are understood on $\mathrm{Int}(I)$.

Take $a, b \in Int(I)$ with $a < b$ and assume that (3) has a fundamental system of solutions, say $\{z_0, z_1\}$, which form an ECT-System on (a, b). We write it as in [6], from the functions w_0 and w_1:

$$z_0(t) = w_0(t), \quad z_1(t) = w_0(t)\int_a^t w_1(s)ds. \tag{4}$$

The next lemma shows the relation between convexity respect to this ECT-System and approximation by D^kL_n.

Lemma 3 *Let $f \in C^k(I)$. If*

$$\limsup_{n\to\infty} \lambda_n \left(D^kL_nf(t) - D^kf(t)\right) \geq 0, \ t \in (a, b),$$

then D^kf is convex on (a, b) with respect to z_0, z_1.

Proof Assuming the contrary, there exist $a < t_1 < t_0 < t_2 < b$ such that $D^k f(t_0) > z(t_0)$, being $z = z(t)$ the unique solution of (3) satisfying $z(t_i) = D^k f(t_i)$, $i = 1, 2$. Let us apply Lemma 1 to (3) and $D^k f - z$, with $g = D^k w$, being $w \in C^k(I)$ such that $D^2 w(t) = e_k(t)/p(t)$ for all $t \in [t_1, t_2]$. Then there exist a solution of (3), say $\tilde{z}$, and a constant $\alpha < 0$ verifying that for all $t \in [t_1, t_2]$

$$D^k f(t) - z(t) \leq \alpha D^k w(t) + \tilde{z}(t)$$

and for some $s \in (t_1, t_2)$

$$D^k f(s) - z(s) = \alpha D^k w(s) + \tilde{z}(s).$$

Now if we take $y, \tilde{y} \in C_B^k(I)$, solutions of (2) on (t_1, t_2) such that $D^k y(t) = z(t)$ and $D^k \tilde{y}(t) = \tilde{z}(t)$, then using c), Lemma 2,

$$\lambda_n \left(D^k L_n f(s) - D^k f(s) \right) \leq \alpha \lambda_n \left(D^k L_n w(s) - D^k w(s) \right)$$

$$+ \lambda_n \left(D^k L_n \tilde{y}(s) - D^k \tilde{y}(s) \right) + \lambda_n \left(D^k L_n y(s) - D^k y(s) \right) + o(1).$$

Using b), Lemma 2 for y and $\tilde{y}$ and the asymptotic condition A) for w we obtain

$$\lambda_n \left(D^k L_n f(s) - D^k f(s) \right) \leq \alpha k! + o(1)$$

what is a contradiction that ends the proof. □

Now we prove the main result. We shall obtain some information about functions f that verify $D^k L_n f(x) - D^k f(x) = o(\lambda_n^{-1})$ and $D^k L_n f(x) - D^k f(x) = O(\lambda_n^{-1})$ for $x \in (a, b)$, though we shall consider a more general framework. For this purpose we define $\varphi_x(t) := (t - x)^{k+2}/(k + 2)!$ and $\mu_n(x) := D^k L_n \varphi_x(x)$. Notice that from A) $\mu_n(x) = O(\lambda_n^{-1})$, specifically

$$\lambda_n \mu_n(x) = p(x) + o(1). \tag{5}$$

Firstly we prove the following technical lemma which shall be used in the proof of Theorem 1. In both of them we use the functions $w_2 := 1/w_0 w_1$ and $W_2(t) := \int_a^t w_2(s) ds$.

