Monografías de la Academia de Ciencias de Zaragoza. **20:** 9–58, (2002).

On Variation–diminishing Schoenberg Operators: New Quantitative Statements *

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Abstract

We give quantitative results for variation–diminishing splines, focusing on the case of equidistant knots. New direct inequalities are obtained, both in terms of the classical second modulus of continuity and in terms of the second Ditzian–Totik modulus. These new results are based upon a detailed analysis of the second moments and very recent theorems for positive linear operator approximation. The potential for simultaneous approximation is described by means of an estimate involving both the first and the second classical modulus of continuity. The topic of global smoothness preservation is also addressed. Furthermore, we discuss the degree of simultaneous approximation in the multivariate case, namely for Boolean sums and tensor products of Schoenberg splines.

Keywords: Variation–diminishing splines, degree of approximation, simultaneous approximation, global smoothness preservation, Boolean sums, tensor products.
2000 MSC: 41A15, 41A25, 41A28, 41A36, 41A63, 65D07, 65D17.

1. Introduction

Consider the knot sequence $\Delta_n = \{x_i\}_{-k}^{n+k}$ $(n > 0,\ k > 0)$, with

$$x_{-k} = x_{-k+1} = \ldots = x_0 = 0 < x_1 < \ldots < x_n = \ldots = x_{n+k} = 1.$$

Dedicated to Prof. D.D. Stancu (February 11, 1927) on the occasion of his 75th birthday

For a function $f \in \mathbb{R}^{[0,1]}$, the variation–diminishing spline of degree k w.r.t. Δ_n is given by

$$S_{\Delta_n,k}f(x) := \sum_{j=-k}^{n-1} f(\xi_{j,k}) \cdot N_{j,k}(x) \quad \text{for } 0 \le x < 1 \text{ and}$$
$$S_{\Delta_n,k}f(1) := \lim_{\substack{y\to 1\\ y<1}} S_{\Delta_n,k}f(y),$$

with the nodes (Greville abscissas) $\xi_{j,k} := \frac{x_{j+1} + \ldots + x_{j+k}}{k}$, $-k \le j \le n-1$, and the normalized B–splines as fundamental functions

$$N_{j,k}(x) := (x_{j+k+1} - x_j)[x_j, x_{j+1}, \ldots, x_{j+k+1}](\cdot - x)_+^k.$$

This method of approximation was introduced by Schoenberg [72] in 1965 as a "natural" extension of the classical Bernstein polynomial approximation; an important predecessor is a paper by Curry and Schoenberg [18] written in 1945 and completed by 1947, but "for no good reason" not published until 1966. One further key article on the method is one by Marsden and Schoenberg [58] which appeared in Romania, Schoenberg's native country, in 1966. Due to the early work of Marsden [55], [56] on the subject, Schoenberg's variation–diminishing splines (colloquially just denoted as "Schoenberg splines") became known to the mathematical community in the early 1970's and immediately attracted considerable interest. Before continuing these short historical remarks, we list some of their most important properties:

P1) $S_{\Delta_n,k}$ is a positive linear operator which reproduces linear functions, i.e.,

$$\sum_{j=-k}^{n-1} N_{j,k}(x) = 1, \quad 0 \le x \le 1,$$
$$\sum_{j=-k}^{n-1} \xi_{j,k} \cdot N_{j,k}(x) = x, \quad 0 \le x \le 1.$$

P2) **Theorem 1** *(see [55, Theorem 3]) A necessary and sufficient condition that*

$$\lim S_{\Delta_n,k}f(x) = f(x), \text{ uniformly in } [0,1]$$

for every $f \in C[0,1]$, is that

$$\lim \frac{\|\Delta_n\|}{k} = 0.$$

P3) $S_{\Delta_n,k}$ is a discretely defined operator, which maps $\mathbb{R}^{[0,1]}$ into that subspace of $C^{k-1}[0,1]$ containing all functions which are on each interval $[x_i, x_{i+1}]$ a polynomial of degree at most k;

P4) Besides of the Bernstein operators, $S_{\Delta_n,k}$ also generalizes piecewise linear interpolation at the knots of Δ_n;

P5) $S_{\Delta_n,k}$ has the convex hull property and interpolates at the endpoints;

P6) $S_{\Delta_n,k}$ has the variation–diminishing property, i.e., $V(S_{\Delta_n,k}f - l) \leq V(f - l)$ on $[0, 1]$, for all linear functions l, where $V(g)$ denotes the number of sign changes of the function g.

We said before that this method attracted interest in the mathematical community already in the early 1970's. The reader ought to consult the book of DeVore [21], and papers by Leviatan [53], Meyer and Thomas [60], Scherer [71] (for an L_p modification), and by Coman and Frenţiu [14], [15] (for multivariate approaches) in order to confirm our statement. An important contribution from the period 1970–1975 is due to Munteanu and Schumaker [62]. We will cite their article on several occasions in the sequel.

During the late 1970's, the 80's and the 90's further contributions concerning modifications and generalizations of Schoenberg's original method were given, both for the univariate and multivariate cases. With a few exceptions the results given there were of a positive nature. It should not be overlooked, though, that the behaviour in the vicinities of the endpoints 0 and 1 is somewhat poor due to the coalescence of the knots there. We will also discuss this below. Since the present note is not intended to be a survey paper, we have chosen to add several references to the bibliography which are not explicitely cited in the text, but should provide the reader with an idea of the continuing interest among approximation theorists. We make no claim for completeness.

However, this introduction is not yet finished. Schoenberg's variation–diminishing spline operator is in much use in Computer–Aided Geometric Design and has become an indispensible tool there. In CAGD the method has an early history of its own. In his most interesting thesis Riesenfeld [69] introduced Schoenberg splines to the field, having Gordon as his principal advisor. See [46] and [4] in order to confirm that Gordon was always the driving force behind introducing B–spline methods into CAGD at a very early stage of its development. These historical facts seem to be frequently overlooked (or neglected). For more details in regard to their use in CAGD see the books by Farin [26] and by Hoschek and Lasser [47] where more references can be found.

In the present note we will supplement the quantitative information available on Schoenberg's method. In doing so we will in part follow the organization of the Munteanu and Schumaker paper, but also cover further aspects. We will consequently use second order moduli of various types in our assertions. In the late 1960's and early 70's, that is, at the time of writing of the fundamental papers on the subject, these were quantities not too well understood and hardly ever used. The estimates given here are to the most part based upon very recent general results for positive linear and, more generally, convex operators of various orders. This will enable us to also provide new statements on the degree of simultaneous approximation for first and second order derivatives in both the univariate and certain bivariate cases. There will be an emphasis on small explicit constants.

In the foreground of our considerations will mostly be the mesh gauge (rather than the degree) of the splines. Sometimes we will restrict ourselves to the case of equidistant knots $x_j = \frac{j}{n},\ 0 \le j \le n$, because we have not found corresponding statements for the general case which reduce to the "equidistant ones" we are able to give. Throughout this paper we will always denote the $k-$th degree Schoenberg splines with equidistant knots $x_j = \frac{j}{n},\ 0 \le j \le n$, by $S_{n,k}$.

2. The second moments

As for any positive linear operator, the second moments $(S_{\Delta_n,k}(e_1-x)^2)(x),\ x \in [0,1],\ e_i(t) = t^i$ for $i \ge 0$, play an important role for the quantitative behaviour of $S_{\Delta_n,k}$. It is thus instructive to have an idea of where the graph of the function

$$[0,1] \ni x \mapsto (S_{\Delta_n,k}(e_1-x)^2)(x) \in \mathbb{R}$$

is located. For $\xi_{j,k} \le x \le \xi_{j+1,k}$ we have

$$0 \le (x-\xi_{j,k})(\xi_{j+1,k}-x) \le (S_{\Delta_n,k}(e_1-x)^2)(x) \le (B_k(e_1-x)^2)(x) = \frac{x(1-x)}{k}, \qquad (1)$$

$n, k \ge 1$, where B_k is the $k-$th Bernstein operator given by

$$B_k f(x) = \sum_{i=0}^{k} f\left(\frac{i}{k}\right)\binom{k}{i} x^i (1-x)^{k-i},\ x \in [0,1].$$

The second inequality in (1) follows from the fact that, for x fixed, the graph of $S_{\Delta_n,k}(e_1-x)^2$ lies in the convex hull of its convex control polygon. The third inequality is a consequence of an observation made by Goodman and Sharma [43, Theorem 1], namely that, for a convex function f, one has

$$S_{\Delta_n,k} f(t) \le B_k f(t),\ t \in [0,1].$$

One further exact representation is

$$(S_{\Delta_n,1}(e_1-x)^2)(x) = (x-x_j)(x_{j+1}-x),\ x \in [x_j, x_{j+1}],\ 0 \le j \le n-1. \qquad (2)$$

In the equidistant case this reduces to

$$(S_{n,1}(e_1-x)^2)(x) = \frac{\{nx\}(1-\{nx\})}{n^2},\ x \in [0,1],$$

where $y = [y] + \{y\}$, i.e.,$\{y\}$ is the fractional part of y (see [54]).

We continue to discuss the general case. As shown by DeVore [21]

$$\begin{aligned} 0 &\leq (S_{\Delta_n,k}(e_1 - x)^2)(x) \\ &= \sum_{j=-k}^{n-1} \frac{1}{k^2} \cdot \frac{1}{k-1} \sum_{1\leq r<s\leq k} (x_{j+r} - x_{j+s})^2 \cdot N_{j,k}(x) \\ &\leq \alpha^2_{\Delta_{n,k}} := \frac{1}{k} \cdot \max_{-k\leq j\leq n} (x_{j+k} - x_j)^2. \end{aligned}$$

The above equation is not very instructive. It was shown by Marsden [56] that one has

$$0 \leq (S_{\Delta_n,k}(e_1 - x)^2)(x) \leq \min\left\{\frac{1}{2k}, \frac{(k+1)\|\Delta_n\|^2}{12}\right\},\ 0 \leq x \leq 1, \tag{3}$$

where $\|\Delta_n\| := \max_j (x_{j+1} - x_j)$ is the mesh gauge.

However, the upper bound is not a pointwise one. Such pointwise bound is, for example, needed for expressing the fact that one has interpolation at the endpoints.