Lemma 4 *Let $h \in C[a, b]$ and $H \in C^k(I)$ be such that for all $t \in (a, b)$, $D^k H(t) = w_0(t) \int_a^t h(s) w_1(s) ds$. Then, for $x \in (a, b)$,*

$$\limsup_{n \to \infty} \frac{D^k L_n H(x) - D^k H(x)}{\mu_n(x)} \leq \limsup_{t \to x} \frac{h(t) - h(x)}{W_2(t) - W_2(x)}$$

and

$$\liminf_{t \to x} \frac{h(t) - h(x)}{W_2(t) - W_2(x)} \leq \liminf_{n \to \infty} \frac{D^k L_n H(x) - D^k H(x)}{\mu_n(x)}.$$

Proof We shall only prove the first inequality; the other one works similarly. Let $x \in (a,b)$. We assume that there exists a real number m such that

$$\limsup_{t\to x} \frac{h(t)-h(x)}{W_2(t)-W_2(x)} < m$$

because if this is not the case there is nothing to prove. Then for some $\delta > 0$ whenever $|t-x| < \delta$,

$$\frac{h(t)-h(x)}{W_2(t)-W_2(x)} < m.$$

So for a sufficiently small δ we have

$$\frac{h(t)-h(x)}{(t-x)w_2(x)} < m.$$

Multiplying by $w_1(t)$ and integrating we have

$$\int_x^t (h(s)-h(x))\, w_1(s)ds < mw_2(x)\int_x^t (s-x)w_1(s)ds,$$

which, taking into account that

$$\frac{D^kH(t)}{w_0(t)} - \frac{D^kH(x)}{w_0(x)} = \int_x^t h(s)w_1(s)ds,$$

provides

$$\frac{D^kH(t)}{w_0(t)} - \frac{D^kH(x)}{w_0(x)} - h(x)\int_x^t w_1(s)ds < mw_2(x)\int_x^t (s-x)w_1(s)ds.$$

Multiplying now by $w_0(t)$ and considering $W_1(t) := \int_a^t w_1(s)ds$ we obtain

$$D^kH(t) - \frac{D^kH(x)w_0(t)}{w_0(x)} - h(x)\left(z_1(t) - W_1(x)w_0(t)\right)$$

$$< mw_2(x)w_0(t)\int_x^t (s-x)w_1(s)ds.$$

Equivalently, taking $y_0, y_1, Y \in C^k(I)$ such that their k-th derivatives coincide respectively with z_0, z_1 and $w_2(x)w_0(t)\int_x^t(s-x)w_1(s)ds$ in the neighborhood of the point x we are dealing with,

$$D^kH(t) - \frac{D^kH(x)D^ky_0(t)}{w_0(x)} - h(x)\left(D^ky_1(t) - W_1(x)D^ky_0(t)\right) < mD^kY(t).$$

Applying c), Lemma 2

$$D^kL_nH(x) - \frac{D^kH(x)D^kL_ny_0(x)}{w_0(x)} - h(x)\left(D^kL_ny_1(x) - W_1(x)D^kL_ny_0(x)\right)$$

$$\le mD^kL_nY(x) + o(\lambda_n^{-1}).$$

Introducing the zero terms $-D^kH(x)+\frac{D^kH(x)}{z_0(x)}D^ky_0(x)$ and $-D^ky_1(x)+D^ky_0(x)W_1(x)$ (notice that $z_1(x)=z_0(x)W_1(x)$), and regrouping,

$$D^kL_nH(x)-D^kH(x)-\frac{D^kH(x)}{z_0(x)}\left(D^kL_ny_0(x)-D^ky_0(x)\right)$$

$$-h(x)\left(D^kL_ny_1(x)-D^ky_1(x)-W_1(x)\left(D^kL_ny_0(x)-D^ky_0(x)\right)\right)$$

$$\le mD^kL_nY(x)+o(\lambda_n^{-1}).$$

Applying b), Lemma 2 to the functions y_0 and y_1, and hypothesis A) to Y,

$$\lambda_n\left(D^kL_nH(x)-D^kH(x)\right)\le mD^k\left(pD^2Y\right)(x)+o(1)$$

$$=mp(x)w_2(x)w_0(x)w_1(x)+o(1)=mp(x)+o(1),$$

where for the last equalities we have done some calculations taking into account the definitions of Y, w_2 and that p is a polynomial of degree less than or equal to 2. Finally, using (5) and taking $\limsup_{n\to\infty}$ we obtained

$$\limsup_{n\to\infty}\frac{D^kL_nH(x)-D^kH(x)}{\mu_n(x)}\le m,$$

and the proof is over. □

Theorem 1 *Let $f\in C^k(I)$ and suppose that ψ is a finitely valued function in $L_1[a,b]$ for which*

$$\liminf_{n\to\infty}\frac{D^kL_nf(x)-D^kf(x)}{\mu_n(x)}\le\psi(x)\le\limsup_{n\to\infty}\frac{D^kL_nf(x)-D^kf(x)}{\mu_n(x)}.$$

Then there exist two constants α_0 and α_1 such that for all $t\in(a,b)$,

$$D^kf(t)=\alpha_0z_0(t)+\alpha_1z_1(t)+w_0(t)\int_a^t w_1(s)\int_a^s\psi(v)w_2(v)dvds.$$

Proof Let $G\in C^k(I)$ such that for all $t\in(a,b)$

$$D^kG(t)=D^kf(t)-w_0(t)\int_a^t w_1(s)\int_a^s\psi(v)w_2(v)dvds.$$

We shall prove that D^kG is convex and concave in (a,b) with respect to z_0 and z_1.

For $q\in\mathbb{N}$ let m_q and M_q be respectively the minor and major functions of ψ with respect to w_2, such that

$$\left|m_q(t)-\int_a^t\psi(s)w_2(s)ds\right|<\frac{1}{q},\ t\in(a,b),$$

$$\left|M_q(t)-\int_a^t\psi(s)w_2(s)ds\right|<\frac{1}{q},\ t\in(a,b),$$

whose existence is well known from the theory of Lebesgue integration (see for instance [12]). In particular it follows that

$$\limsup_{t\to x} \frac{m_q(t) - m_q(x)}{W_2(t) - W_2(x)} \le \psi(x) \le \liminf_{t\to x} \frac{M_q(t) - M_q(x)}{W_2(t) - W_2(x)}.$$

From Lemma 4 and the hypothesis, if we consider $\tilde{m}_q \in C^k(I)$ such that for all $t \in (a,b)$ $D^k\tilde{m}_q(t) = w_0(t)\int_a^t m_q(s)w_1(s)ds$, we have that

$$\limsup_{n\to\infty} \frac{D^kL_n\tilde{m}_q(x) - D^k\tilde{m}_q(x)}{\mu_n(x)} \le \limsup_{t\to x} \frac{m_q(t) - m_q(x)}{W_2(t) - W_2(x)}$$

$$\le \psi(x) \le \limsup_{n\to\infty} \frac{D^kL_nf(x) - D^kf(x)}{\mu_n(x)},$$

so

$$\limsup_{n\to\infty} \frac{D^kL_n\left(f - \tilde{m}_q\right)(x) - D^k\left(f - \tilde{m}_q\right)(x)}{\mu_n(x)} \ge 0.$$

From (5) and Lemma 3, we deduce that for all $q \in \mathbb{N}$ $D^k\left(f - \tilde{m}_q\right)$ is convex in (a,b) with respect to z_0 and z_1. Letting q tend to infinity we conclude that this also holds for D^kG. Analogously from M_q we obtain that D^kG is concave in (a,b) with respect to z_0 and z_1. □

Remark This theorem recovers the converse result of b), Lemma 2 that was stated in [5]. Indeed, if $D^kL_nf(x) - D^kf(x) = o(\lambda_n^{-1})$, then $D^kL_nf(x) - D^kf(x) = o(\mu_n(x))$ and the theorem applies with $\psi \equiv 0$.