For the case of equidistant knots we will give such inequalities in this section. We will restrict ourselves first to a discussion of the cases $k \in \{1, 2, 3\},\ n \geq 2$ and present in detail the case $k = 3$.

To this end, we have to estimate the quantity $\dfrac{(S_{n,3}(e_1 - x)^2)(x)}{x(1-x)}$.

For the case $k = 3$ and equidistant knots we get the Greville abscissas

$$\begin{aligned} &\xi_{-3,3} = 0,\ \ \xi_{-2,3} = \frac{1}{3n},\ \ \xi_{-1,3} = \frac{1}{n}, \\ &\xi_{j,3} = x_{j+2} = \frac{j+2}{n},\ \ j = 0, \ldots, n-4, \\ &\xi_{n-3,3} = \frac{n-1}{n},\ \ \xi_{n-2,3} = 1 - \frac{1}{3n},\ \ \xi_{n-1,3} = 1. \end{aligned}$$

For $0 \leq x \leq \frac{1}{n}$ we have $S_{n,3}f(x) = \sum_{j=-3}^{0} f(\xi_{j,3}) \cdot N_{j,3}(x)$. The divided differences which we are interested in are equal to

$$\begin{aligned} \left[0,0,0,0,\frac{1}{n}\right](\cdot - t)^3_+ &= n^4 \cdot \left(\frac{1}{n} - t\right)^3_+, \\ \left[0,0,0,\frac{1}{n},\frac{2}{n}\right](\cdot - t)^3_+ &= \frac{n^4}{8} \cdot \left(\frac{2}{n} - t\right)^3_+ - n^4 \cdot \left(\frac{1}{n} - t\right)^3_+, \\ \left[0,0,\frac{1}{n},\frac{2}{n},\frac{3}{n}\right](\cdot - t)^3_+ &= \frac{n^4}{18} \cdot \left(\frac{3}{n} - t\right)^3_+ - \frac{n^4}{4} \cdot \left(\frac{2}{n} - t\right)^3_+ + \frac{n^4}{2} \cdot \left(\frac{1}{n} - t\right)^3_+, \\ \left[0,\frac{1}{n},\frac{2}{n},\frac{3}{n},\frac{4}{n}\right](\cdot - t)^3_+ &= \frac{n^4}{24} \cdot \left(\frac{4}{n} - t\right)^3_+ - \frac{n^4}{6} \cdot \left(\frac{3}{n} - t\right)^3_+ + \frac{n^4}{4} \cdot \left(\frac{2}{n} - t\right)^3_+ - \frac{n^4}{6} \cdot \left(\frac{1}{n} - t\right)^3_+. \end{aligned}$$

For $x \in [0, \frac{1}{n}]$, the first four B–splines of the basis have the form

$$\begin{aligned} N_{-3,3}(x) &= n^3 \cdot \left(\frac{1}{n^3} - \frac{3x}{n^2} + \frac{3x^2}{n} - x^3 \right), \\ N_{-2,3}(x) &= n^3 \cdot \left(\frac{3x}{n^2} - \frac{9x^2}{2n} + \frac{7x^3}{4} \right), \\ N_{-1,3}(x) &= n^3 \cdot \left(\frac{3x^2}{2n} - \frac{11x^3}{12} \right), \\ N_{0,3}(x) &= n^3 \cdot \frac{x^3}{6}. \end{aligned}$$

For $0 \le x \le \frac{1}{n}$ it follows that

$$(S_{n,3}(e_1 - x)^2)(x) = \frac{x}{3n} - \frac{nx^3}{18},$$

whence

$$\frac{(S_{n,3}(e_1 - x)^2)(x)}{x(1-x)} = \frac{\frac{1}{3n} - \frac{nx^2}{18}}{1 - x} \le \frac{\frac{1}{3n} - \frac{nx^2}{18}}{1 - \frac{1}{n}} = \frac{6 - n^2x^2}{18(n-1)} \le \frac{1}{3(n-1)}. \tag{4}$$

Analogously, for $0 \le x \le \frac{1}{n}$, one can prove that

$$\frac{(S_{n,2}(e_1 - x)^2)(x)}{x(1-x)} = \frac{\frac{1}{2n} - \frac{x}{4}}{1 - x} \le \frac{1}{2n} \quad \text{and} \tag{5}$$

$$\frac{(S_{n,1}(e_1 - x)^2)(x)}{x(1-x)} = \frac{-x + \frac{1}{n}}{1 - x} \le \frac{1}{n}. \tag{6}$$

Because of the symmetry of the B–spline basis and the symmetry of the function $(e_1 - x)^2$, we also get the above inequalities for $1 - \frac{1}{n} \le x \le 1$. Furthermore, using (3), for arbitrary k and $\frac{1}{n} \le x \le 1 - \frac{1}{n}$ there holds:

$$\frac{(S_{n,k}(e_1 - x)^2)(x)}{x(1-x)} \le \frac{(k+1) \cdot \|\Delta_n\|^2}{12x(1-x)} \le \frac{k+1}{12} \cdot \frac{1}{n^2} \cdot \max_{\frac{1}{n} \le x \le 1 - \frac{1}{n}} \frac{1}{x(1-x)} = \frac{k+1}{12(n-1)}.$$

For $k \in \{1, 2\}$ and $n \ge 2$, one has $\frac{k+1}{12(n-1)} \le \frac{1}{kn}$, thus

$$\frac{(S_{n,k}(e_1 - x)^2)(x)}{x(1-x)} \le \frac{1}{kn}, \text{ for } x \in [0, 1].$$

For $k = 3$ and $n \ge 2$, one can only get

$$\frac{(S_{n,3}(e_1 - x)^2)(x)}{x(1-x)} \le \frac{1}{3(n-1)}, \text{ for } x \in [0, 1].$$

We will consider next the case of general k and n and recall that $S_{1,k}f(x) \equiv B_k f(x)$, where $B_k f$ is the Bernstein polynomial of f of degree k. In this case we have

$$(B_k(e_1 - x)^2)(x) = \frac{x(1-x)}{k} = \frac{x(1-x)}{n+k-1} = \frac{1}{2} \cdot \frac{\min\{2x(1-x), \frac{k}{n}\}}{n+k-1}.$$

So it is natural to ask whether it is possible to find a constant $A > 0$ as small as possible such that

$$(S_{n,k}(e_1 - x)^2)(x) \leq \frac{A \cdot \min\{2x(1-x), \frac{k}{n}\}}{n+k-1} \tag{7}$$

holds for all natural n and k. We give a positive answer to this question but do not assert to find the optimal value of A in (7). Moreover, we show that it is not possible to find a positive constant B such that

$$\frac{(S_{n,k}(e_1 - x)^2)(x)}{x(1-x)} \leq \frac{B}{k(n-1)} \tag{8}$$

holds for all k and n, although this is fulfilled for $k \in \{1, 2, 3\}$ as it was proved above.

To show (7) we rely on the technique in [57]. In principle, the results presented there for the second moments are correct, but the main tool – Marsden's function $f_2(x, y)$ in [57, p. 1089] – is not uniquely defined in certain cases. The corrections have recently been made by us together with a new proof which will be presented elsewhere. Here we will restrict ourselves to state the correct definitions. Following [57] we denote

$$E_2(x) := (S_{n,k}(e_1 - x)^2)(x) = \sum_{j=-k}^{n-1} \frac{f_2(\xi_{j,k})}{k-1} \cdot N_{j,k}(x), \; k \geq 2, \tag{9}$$

where

$$\begin{aligned} f_2(\xi_{j,k}) &:= \xi_{j,k}^2 - \eta_{j,k}, \\ \eta_{j,k} &:= \binom{k}{2}^{-1} \cdot \sum_{j<i_1<i_2<j+k+1} x_{i_1} \cdot x_{i_2}, \end{aligned}$$

(see [57, p. 1084], where to our notations $\xi_{j,k}$, $\eta_{j,k}$ and $f_2(\xi_{j,k})$ correspond there ξ_j, $\xi_{j,k}$ and $f_2(x, \xi_j)$, respectively; the function f_2 does not actually depend on x).

The crucial representation is

$$g_2(y) := \frac{f_2(y)}{k-1} \text{ for } y \in [0,1], \; k \geq 2, \tag{10}$$

where the corrected form of g_2 (not to be confused with Marsden's g_2) is

$$g_2(y) = \begin{cases} \frac{1}{k-1} \cdot \left(-y^2 + \frac{1}{3} \cdot y\sqrt{8\frac{k}{n} \cdot y + \frac{1}{n^2}}\right), & \text{for } 0 \leq y \leq \min\left\{\frac{k+1}{2n}, \frac{n-1}{2k}\right\}, \\ \frac{1}{k-1} \cdot \left(y - y^2 - \frac{n^2-1}{6nk}\right), & \text{for } \frac{n-1}{2k} \leq y \leq \frac{1}{2}, \\ \frac{1}{k-1} \cdot \frac{(k+1)(k-1)}{12n^2}, & \text{for } \frac{k+1}{2n} \leq y \leq \frac{1}{2}, \\ g_2(1-y), & \text{for } \frac{1}{2} \leq y \leq 1. \end{cases}$$

The function g_2 is continuous on $[0, 1]$.

In order to compute a constant A in (7) we have to consider in the sequel the following three cases (for $n \geq 2$, because for $n = 1$ we get the Bernstein operator, which was already discussed):

Case 1: $k = n - 1$

$$g_2(y) = \begin{cases} \frac{1}{k-1} \cdot \left(-y^2 + \frac{1}{3} \cdot y \sqrt{8\frac{k}{n} \cdot y + \frac{1}{n^2}} \right), & \text{for } 0 \leq y \leq \frac{1}{2}, \\ g_2(1-y), & \text{for } \frac{1}{2} \leq y \leq 1. \end{cases}$$

Case 2: $k < n - 1$

$$g_2(y) = \begin{cases} \frac{1}{k-1} \cdot \left(-y^2 + \frac{1}{3} \cdot y \sqrt{8\frac{k}{n} \cdot y + \frac{1}{n^2}} \right), & \text{for } 0 \leq y \leq \frac{k+1}{2n}, \\ \frac{1}{k-1} \cdot \frac{(k+1)(k-1)}{12n^2}, & \text{for } \frac{k+1}{2n} \leq y \leq \frac{1}{2}, \\ g_2(1-y), & \text{for } \frac{1}{2} \leq y \leq 1. \end{cases}$$

Case 3: $k > n - 1$

$$g_2(y) = \begin{cases} \frac{1}{k-1} \cdot \left(-y^2 + \frac{1}{3} \cdot y \sqrt{8\frac{k}{n} \cdot y + \frac{1}{n^2}} \right), & \text{for } 0 \leq y \leq \frac{n-1}{2k}, \\ \frac{1}{k-1} \cdot \left(y - y^2 - \frac{n^2-1}{6nk} \right), & \text{for } \frac{n-1}{2k} \leq y \leq \frac{1}{2}, \\ g_2(1-y), & \text{for } \frac{1}{2} \leq y \leq 1. \end{cases}$$

We have now the necessary ingredients in order to prove the following

Theorem 2 *For* $n \geq 1,\ k \geq 1,\ x \in [0,1]$ *we have*

$$(S_{n,k}(e_1 - x)^2)(x) \leq 1 \cdot \frac{\min\{2x(1-x), \frac{k}{n}\}}{n+k-1}.$$

Proof:

For brevity we write again $E_2(x) = (S_{n,k}(e_1 - x)^2)(x)$.

a) $n = 1,\ k \geq 1$. This is the Bernstein operator case in which we have

$$E_2(x) = \frac{1}{2} \cdot \frac{\min\{2x(1-x), \frac{k}{n}\}}{n+k-1}.$$

b) $n \geq 2,\ k = 1$. This is piecewise linear interpolation at $\frac{l}{n},\ 0 \leq l \leq n$. Here (see (2)),

$$E_2(x) = \left(x - \frac{l}{n} \right) \left(\frac{l+1}{n} - x \right) \quad \text{for } x \in \left[\frac{l}{n}, \frac{l+1}{n} \right].$$

For $l = 0$ one has

$$E_2(x) = x \left(\frac{1}{n} - x \right) \leq E_2 \left(\frac{1}{2n} \right) = \frac{1}{4n^2} \leq \frac{1}{2} \cdot \frac{\min\{2x(1-x), \frac{1}{n}\}}{n} \quad \text{for } x \in \left[0, \frac{1}{n} \right].$$

For $1 \le l \le n-2$, i.e., $x \in \left[\frac{1}{n}, \frac{n-1}{n}\right]$, we have

$$E_2(x) \le \frac{1}{4n^2} \le \frac{1}{2n^2} = \frac{1}{2} \cdot \frac{\min\{2x(1-x), \frac{1}{n}\}}{n}.$$

The case $l = n-1$ is symmetric to $l = 0$.

c) $n \ge 2,\ k \in \{2,3\}$. First we observe that

$$\min\left\{2x(1-x), \frac{k}{n}\right\} = \begin{cases} 2x(1-x), & \text{for } 0 \le x \le \frac{1}{n}, \\ 2x(1-x), & \text{for } \frac{1}{n} \le x \le \frac{1}{2} \text{ and } 2 \le n \le 2k, \\ 2x(1-x), & \text{for } \frac{1}{n} \le x \le \frac{k}{n\left(1+\sqrt{1-\frac{2k}{n}}\right)} \text{ and } n \ge 2k+1, \\ \frac{k}{n}, & \text{for } \frac{k}{n\left(1+\sqrt{1-\frac{2k}{n}}\right)} \le x \le \frac{1}{2} \text{ and } n \ge 2k+1. \end{cases}$$

Case 1: $0 \le x \le \frac{1}{n}$

From (4) and (5) it follows

$$\begin{aligned} E_2(x) &\le 2x(1-x) \cdot \frac{1}{kn} \cdot \frac{1}{2(1-x)} \le 2x(1-x) \cdot \frac{1}{kn} \cdot \frac{1}{2(1-\frac{1}{n})} \\ &= \frac{2x(1-x)}{n+k-1} \cdot \frac{n+k-1}{2k(n-1)}. \end{aligned}$$

We need now a constant a such that for all $n \ge 2$ and $k \in \{2,3\}$ there holds

$$\frac{n+k-1}{2k(n-1)} \le a$$

which is equivalent to

$$n \ge \frac{k-1+2ak}{2ak-1}.$$

We impose

$$\frac{k-1+2ak}{2ak-1} = 2,$$

which implies $a = \frac{k+1}{2k}$. Thus we get $a = \frac{3}{4}$ for $k = 2$, and $a = \frac{2}{3}$ for $k = 3$, respectively.

Case 2: $(\frac{1}{n} \le x \le \frac{1}{2}$ and $2 \le n \le 2k)$ or $(\frac{1}{n} \le x \le \frac{k}{n\left(1+\sqrt{1-\frac{2k}{n}}\right)}$ and $n \ge 2k+1)$.

Under these assumptions we can always write

$$\frac{2\frac{1}{n}\left(1-\frac{1}{n}\right)}{n+k-1} \le \frac{2x(1-x)}{n+k-1}.$$

From Marsden's paper [57] we know that

$$E_2(x) \le \frac{k+1}{12n^2}. \tag{11}$$

We need now to find again a constant a such that the inequality

$$\frac{k+1}{12n^2} \leq a \cdot \frac{2(n-1)}{n^2(n+k-1)}$$

holds for all $n \geq 2$ and $k \in \{2,3\}$. We thus get $a = \frac{k^2+2k+1}{24}$, which gives $a = \frac{3}{8}$ for $k = 2$, and $a = \frac{2}{3}$ for $k = 3$.

<u>Case 3</u>: $\frac{k}{n\left(1+\sqrt{1-\frac{2k}{n}}\right)} \leq x \leq \frac{1}{2}$ and $n \geq 2k+1$.
Inequality (11) holds also in this case. We require

$$\frac{k+1}{12n^2} \leq a \cdot \frac{k}{n(n+k-1)}$$

for $n \geq 2$, which leads to $a = \frac{(k+1)^2}{24k}$. Whence we get $a = \frac{3}{16}$ for $k = 2$, and $a = \frac{2}{9}$ for $k = 3$.

Taking the maximum over all the constants a which have been computed, we find that for $n \geq 2$ and $k \in \{2,3\}$

$$E_2(x) \leq \frac{3}{4} \cdot \frac{\min\{2x(1-x), \frac{k}{n}\}}{n+k-1}.$$

d) $n \geq 2,\ k \geq 4$. Here we proceed differently using the function g_2 from above. It is our aim to show that in this case we have

$$g_2(y) \leq h_2(y) := \frac{\min\{2y(1-y), \frac{k}{n}\}}{n+k-1},\ y \in [0,1]. \tag{12}$$

Since $h_2(y)$ is a concave function, one has

$$S_{n,k}(h_2(\cdot); y) \leq h_2(y), \quad y \in [0,1].$$

Due to the positivity of $S_{n,k}$ we also have

$$S_{n,k}(g_2(\cdot); y) \leq S_{n,k}(h_2(\cdot); y), \quad \text{or}$$

$$\sum_{j=-k}^{n-1} \frac{f_2(\xi_{j,k})}{k-1} \cdot N_{j,k}(y) \leq h_2(y) \quad \text{for all } y \in [0,1]. \tag{13}$$

Setting $y = x$ and combining (9), (12) and (13) then shows that for $n \geq 2$ and $k \geq 4$

$$E_2(x) \leq 1 \cdot \frac{\min\{2x(1-x), \frac{k}{n}\}}{n+k-1}.$$

Hence it remains to prove (12).

<u>Case $k = n-1$</u>:

We may consider $n \geq 5$, because $n < 5$ (that is $k \leq 3$) was considered already. Since

$$8\frac{k}{n}y + \frac{1}{n^2} \leq 4\frac{n-1}{n} + \frac{1}{n^2} = \left(2 - \frac{1}{n}\right)^2$$

we obtain

$$g_2(y) \leq \frac{1}{k-1}\left[-y^2 + \frac{y}{3} \cdot \left(2 - \frac{1}{n}\right)\right] \leq \frac{y(1-y)}{3(k-1)}\left(2 - \frac{1}{n}\right) = \frac{y(1-y)}{n-1} \cdot a(n),$$

where

$$a(n) := \frac{n-1}{n-2} \cdot \frac{1}{3} \cdot \left(2 - \frac{1}{n}\right) \leq a(5) = \frac{4}{5}.$$

Finally it follows

$$g_2(y) \leq \frac{4}{5} \cdot \frac{y(1-y)}{n-1} = \frac{4}{5} \cdot \frac{\min\{2y(1-y), \frac{k}{n}\}}{n+k-1} \tag{14}$$

for all $y \in [0,1]$.

Case $k < n-1$:

We may consider $n \geq 6$, because $k \geq 4$.

For $0 \leq y \leq \frac{k+1}{2n}$ we obtain successively

$$\begin{aligned} g_2(y) &\leq \frac{1}{k-1}\left(-y^2 + \frac{y}{3}\sqrt{8\frac{k}{n} \cdot \frac{k+1}{2n} + \frac{1}{n^2}}\right) = \frac{1}{k-1}\left(-y^2 + \frac{y}{3} \cdot \frac{2k+1}{n}\right) \\ &\leq \frac{y(1-y)}{k-1} \cdot \frac{2k+1}{3n} = \frac{2y(1-y)}{n+k-1} \cdot \left[\left(\frac{n}{k-1} + 1\right) \cdot \frac{2k+1}{6n}\right] \\ &\leq \frac{2y(1-y)}{n+k-1} \cdot \left[\frac{1}{3} + \frac{1}{2(k-1)} + \frac{2n-1}{6n}\right] \leq \frac{29}{36} \cdot \frac{2y(1-y)}{n+k-1}. \end{aligned}$$

For $0 \leq y \leq \frac{k+1}{2n}$ we also have

$$g_2(y) \leq \frac{y(1-y)}{k-1} \cdot \frac{2k+1}{3n} \leq \frac{35}{64} \cdot \frac{k}{n(n+k-1)}.$$

We conclude now that for $0 \leq y \leq \frac{k+1}{2n}$ one has

$$g_2(y) \leq \max\left\{\frac{29}{36}, \frac{35}{64}\right\} \cdot \frac{\min\{2y(1-y), \frac{k}{n}\}}{n+k-1} = \frac{29}{36} \cdot \frac{\min\{2y(1-y), \frac{k}{n}\}}{n+k-1}. \tag{15}$$

Further we consider $y \in [\frac{k+1}{2n}, \frac{1}{2}]$ and observe that for the continuous function g_2 we can write

$$g_2(y) = g_2\left(\frac{k+1}{2n}\right) = \lim_{z \nearrow \frac{k+1}{2n}} g_2(z) \leq \lim_{z \nearrow \frac{k+1}{2n}} \frac{29}{36} \cdot \frac{\min\{2y(1-y), \frac{k}{n}\}}{n+k-1},$$

for all $y \in [\frac{k+1}{2n}, \frac{1}{2}]$.