3. Applications

In this section we apply the previous result to the Bernstein and Szász-Mirakjan operators defined as follows respectively on $C[0,1]$ and $C[0,\infty)$:

$$B_nf(t) = \sum_{p=0}^{n} f\left(\frac{p}{n}\right)\binom{n}{p} t^p(1-t)^{n-p},$$

$$S_nf(t) = e^{-nt}\sum_{p=0}^{\infty} f\left(\frac{p}{n}\right)\frac{n^pt^p}{p!}.$$

It is very well-known (see [8], [10]) that they are convex of any order , i.e. B) holds true for any value of $k \in \mathbb{N}_0$. The validity of A) for B_n with $k = 0$, $\lambda_n = 2n$, $p(t) = t(1-t)$, and for S_n with $k = 0$, $\lambda_n = 2n$, $p(t) = t$ follows from classical results of Voronovskaya [14] and Szász [13]. Specifically, under the aforementioned conditions,

$$\lim_{n\to+\infty} 2n\left(B_ng(x) - g(x)\right) = x(1-x)D^2g(x), \tag{6}$$

and

$$\lim_{n\to+\infty} 2n\left(S_n g(x) - g(x)\right) = xD^2 g(x). \tag{7}$$

Roughly speaking, one can apply the differential operator D^k for any $k \in \mathbb{N}$ to both sides of the identities, what yields that A) holds true for B_n, S_n and all $k \in \mathbb{N}$ taking for λ_n and p the corresponding values above (see [4], [7], [1]).

Hence we can apply our result to these operators. The following table contains for each operator and for $k > 0$ the values of λ_n and $p(t)$, and a choice for $w_0(t)$ and $w_1(t)$. We do not apply our result to the case $k = 0$ because this can be done from [3].

	B_n	S_n
λ_n	$2n$	$2n$
$p(t)$	$t(1-t)$	t
$w_0(t)$	$1/t^{k-1}$	1
$w_1(t)$	$t^{k-2}/(1-t)^k$	$1/t^k$
$w_2(t)$	$t(1-t)^k$	t^k

From Theorem 1, the following corollaries are easily obtained.

Corollary 1 *Let $k \in \mathbb{N}$, $0 < a < b < 1$, $f \in C^k[0,1]$, $\mu_n(x) = D^k B_n \varphi_x(x)$ and suppose that ψ is a finitely valued function in $L_1[a,b]$ such that*

$$\liminf_{n\to\infty} \frac{D^k B_n f(x) - D^k f(x)}{\mu_n(x)} \le \psi(x) \le \limsup_{n\to\infty} \frac{D^k B_n f(x) - D^k f(x)}{\mu_n(x)}.$$

Then there exist two constants α_0 and α_1 such that for all $t \in (a,b)$, one has

$$D^1 f(t) = \alpha_0 + \alpha_1 \log\frac{t}{1-t} + \int_a^t \frac{1}{s(1-s)} \int_a^s \psi(v) v(1-v)\,dv\,ds,$$

for $k = 1$ and

$$D^k f(t) = \frac{\alpha_0}{t^{k-1}} + \frac{\alpha_1}{(1-t)^{k-1}} + \frac{1}{t^{k-1}} \int_a^t \frac{s^{k-2}}{(1-s)^k} \int_a^s \psi(v) v(1-v)^k\,dv\,ds,$$

for $k > 1$.

Corollary 2 *Let $k \in \mathbb{N}$, $0 < a < b$, $f \in C_B^k[0,\infty)$, $\mu_n(x) = D^k S_n \varphi_x(x)$ and suppose that ψ is a finitely valued function in $L_1[a,b]$ such that*

$$\liminf_{n\to\infty} \frac{D^k S_n f(x) - D^k f(x)}{\mu_n(x)} \le \psi(x) \le \limsup_{n\to\infty} \frac{D^k S_n f(x) - D^k f(x)}{\mu_n(x)}.$$

Then there exist two constants α_0 and α_1 such that for all $t \in (a,b)$, one has

$$D^1 f(t) = \alpha_0 + \alpha_1 \log t + \int_a^t \frac{1}{s} \int_a^s \psi(v) v\,dv\,ds,$$

for $k = 1$ and

$$D^k f(t) = \alpha_0 + \frac{\alpha_1}{t^{k-1}} + \int_a^t \frac{1}{s^k} \int_a^s \psi(v) v^k\,dv\,ds,$$

for $k > 1$.

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