Hence (15) holds for $y \in [0,1]$.

Case $k > n-1$:
It follows immediately that $\min\{2y(1-y), \frac{k}{n}\} = 2y(1-y)$.
For $0 \le y \le \frac{n-1}{2k}$ we obtain successively

$$\begin{aligned} g_2(y) &\le \frac{1}{k-1}\left(-y^2 + \frac{y}{3}\sqrt{8\frac{k}{n}\cdot\frac{n-1}{2k} + \frac{1}{n^2}}\right) \\ &= \frac{1}{k-1}\left[-y^2 + \frac{y}{3}\left(2-\frac{1}{n}\right)\right] \le \frac{y(1-y)}{k-1}\frac{1}{3}\left(2-\frac{1}{n}\right) \\ &= \frac{2y(1-y)}{n+k-1}\cdot\frac{1}{6}\left(2-\frac{1}{n}\right)\left(1+\frac{n}{k-1}\right) \\ &\le \frac{7}{9}\cdot\frac{2y(1-y)}{n+k-1}. \end{aligned}$$

If $y \in [\frac{n-1}{2k}, \frac{1}{2}]$ then we have

$$g_2(y) = \frac{1}{k-1}\left(y - y^2 - \frac{n^2-1}{6nk}\right).$$

We have to show that

$$\frac{1}{k-1}\left(y - y^2 - \frac{n^2-1}{6nk}\right) \le \frac{2y(1-y)}{n+k-1},$$

the latter being equivalent to

$$\frac{n-k+1}{n+k-1}\cdot y(1-y) \le \frac{n^2-1}{6nk}.$$

For $y \in [\frac{n-1}{2k}, \frac{1}{2}]$ the left hand side does not exceed

$$\frac{n-k+1}{4(n+k-1)} \le \frac{n^2-1}{6nk} \quad \text{for } n \ge 2 \text{ and } k > n-1.$$

Thus

$$g_2(y) \le 1\cdot\frac{2y(1-y)}{n+k-1} \tag{16}$$

for all $y \in [0,1]$, and the proof of Theorem 2 is complete. □

Remark 3 *We prove here that (8) is not possible. Suppose (8) holds for some $B > 0$. From [57, Theorem 2] we get*

$$\lim_{(n+k+1)\to\infty} (n+k+1)(S_{n,k}(e_1-x)^2)(x) = \frac{1}{12}t(t+1),$$

for $t := \lim_{(n+k+1)\to\infty} \frac{k}{n}$, $0 \le t \le 1$ *and* $\frac{t}{2} \le x \le 1-\frac{t}{2}$.

For $k = n$ we obtain $t = 1$ and $x = \frac{1}{2}$. In this case one gets

$$\frac{B}{4} \cdot \lim_{(n+k+1)\to\infty} \left[\frac{n+k+1}{k(n-1)}\right] = \frac{B}{4} \cdot \lim_{n\to\infty} \left[\frac{2n+1}{n(n-1)}\right] \geq \frac{1}{12} \cdot 2 = \frac{1}{6}.$$

The latter inequality is not true. □

3. New direct inequalities

In this section we prove direct inequalities for arbitrary functions in $C[0,1]$.

3.1 Uniform estimates in terms of the classical second order modulus of smoothness

While there are many estimates in terms of the first order modulus of smoothness available in the literature – starting with the ones by Marsden and Schoenberg ([72], [58], [55], [56]) and by Munteanu and Schumaker [62] – the first estimates with ω_2 were given by Esser in [25] and later further improved by Gonska [31]. One advantage of the use of ω_2 is the fact that this quantity annihilates linear functions. The desirability to have estimates in terms of such a quantity was already observed at the end of the paper by Marsden and Schoenberg [58] where

$$\omega_1^*(f;\delta) := \inf_{c\in\mathbb{R}} \omega_1(f - ce_1;\delta)$$

was used. As was noted by one of the present authors in [33, p. 17] there is no constant $c > 0$ such that

$$\omega_1^*(f;\delta) \leq c \cdot \omega_2(f;\delta) \text{ for all } f \in C[0,1] \text{ and all } \delta > 0.$$

The following elegant general result of Păltănea is the key for our subsequent applications to Schoenberg splines.

Lemma 4 *(see [68, Corollary 3.1]) Let $K = [a,b]$ be a compact interval of the real axis and K' a compact subinterval of K. If $L : C(K) \to C(K')$ is a positive linear operator, then for $f \in C(K)$, $x \in K'$, and each $0 < h \leq \frac{1}{2}$length(K), the following holds:*

$$\begin{aligned} |(Lf)(x) - f(x)| \;\leq\; & |(Le_0)(x) - 1| \cdot |f(x)| + \frac{1}{h} \cdot |(L(e_1 - x))(x)| \cdot \omega_1(f;h) \\ & + \left[(Le_0)(x) + \frac{1}{2h^2} \cdot (L(e_1 - x)^2)(x)\right] \cdot \omega_2(f;h). \end{aligned} \quad (17)$$

Remark 5 *Condition $h \leq \frac{1}{2} \cdot$length(K) in the above can be eliminated for operators which preserve linear functions.*

Thus we can state

Theorem 6 *For all $f \in C[0,1]$, $x \in [0,1]$ and $h > 0$, there holds*

$$|S_{\Delta_n,k}f(x) - f(x)| \leq \left(1 + \frac{1}{2h^2} \cdot \min\left\{\frac{1}{2k}, \frac{(k+1)\|\Delta_n\|^2}{12}\right\}\right) \cdot \omega_2(f;h). \tag{18}$$

Proof:

Applying (17) and taking into account that $S_{\Delta_n,k}$ reproduces linear functions yields

$$|S_{\Delta_n,k}f(x) - f(x)| \leq \left[1 + \frac{1}{2h^2} \cdot (S_{\Delta_n,k}(e_1 - x)^2)(x)\right] \cdot \omega_2(f;h).$$

Since

$$(S_{\Delta_n,k}(e_1 - x)^2)(x) \leq \min\left\{\frac{1}{2k}, \frac{(k+1)\|\Delta_n\|^2}{12}\right\},$$

the statement of our theorem follows. □

To achieve the goal of this subsection there are two meaningful choices for the parameter h in (18), namely in terms of the degree k and in terms of the mesh gauge $\|\Delta_n\|$. A direct application of Theorem 6 yields in these cases:

Corollary 7 *For all $f \in C[0,1]$, $x \in [0,1]$, one has the following uniform estimates*

$$\|S_{\Delta_n,k}f - f\|_\infty \leq \frac{5}{4} \cdot \omega_2\left(f; \frac{1}{\sqrt{k}}\right), \text{ and} \tag{19}$$

$$\|S_{\Delta_n,k}f - f\|_\infty \leq \left(1 + \frac{k+1}{24}\right) \cdot \omega_2\left(f; \|\Delta_n\|\right). \tag{20}$$

Remark 8 *From (19) and (20), using the properties of the moduli, one gets for $f \in C^1[0,1]$, $x \in [0,1]$, that*

$$\|S_{\Delta_n,k}f - f\|_\infty \leq \frac{5}{4\sqrt{k}} \cdot \omega_1\left(f'; \frac{1}{\sqrt{k}}\right), \text{ and} \tag{21}$$

$$\|S_{\Delta_n,k}f - f\|_\infty \leq \left(1 + \frac{k+1}{24}\right) \cdot \|\Delta_n\| \cdot \omega_1\left(f'; \|\Delta_n\|\right), \tag{22}$$

respectively.

We listed inequalities (21) and (22) here, because they improve the corresponding ones by Munteanu and Schumaker [62, (2.19) and (2.18), respectively] (the second one, however, only for $k \geq 2$).

The inequality

$$(S_{\Delta_n,k}(e_1 - x)^2)(x) \leq \min\left\{\frac{1}{2k}, \frac{(k+1)\|\Delta_n\|^2}{12}\right\}$$

is not quite satisfactory because it does not reflect the fact that

$$(S_{\Delta_n,k}(e_1 - x)^2)(x) = 0 \text{ for } x \in \{0,1\}.$$

For the case of equidistant knots the situation is different as we showed in Section 2. The pointwise inequalities from there will be employed in Sections 3.2 and 3.3.

In the sequel we use the following particular case of a very recent result by Gonska and Păltănea:

Lemma 9 *(see [37]) If $L : C[0,1] \to C[0,1]$ is a linear positive operator reproducing linear functions, then we have*

$$|L(f,x) - f(x)| \le \left[1 + \frac{7}{4} \cdot \frac{(L(e_1 - x)^2)(x)}{(h\varphi(x))^2}\right] \omega_2^{\varphi}(f;h), \tag{23}$$

for all $f \in C[0,1]$, $x \in (0,1)$ and $h \in (0,1]$.

Here

$$\omega_2^{\varphi}(f;h) = \sup\{|\Delta^2_{\rho\varphi(x)} f(x)|, \;\; x \pm \rho\varphi(x) \in [0,1], \;\; 0 < \rho \le h\}$$

is the second order Ditzian–Totik modulus, with $\varphi(x) = \sqrt{x(1-x)}$ and $\Delta^2_{\eta} f(y) = f(y-\eta) - 2f(y) + f(y+\eta)$, if $\eta > 0$, $y \pm \eta \in [0,1]$, $f \in \mathbb{R}^{[0,1]}$.

Applying Lemma 9 we first consider the three cases in which we have an exact representation of $(S_{n,k}(e_1 - x)^2)(x)$ close to the endpoints.

Theorem 10 *For all $f \in C[0,1]$, $x \in [0,1]$, $h \in (0,1]$ and $n \ge 2$, one has*

$$|S_{n,3} f(x) - f(x)| \;\le\; \left[1 + \frac{7}{4} \cdot \frac{1}{h^2} \cdot \frac{1}{3(n-1)}\right] \omega_2^{\varphi}(f;h), \text{ and}$$

$$|S_{n,k} f(x) - f(x)| \;\le\; \left[1 + \frac{7}{4} \cdot \frac{1}{h^2} \cdot \frac{1}{kn}\right] \omega_2^{\varphi}(f;h), \text{ for } k \in \{1,2\}.$$

Thus it follows immediately

Corollary 11

$$\|S_{n,3} f - f\|_{\infty} \;\le\; \frac{19}{12} \cdot \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n-1}}\right),$$

$$\|S_{n,2} f - f\|_{\infty} \;\le\; \frac{15}{8} \cdot \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right), \text{ and}$$

$$\|S_{n,1} f - f\|_{\infty} \;\le\; \frac{11}{4} \cdot \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right).$$

In the general case we apply again Lemma 9 and use Theorem 2.

Theorem 12 *For all $f \in C[0,1]$, $x \in [0,1]$, $h \in (0,1]$ and $n,k \ge 1$ one has:*

i) If $\frac{k}{n} \ge \frac{1}{2}$, then

$$|S_{n,k} f(x) - f(x)| \le \left[1 + \frac{7}{2} \cdot \frac{1}{h^2(n+k-1)}\right] \cdot \omega_2^{\varphi}(f;h).$$

ii) If $\frac{k}{n} < \frac{1}{2}$, then

$$|S_{n,k}f(x) - f(x)| \leq \left[1 + \frac{7}{4} \cdot \frac{k}{h^2 \cdot n(n+k-1)}\right] \cdot \omega_2^{\varphi}(f;h).$$

In particular we get

Corollary 13 *i) If $\frac{k}{n} \geq \frac{1}{2}$, then*

$$\begin{aligned} \|S_{n,k}f - f\|_\infty &\leq \left[1 + \frac{7}{2} \cdot \frac{n}{n+k-1}\right] \cdot \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right), \\ \|S_{n,k}f - f\|_\infty &\leq \left[1 + \frac{7}{2} \cdot \frac{k}{n+k-1}\right] \cdot \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{k}}\right), \\ \|S_{n,k}f - f\|_\infty &\leq \frac{9}{2} \cdot \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n+k-1}}\right). \end{aligned}$$

ii) If $\frac{k}{n} < \frac{1}{2}$, then

$$\begin{aligned} \|S_{n,k}f - f\|_\infty &\leq \left[1 + \frac{7}{4} \cdot \frac{k}{n+k-1}\right] \cdot \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{n}}\right), \\ \|S_{n,k}f - f\|_\infty &\leq \left[1 + \frac{7}{4} \cdot \frac{k^2}{n(n+k-1)}\right] \cdot \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{k}}\right), \\ \|S_{n,k}f - f\|_\infty &\leq \frac{11}{4} \cdot \omega_2^{\varphi}\left(f; \sqrt{\frac{k}{n(n+k-1)}}\right). \end{aligned}$$

Remark 14 *We recall here that Schoenberg's original intention was to introduce a natural "spline extension" of the Bernstein polynomials. This was definitely achieved. Since then impressive progress was made in the investigation of Bernstein operators. One particular highlight is the result of Knoop and Zhou [51]. They showed that for the second order Ditzian–Totik modulus one has*

$$\|B_k f - f\|_\infty \approx \omega_2^{\varphi}\left(f; \frac{1}{\sqrt{k}}\right), \ k \to \infty.$$

The authors are not aware of any corresponding result for the $S_{\Delta_{n},k}$'s which generalizes the assertion of Knoop and Zhou. We feel that the proof of a strong converse inequality (in what form soever) would be a significant and most valuable contribution to both Approximation Theory and CAGD.

3.3 Pointwise inequalities

For the case of equidistant knots Lemma 4 can also be used to give pointwise inequalities. We have

Theorem 15 *For all $f \in C[0,1]$, $x \in [0,1]$, $h \in (0,1]$ and $n, k \geq 1$ one has:*

$$|S_{n,k}f(x) - f(x)| \leq \left[1 + \frac{1}{2h^2} \cdot \frac{\min\{2x(1-x); \frac{k}{n}\}}{n+k-1}\right] \cdot \omega_2(f; h). \tag{24}$$

In particular, one has

$$|S_{n,k}f(x) - f(x)| \leq \frac{3}{2} \cdot \omega_2\left(f; \sqrt{\frac{2x(1-x)}{n+k-1}}\right). \tag{25}$$

Proof:

The first inequality is an immediate consequence of Theorem 2 and Lemma 4; for the second one we consider two cases:

Case 1: $\frac{k}{n} \geq \frac{1}{2}$. In this case we have for $x \in [0,1]$

$$\min\left\{2x(1-x); \frac{k}{n}\right\} = 2x(1-x).$$

Putting $h = \sqrt{\frac{2x(1-x)}{n+k-1}} \leq 1$ yields

$$|S_{n,k}f(x) - f(x)| \leq \frac{3}{2} \cdot \omega_2\left(f; \sqrt{\frac{2x(1-x)}{n+k-1}}\right). \tag{26}$$

Case 2: $\frac{k}{n} < \frac{1}{2}$.

Depending on the position of x we have two possibilities.

For $x \in \left[0, \frac{1}{2}\left(1 - \sqrt{1 - \frac{2k}{n}}\right)\right] \cup \left[\frac{1}{2}\left(1 + \sqrt{1 - \frac{2k}{n}}\right), 1\right]$ it follows $\min\left\{2x(1-x); \frac{k}{n}\right\} = 2x(1-x)$, thus (26) also holds in this case.

For $x \in \left(\frac{1}{2}\left(1 - \sqrt{1 - \frac{2k}{n}}\right), \frac{1}{2}\left(1 + \sqrt{1 - \frac{2k}{n}}\right)\right)$ it follows $\min\left\{2x(1-x); \frac{k}{n}\right\} = \frac{k}{n}$. In this case Theorem 15 implies

$$|S_{n,k}f(x) - f(x)| \leq \left[1 + \frac{1}{2h^2} \cdot \frac{k}{n(n+k-1)}\right] \cdot \omega_2(f; h). \tag{27}$$

Setting $h = \sqrt{\frac{k}{n(n+k-1)}}$ we obtain

$$|S_{n,k}f(x) - f(x)| \leq \frac{3}{2} \cdot \omega_2\left(f; \sqrt{\frac{k}{n(n+k-1)}}\right) \leq \frac{3}{2} \cdot \omega_2\left(f; \sqrt{\frac{2x(1-x)}{n+k-1}}\right). \tag{28}$$

This concludes the proof. □

Remark 16 *(i) Case 1 in the proof of Theorem 15, namely $\frac{k}{n} \geq \frac{1}{2}$, is the one similar to that of the Bernstein operators B_k. For these we obtain*

$$|B_k f(x) - f(x)| \leq \frac{3}{2} \cdot \omega_2\left(f; \sqrt{\frac{2x(1-x)}{k}}\right).$$

(ii) *The case* $\frac{k}{n} < \frac{1}{2}$ *(where the piecewise polynomial degree* k *is small in comparison to the number* $n-1$ *of interior knots) is more similar to "true spline interpolation". In the middle of the interval* $[0,1]$ *we used the inequality* $\frac{k}{n} \leq 2x(1-x)$ *in order to arrive at (25), thus loosing one power of* $\frac{1}{n}$ *under the square root in the special situation* $\frac{k}{n} = \frac{k_n}{n} = \mathcal{O}(\frac{1}{n}),\ n \to \infty$.

There is a second possibility to prove pointwise inequalities. This bridges the gap between pointwise ones in terms of the classical second modulus and uniform estimates using the Ditzian–Totik modulus. In order to indicate what can be done in this direction, we give without proof the following

Theorem 17 *Under the conditions of Theorem 15 one has*

$$|S_{n,k}f(x) - f(x)| \leq 2 \cdot c\left(\lambda, \left(\frac{1}{2}\right)^{1-\lambda}\right) \cdot \omega_2^{\varphi^\lambda}\left(f; \frac{\varphi(x)^{1-\lambda}}{\sqrt{n+k-1}}\right).$$

Here, $\varphi(x) = \sqrt{x(1-x)},\ 0 \leq \lambda \leq 1$*, the constant* $c(\lambda, t_0)$ *is chosen such that* $K_2^{\varphi^\lambda}(f; t^2) \leq c(\lambda, t_0) \cdot \omega_2^{\varphi^\lambda}(f; t)$ *for* $0 \leq t \leq t_0$*;* $K_2^{\varphi^\lambda}(f; t^2) := \inf\{\|f - g\|_\infty + t^2 \cdot \|\varphi^{2\lambda} \cdot g''\|_\infty\},\ t \geq 0$*, where the infimum is taken over all* g *such that* $g' \in AC_{loc}[0,1]$ *and* $\|\varphi^{2\lambda} \cdot g''\|_\infty < \infty$*, and* $\omega_2^{\varphi^\lambda}(f; t) := \sup_{0 \leq h \leq t} \|\Delta^2_{h\varphi^\lambda} f\|_\infty$ *with*

$$\Delta^2_{h\varphi^\lambda} f(x) := \begin{cases} f(x - h\varphi^\lambda(x)) - 2f(x) + f(x + h\varphi^\lambda(x)), & \text{if } [x - h\varphi^\lambda(x), x + h\varphi^\lambda(x)] \subseteq [0,1]; \\ 0, & \text{otherwise.} \end{cases}$$

For details on the technique employed here see [38, Theorem 4.1] or [22, 27, 28]. Refinements of Theorem 17 are possible.

4. Approximation of derivatives

While, for functions $f \in C^r[0,1]$, the rth derivatives of the corresponding Bernstein polynomials converge uniformly to the rth derivative of the function f on $[0,1]$, this fact does not hold for variation–diminishing spline approximations in general, except for $r = 1$ (see, for example, [55, Theorem 9]).

For the special case of equidistant knots,

$$n > 1 \text{ and } x_j = \frac{j}{n},\ 0 \leq j \leq n,$$

Marsden [55, Section 10] noted that the pth derivative $(1 < p \leq r)$ of the spline approximation of degree k to $f(x)$ converges to $f^{(p)}(x)$ as

$$\frac{\|\Delta_n\|}{k} \to 0 \ \text{ or, equivalently } \ k + n \to \infty$$

if and only if $0 < x < 1$. The convergence is uniform on compact subintervals of $(0,1)$.

Of course, the latter statement assumes that all quantities in question are defined. For example, one needs $k-1 \geq p$ here in order to have sufficiently many derivatives of the splines available.

In the sequel we present quantitative estimates concerning the degree of simultaneous approximation for the first and second derivative. In order to do so, we need some general settings.

Again $K = [a,b]$ is a compact interval of the real axis and $K' \subset K$. We consider the Banach space $X = C^r(K)$ endowed with the norm $\|g\|_X := \max_{0\leq j\leq r}(\|D^j g\|_K)$. Here $\|\cdot\|_K$ denotes the Chebyshev norm in $C(K) := C^0(K)$ and D^j is the $j-$th differential operator.

Let $\mathcal{K}_K^i := \{f \in C(K) : [x_0,\ldots,x_i;f] \geq 0 \text{ for any } x_0 < \ldots < x_i \in K\}$, where $[x_0,\ldots,x_i;f]$ is an $i-$th order divided difference of f. Note that $\mathcal{K}_K^0$ is the set of all positive functions on K, $\mathcal{K}_K^1$ is the set of non–decreasing functions, and $\mathcal{K}_K^2$ represents the usual convex functions on the same interval.

Knoop and Pottinger [50] generalized the convexity notion for operators as follows: an operator $L : V \to C(K')$ is called almost convex of order $r-1$ $(r \geq 0)$ if there exist $p \geq 0$ integers i_j, $1 \leq j \leq p$, satisfying $0 \leq i_1 < \ldots < i_p < r$ such that

$$f \in \left(\bigcap_{j=1}^{p} \mathcal{K}_K^{i_j}\right) \cap \mathcal{K}_K^r \cap V \text{ implies } Lf \in \mathcal{K}_{K'}^r.$$

Here, the empty intersection $\left(\bigcap_{j=1}^{0}\ldots\right)$ is taken by definition to be the entire subspace V.

The main result that we use in the sequel is the following quantitative Korovkin–type theorem on simultaneous approximation given by Kacsó (see [48], [49]), improving an earlier similar result of Gonska [32]:

Theorem 18 *Let $r \in \mathbb{N}_0$ and the operator $L : C^r(K) \to C^r(K')$ be almost convex of order $r-1$. If $L(\Pi_{r-1}) \subseteq \Pi_{r-1}$, then for all $f \in C^r(K)$, $x \in K'$ and $0 < h \leq \frac{1}{2}$length(K) there holds:*

$$\begin{aligned}|D^r Lf(x) - D^r f(x)| \leq{}& \left|\frac{1}{r!}D^r Le_r(x) - 1\right| \cdot |D^r f(x)| + \frac{1}{h}\cdot|\gamma_L(x)|\cdot\omega_1(D^r f;h)\\ &+ \left[D^r L\left(\frac{1}{r!}e_r\right)(x) + \frac{1}{2h^2}\cdot\beta_L(x)\right]\cdot\omega_2(D^r f;h),\end{aligned}$$

where

$$\gamma_L(x) := D^r L\left(\frac{1}{(r+1)!}\,e_{r+1} - \frac{1}{r!}\,x\cdot e_r\right)(x), \tag{29}$$

$$\beta_L(x) := D^r L\left(\frac{2}{(r+2)!}\,e_{r+2} - \frac{2}{(r+1)!}\,x\cdot e_{r+1} + \frac{1}{r!}\,x^2\cdot e_r\right)(x). \tag{30}$$

In regard to the representation and the behaviour of the spline functions, Marsden proved the following results:

Lemma 19 *(see [55, Lemmas 1,2])*

(i) Let $f \in C^1[0,1]$ *and* $k > 1$. *Then*

$$\begin{aligned} DS_{\Delta_n,k}f(x) &= \sum_{j=1-k}^{n-1} \frac{f(\xi_{j,k}) - f(\xi_{j-1,k})}{\xi_{j,k} - \xi_{j-1,k}} \cdot N_{j,k-1}(x) \\ &= \sum_{j=1-k}^{n-1} Df(\theta_{j,k}) \cdot N_{j,k-1}(x), \ \xi_{j-1,k} < \theta_{j,k} < \xi_{j,k}. \end{aligned} \tag{31}$$

(ii) Let $f \in C^2[0,1]$ *and* $k > 1$. *Then*

$$D^2 S_{\Delta_n,k}f(x) = \sum_{j=2-k}^{n-1} D^2 f(\eta_{j,k}) \cdot \frac{\xi_{j,k} - \xi_{j-2,k}}{2(\xi_{j,k-1} - \xi_{j-1,k-1})} \cdot N_{j,k-2}(x), \ \xi_{j-2,k} < \eta_{j,k} < \xi_{j,k}. \tag{32}$$

Lemma 20 *(see [55, Theorem 10]) Let* $f \in C^3[0,1]$ *and* $k > 2$. *Then*

(i) If $Df(x) \geq 0$ *on* $[0,1]$, *then* $DS_{\Delta_n,k}f(x) \geq 0$ *on* $[0,1]$.

(ii) If $D^2 f(x) \geq 0$ *on* $[0,1]$, *then* $D^2 S_{\Delta_n,k}f(x) \geq 0$ *on* $[0,1]$.

However,

(iii) If $D^3 f(x) \geq 0$ *on* $[0,1]$, $D^3 S_{\Delta_n,k}f(x)$ *need not be nonnegative.*

Lemma 21 *(see [55, Theorem 11]) Let* $f \in C^2[0,1]$, *and let* $x_j = \frac{j}{n}$, $0 < j < n$, *be the interior knots of* Δ_n. *Let* $k + n \to \infty$, $\liminf n > 1$, *and* $\liminf k > 1$. *If*

$$\lim \left(\frac{k-1}{k} \right) = R$$

exists, then

$$\begin{aligned} \lim D^2 S_{n,k}f(0) &= \frac{3R}{2} D^2 f(0), \\ \lim D^2 S_{n,k}f(1) &= \frac{3R}{2} D^2 f(1), \\ \lim D^2 S_{n,k}f(x) &= D^2 f(x), \quad \textit{for } 0 < x < 1. \end{aligned}$$

The convergence is uniform on compact subsets of $(0,1)$.

The most elegant case in the above is attained for $\frac{3R}{2} = 1$, that is for $k = 3$. This is why for the second derivative cubic splines with equidistant knots play a special role.

For the first order derivatives we can prove the following

Theorem 22 *Let $f \in C^1[0,1]$, $x \in [0,1]$ and $h > 0$. Then, for $n \geq 1$, $k \geq 2$, the following estimate holds:*

$$|DS_{\Delta_n,k}f(x) - Df(x)| \tag{33}$$
$$\leq \frac{1}{h} \cdot \|\Delta_n\| \cdot \omega_1(Df;h) + \left[1 + \frac{1}{2h^2}\left(1+\sqrt{\frac{k}{12}}\right)^2 \cdot \|\Delta_n\|^2\right] \omega_2(Df;h).$$

Proof:

The above statement will be derived using the result in Theorem 18; to that end we need upper bounds for the quantities appearing there (with $r = 1$).

Using (31) we get immediately

$$DS_{\Delta_n,k}e_1(x) = \sum_{j=1-k}^{n-1} De_1(\theta_{j,k}) \cdot N_{j,k-1}(x) = \sum_{j=1-k}^{n-1} N_{j,k-1}(x) = 1,\ \xi_{j-1,k} < \theta_{j,k} < \xi_{j,k},$$

thus

$$|DS_{\Delta_n,k}e_1(x) - 1| = |DS_{n,k}e_1(x) - 1| = 0.$$

For $\gamma_{S_{\Delta_n,k}}(x)$ we obtain successively

$$\begin{aligned}
|\gamma_{S_{\Delta_n,k}}(x)| &:= \left|DS_{\Delta_n,k}\left(\frac{e_2}{2} - xe_1\right)(x)\right| = \left|\frac{1}{2}DS_{\Delta_n,k}e_2(x) - xDS_{\Delta_n,k}e_1(x)\right| \\
&= \left|\frac{1}{2} \cdot 2\sum_{j=1-k}^{n-1} N_{j,k-1}(x) \cdot \theta_{j,k} - \sum_{j=1-k}^{n-1} N_{j,k-1}(x) \cdot \xi_{j,k-1}\right| \\
&= |\sum_{j=1-k}^{n-1} N_{j,k-1}(x)(\theta_{j,k} - \xi_{j,k-1})| \\
&\leq \sum_{j=1-k}^{n-1} N_{j,k-1}(x)|\theta_{j,k} - \xi_{j,k-1}|.
\end{aligned}$$

Since

$$\xi_{j-1,k} \leq \xi_{j,k-1} \leq \xi_{j,k},\ -k < j < n \text{ and}$$
$$\xi_{j-1,k} < \theta_{j,k} < \xi_{j,k},$$

it follows that

$$|\theta_{j,k} - \xi_{j,k-1}| \leq \xi_{j,k} - \xi_{j-1,k} = \frac{x_{j+k} - x_j}{k} \leq \|\Delta_n\|.$$

Substituting this in the above yields

$$|\gamma_{S_{\Delta_n,k}}(x)| \leq \|\Delta_n\| \sum_{j=1-k}^{n-1} N_{j,k-1}(x) = \|\Delta_n\|.$$

In order to obtain an upper bound for $\beta_{S_{\Delta_n,k}}(x)$ we apply formula (31) for the function $f = \frac{e_3}{3} - xe_2 + x^2 e_1$ and write successively

$$\begin{aligned}
\beta_{S_{\Delta_n,k}}(x) &:= DS_{\Delta_n,k}\left(\frac{e_3}{3} - xe_2 + x^2 e_1\right)(x) \\
&= \sum_{j=1-k}^{n-1} (\theta_{j,k}^2 - 2x\theta_{j,k} + x^2) \cdot N_{j,k-1}(x) = \sum_{j=1-k}^{n-1} (\theta_{j,k} - x)^2 \cdot N_{j,k-1}(x) \\
&= \sum_{j=1-k}^{n-1} (\theta_{j,k} - \xi_{j,k-1} + \xi_{j,k-1} - x)^2 \cdot N_{j,k-1}(x) \\
&\leq \sum_{j=1-k}^{n-1} (\theta_{j,k} - \xi_{j,k-1})^2 \cdot N_{j,k-1}(x) + \sum_{j=1-k}^{n-1} (\xi_{j,k-1} - x)^2 \cdot N_{j,k-1}(x) \\
&\qquad +2 \sum_{j=1-k}^{n-1} |\theta_{j,k} - \xi_{j,k-1}| \cdot |\xi_{j,k-1} - x| \cdot N_{j,k-1}(x) \\
&\leq \|\Delta_n\|^2 + S_{\Delta_n,k-1}((e_1 - x)^2; x) + 2\|\Delta_n\| \cdot S_{\Delta_n,k-1}(|e_1 - x|; x) \\
&\leq \|\Delta_n\|^2 + S_{\Delta_n,k-1}((e_1 - x)^2; x) + 2\|\Delta_n\| \cdot \sqrt{S_{\Delta_n,k-1}((e_1 - x)^2; x)} \\
&\leq \|\Delta_n\|^2 + \frac{k\|\Delta_n\|^2}{12} + 2\sqrt{\frac{k}{12}} \cdot \|\Delta_n\|^2 \\
&= \left(1 + \sqrt{\frac{k}{12}}\right)^2 \cdot \|\Delta_n\|^2.
\end{aligned}$$

In the above we used the Cauchy inequality and (3).

Replacing the above quantities into the general estimate of Theorem 18, we obtain the statement of our theorem. □

Taking $h = \|\Delta_n\|$ in Theorem 22 yields

Corollary 23 *Let $f \in C^1[0,1]$, $x \in [0,1]$. Then, for $n \geq 1$, $k \geq 2$, one has*

$$|DS_{\Delta_n,k} f(x) - Df(x)| \leq \omega_1(Df; \|\Delta_n\|) + \frac{3}{2}\left(1 + \sqrt{\frac{k}{12}}\right)^2 \cdot \omega_2(Df; \|\Delta_n\|). \quad (34)$$

Remark 24 *For the Schoenberg splines $S_{n,k}$ (with equidistant knots), inequality (33) can be given only in terms of the second order modulus of smoothness if $x \in \left[\frac{k-1}{n}, 1 - \frac{k-1}{n}\right]$ instead of $x \in [0,1]$ since, on this smaller interval, $\gamma_{S_{n,k}}(x) = 0$. Thus we get*

$$|DS_{n,k} f(x) - Df(x)| \leq \left[1 + \frac{1}{2h^2} \cdot \frac{1}{n^2}\left(1 + \sqrt{\frac{k}{12}}\right)^2\right] \omega_2(Df; h), \quad (35)$$

for $2 \le k \le \frac{n}{2}+1$ *(the latter inequality following from the requirement* $\frac{k-1}{n} \le 1-\frac{k-1}{n}$*). In particular, for* $h=\dfrac{1}{n}$*, the latter estimate becomes*

$$|DS_{n,k}f(x)-Df(x)| \le \frac{3}{2}\left(1+\sqrt{\frac{k}{12}}\right)^2 \cdot \omega_2\left(Df;\frac{1}{n}\right), \tag{36}$$

for $2 \le k \le \frac{n}{2}+1$.

For splines with $x_j = \frac{j}{n}$, $0 \le j \le n$, and second order derivatives one has uniform convergence on compact subsets of $(0,1)$ only. In this case we can state the following

Theorem 25 *Let* $f \in C^2[0,1]$, $x \in \left[\dfrac{k-1}{n}, 1-\dfrac{k-1}{n}\right]$ *and* $h>0$*. Then there holds:*

$$\begin{aligned}&|D^2S_{n,k}f(x)-D^2f(x)| \qquad (37)\\ &\le \frac{1}{h}\cdot\frac{1}{n}\cdot\omega_1(D^2f;h)+\left[1+\frac{1}{2h^2}\cdot\frac{1}{n^2}\left(1+\sqrt{\frac{k-1}{12}}\right)^2\right]\cdot\omega_2(D^2f;h),\end{aligned}$$

for $3 \le k \le \frac{n}{2}+1$.

Proof:

Putting

$$B_{j,k} := \frac{\xi_{j,k}-\xi_{j-2,k}}{2(\xi_{j,k-1}-\xi_{j-1,k-1})}, \tag{38}$$

formula (32) becomes

$$D^2S_{n,k}f(x) = \sum_{j=2-k}^{n-1} D^2f(\eta_{j,k})\cdot B_{j,k}\cdot N_{j,k-2}(x),\ \xi_{j-2,k}<\eta_{j,k}<\xi_{j,k}. \tag{39}$$

One has

$$\begin{aligned} B_{j,k} &= \frac{1}{2}\cdot\frac{k-1}{k}\cdot\left(1+\frac{x_{j+k}-x_{j-1}}{x_{j+k-1}-x_j}\right)\\ &= \frac{1}{2}\cdot\frac{k-1}{k}\cdot\left(1+\frac{k+1}{k-1}\right),\ \text{for } 1 \le j \le n-k\\ &= 1,\ \text{for } 1 \le j \le n-k.\end{aligned}$$

Thus, for $x \in \left[\dfrac{k-1}{n}, 1-\dfrac{k-1}{n}\right]$, we obtain

$$\left|\frac{1}{2}D^2S_{n,k}e_2(x)-1\right| = \left|\frac{1}{2}\sum_{j=1}^{n-k} 2\cdot B_{j,k}\cdot N_{j,k-2}(x)-1\right| = \left|\sum_{j=1}^{n-k} N_{j,k-2}(x)-1\right| = 0.$$

Here we have taken into account that on $\left[0, \frac{k-1}{n}\right]$ the splines $N_{j,k-2},\ 2-k \le j \le 0$, are zero, and the same is true for $N_{j,k-2}$ on $\left[1-\frac{k-1}{n}, 1\right]$ for $n-k+1 \le j \le n-1$.

Furthermore,

$$\begin{aligned}
|\gamma_{S_{n,k}}(x)| &:= \left|D^2 S_{n,k}\left(\frac{e_3}{3!} - x\frac{e_2}{2}\right)(x)\right| = \left|\frac{1}{3!}D^2 S_{n,k} e_3(x) - x\right| \\
&= \left|\frac{1}{3!}\cdot 6\sum_{j=1}^{n-k} \eta_{j,k}\cdot B_{j,k}\cdot N_{j,k-2}(x) - \sum_{j=1}^{n-k}\xi_{j,k-2}\cdot N_{j,k-2}(x)\right| \quad (\xi_{j-2,k} < \eta_{j,k} < \xi_{j,k}) \\
&= \left|\sum_{j=1}^{n-k} N_{j,k-2}(x)(\eta_{j,k}\cdot B_{j,k} - \xi_{j,k-2})\right| \\
&\le \sum_{j=1}^{n-k} N_{j,k-2}(x)|\eta_{j,k} - \xi_{j,k-2}| \le \sum_{j=1}^{n-k} N_{j,k-2}(x)\cdot\frac{1}{n} = \frac{1}{n}.
\end{aligned}$$

In the above we used the fact that

$$|\eta_{j,k} - \xi_{j,k-2}| \le \begin{cases} \xi_{j,k} - \xi_{j,k-2} = \frac{1}{n}, & \text{if } \eta_{j,k} \ge \xi_{j,k-2}, \\ \xi_{j,k-2} - \xi_{j-2,k} = \frac{1}{n}, & \text{if } \eta_{j,k} < \xi_{j,k-2}. \end{cases}$$

For $\beta_{S_{n,k}}(x),\ x \in \left[\frac{k-1}{n}, 1-\frac{k-1}{n}\right]$, we use formula (32) for the function $f = \frac{2}{4!}e_4 - \frac{2}{3!}xe_3 + \frac{1}{2!}x^2 e_2$ and write successively

$$\begin{aligned}
0 \le \beta_{S_{n,k}}(x) &:= D^2 S_{n,k}\left(\frac{2}{4!}e_4 - \frac{2}{3!}xe_3 + \frac{1}{2!}x^2 e_2\right)(x) \\
&= \sum_{j=1}^{n-k} N_{j,k-2}(x)(\eta_{j,k}^2 - 2x\eta_{j,k} + x^2)\cdot B_{j,k} \quad (\xi_{j-2,k} < \eta_{j,k} < \xi_{j,k}) \\
&= \sum_{j=1}^{n-k} N_{j,k-2}(x)(\eta_{j,k} - x)^2 \\
&= \sum_{j=1}^{n-k} N_{j,k-2}(x)(\eta_{j,k} - \xi_{j,k-2} + \xi_{j,k-2} - x)^2 \\
&\le \sum_{j=1}^{n-k} N_{j,k-2}(x)(\eta_{j,k} - \xi_{j,k-2})^2 + \sum_{j=1}^{n-k} N_{j,k-2}(x)(\xi_{j,k-2} - x)^2 \\
&\quad +2\sum_{j=1}^{n-k} |\eta_{j,k} - \xi_{j,k-2}|\cdot|\xi_{j,k-2} - x|\cdot N_{j,k-2}(x) \\
&\le \frac{1}{n^2} + \frac{k-1}{12n^2} + 2\cdot\frac{1}{n}\cdot\sqrt{\frac{k-1}{12n^2}}
\end{aligned}$$

$$= \frac{1}{n^2}\left(1+\sqrt{\frac{k-1}{12}}\right)^2.$$

In the above we used the Cauchy inequality and (3). An application of Theorem 18 yields the statement of our theorem. □

In particular, for $h = \dfrac{1}{n}$, we get

Corollary 26 *Let $f \in C^2[0,1]$, $x \in \left[\dfrac{k-1}{n}, 1-\dfrac{k-1}{n}\right]$ and $3 \le k \le \frac{n}{2}+1$. Then there holds:*

$$|D^2 S_{n,k} f(x) - D^2 f(x)| \le \omega_1\left(D^2 f; \frac{1}{n}\right) + \frac{3}{2}\left(1+\sqrt{\frac{k-1}{12}}\right)^2 \cdot \omega_2\left(D^2 f; \frac{1}{n}\right) \quad (40)$$

As was mentioned earlier in this note, close to the endpoints there are problems with second order derivatives. We illustrate this for a simple case in the following

Example 27 *Consider $S_{n,3}(e_2; x)$ for $0 \le x \le \frac{1}{n}$. From the representations for $N_{j,3}(x)$, $-3 \le j \le 0$, given above it can be derived that on $[0, \frac{1}{n}]$ one has*

$$D^2 S_{n,3} e_2(x) = 2 - \frac{1}{3} x n,$$

that is

$$D^2 S_{n,3}(e_2; 0) = 2 = D^2 e_2(0),$$

but, independent of n,

$$D^2 S_{n,3}\left(e_2; \frac{1}{n}\right) = \frac{5}{3} < 2 = D^2 e_2\left(\frac{1}{n}\right).$$

This is why it is impossible to prove uniform convergence for the second derivatives on the whole interval $[0,1]$ as $\frac{1}{n} \to 0$.

At the left endpoint $\frac{k-1}{n} = \frac{2}{n}$ of the interval on which we proved uniform convergence we have

$$D^2 S_{n,3}\left(e_2; \frac{2}{n}\right) = 2 = D^2 e_2\left(\frac{2}{n}\right), \ 4 \le n,$$

again due to the general statement. In fact,

$$D^2 S_{n,3}(e_2; x) = D^2 e_2(x)$$

even for all $x \in [\frac{2}{n}, 1-\frac{2}{n}]$.

It is also interesting to note that, while $D^3 e_2(x) = 0$ for all $x \in [0,1]$, the second derivative of $S_{n,3} e_2$ strictly <u>decreases</u> on $[0, \frac{1}{n}]$.

5. Global smoothness preservation

Over the recent years there has been considerable interest in the preservation of global smoothness in various contexts. This intensive research culminated in the recent book by Anastassiou and Gal [3]. Already in the very first article [2] treating this phenomenon under a systematic point of view, global smoothness preservation by Schoenberg operators $S_{\Delta_n,k}$ with respect to the first order modulus ω_1 was investigated. It was shown there, among other things, that

$$\omega_1(S_{\Delta_n,k}f;t) \le 2 \cdot \omega_1(f;t), \quad f \in C[0,1],\ t \ge 0.$$

In this section we present an analogous result for a certain "second" $K-$ functional and the classical second order modulus. To that end we use the following tool given earlier by Cottin and Gonska.

Lemma 28 *(see Theorem 2.2 in [17]) Let $r \ge 0$ and $s \ge 1$ be integers, and let K and K' be given as above. Furthermore, let $L : C^r(K) \to C^r(K')$ be a linear operator having the following properties:*

(i) L is almost convex of orders $r-1$ and $r+s-1$,

(ii) L maps $C^{r+s}(K)$ into $C^{r+s}(K')$,

(iii) $L(\Pi_{r-1}) \subseteq \Pi_{r-1}$ and $L(\Pi_{r+s-1}) \subseteq \Pi_{r+s-1}$

(iv) $L(C^r(K)) \not\subseteq \Pi_{r-1}$.

Then for all $f \in C^r(K)$ and all $\delta \ge 0$ we have

$$K_s(D^rLf;\delta)_{K'} \le \frac{1}{r!} \cdot \|D^rLe_r\| \cdot K_s\left(f^{(r)}; \frac{1}{(r+s)_s} \cdot \frac{\|D^{r+s}Le_{r+s}\|}{\|D^rLe_r\|} \cdot \delta\right)_K. \tag{41}$$

In the above, K_s is the Peetre K–functional of order s, $s \ge 1$, given by

$$K_s(f;\delta) := K(f;\delta;C[0,1],C^s[0,1]) := \inf\{\|f-g\| + \delta \cdot \|g^{(s)}\| : \ g \in C^s[0,1]\},$$

$(a)_b$ denotes the Pochhammer symbol defined by

$$(a)_0 := 1, \ \ (a)_b := \prod_{k=0}^{b-1}(a-k), \ \ a \in \mathbb{R},\ b \in \mathbb{N},$$

and $\Pi_{-1} := \{0\}$.

Now we can state

Theorem 29 *For all $f \in C[0,1]$ and all $\delta \geq 0$, the variation–diminishing splines $S_{\Delta_n,k}$ of degree $k \geq 3$ with $n \geq 2$ satisfy the following estimates:*

$$K_2(S_{\Delta_n,k}f;\delta) \leq K_2\left(f; \frac{k-1}{k} \cdot \left(\frac{1}{2} + \rho(\Delta_n)\right) \cdot \delta\right), \text{ and} \tag{42}$$

$$\omega_2(S_{\Delta_n,k}f;\delta) \leq 3 \cdot \left[1 + \frac{k-1}{4k} \cdot (1 + 2 \cdot \rho(\Delta_n))\right] \cdot \omega_2(f;\delta), \tag{43}$$

where $\rho(\Delta_n) := \dfrac{\|\Delta_n\|}{\min\limits_{0 \leq i \leq n-1}(x_{i+1} - x_i)}$ *is the mesh ratio.*

Proof:

It can be easily verified that, for $r = 0$ and $s = 2$, the assumptions of Lemma 28 are satisfied by $S_{\Delta_n,k}$ with $k \geq 3$. Hence (41) reads now as follows:

$$\begin{aligned} K_2(S_{\Delta_n,k}f;\delta) &\leq \|S_{\Delta_n,k}e_0\| \cdot K_2\left(f; \frac{1}{2} \cdot \frac{\|D^2 S_{\Delta_n,k}e_2\|}{\|S_{\Delta_n,k}e_0\|} \cdot \delta\right) \\ &= K_2\left(f; \frac{1}{2} \cdot \|D^2 S_{\Delta_n,k}e_2\| \cdot \delta\right), \end{aligned} \tag{44}$$

since $\|S_{\Delta_n,k}e_0\| = 1$.

Furthermore, for $k \geq 3$,

$$\begin{aligned} B_{j,k} &= \frac{1}{2} \cdot \frac{k-1}{k} \cdot \left(1 + \frac{x_{j+k} - x_{j-1}}{x_{j+k-1} - x_j}\right) \\ &\leq \frac{1}{2} \cdot \frac{k-1}{k} \cdot \left(1 + \max_{-k+2 \leq j \leq n-1} \frac{x_{j+k} - x_{j-1}}{x_{j+k-1} - x_j}\right) \\ &\leq \frac{1}{2} \cdot \frac{k-1}{k} \cdot \left(1 + \max\left\{2, \frac{k+1}{k-1}\right\} \cdot \rho(\Delta_n)\right) \\ &= \frac{1}{2} \cdot \frac{k-1}{k} \cdot (1 + 2 \cdot \rho(\Delta_n)). \end{aligned}$$

The 2 appearing in $\max\left\{2, \frac{k+1}{k-1}\right\}$ in the above is due to certain special cases when considering equidistant knots.

Thus

$$\begin{aligned} |D^2 S_{\Delta_n,k}f(x)| &\leq \|f''\| \cdot \frac{1}{2} \cdot \frac{k-1}{k} \cdot (1 + 2 \cdot \rho(\Delta_n)) \sum_{j=2-k}^{n-1} N_{j,k-2}(x) \\ &= \|f''\| \cdot \frac{1}{2} \cdot \frac{k-1}{k} \cdot (1 + 2 \cdot \rho(\Delta_n)), \end{aligned}$$

and, in particular,

$$\|D^2 S_{\Delta_n,k}e_2\| \leq \frac{k-1}{k} \cdot (1 + 2 \cdot \rho(\Delta_n)).$$

Substituting this upper bound into (44) yields (42).

For the second statement of our theorem we employ the function $Z_\delta(f)$ from Žuk's paper [81] (see Lemma 1 there), also observing the fact that

$$K_2(f;\delta) = K(f;\delta;C[0,1],C^2[0,1]) = K(f;\delta;C[0,1],W_{2,\infty}[0,1]).$$

Here,

$$W_{2,\infty}[0,1] := \{f \in C[0,1] : f' \text{ absolutely continuous}, \|f''\|_{L_\infty} < \infty\},$$

where

$$\|f''\|_{L_\infty} = \text{vrai} \sup_{x\in[0,1]} |f''(x)|.$$

Let now $f \in C[0,1]$, $0 < \delta \leq \frac{1}{2}$ be arbitrarily given, and let $|h| \leq \delta$. Then for a typical difference figuring in the definition of $\omega_2(S_{\Delta_n,k}f;\delta)$ we have

$$\begin{aligned}
&|S_{\Delta_n,k}f(x-h) - 2S_{\Delta_n,k}f(x) + S_{\Delta_n,k}f(x+h)| \\
&= |\{S_{\Delta_n,k}(f-g;x-h) - 2S_{\Delta_n,k}(f-g;x) + S_{\Delta_n,k}(f-g;x+h)\} \\
&\quad +\{S_{\Delta_n,k}(g;x-h) - 2S_{\Delta_n,k}(g;x) + S_{\Delta_n,k}(g;x+h)\}|,
\end{aligned}$$

where $g \in W_{2,\infty}[0,1]$ may be arbitrarily chosen.

The absolute value of the first term in curly parentheses can be estimated from above by

$$4\|S_{\Delta_n,k}(f-g)\|_\infty \leq 4\|f-g\|_\infty.$$

For the modulus of the second expression in curly brackets we have

$$\begin{aligned}
&|S_{\Delta_n,k}(g;x-h) - 2S_{\Delta_n,k}(g;x) + S_{\Delta_n,k}(g;x+h)| \\
&= |D^2S_{\Delta_n,k}(g;\xi)| \cdot h^2 \text{ (for some } \xi \text{ between } x-h \text{ and } x+h) \\
&\leq \|D^2S_{\Delta_n,k}g\| \cdot h^2 \leq \frac{1}{2} \cdot \frac{k-1}{k} \cdot (1+2\cdot\rho(\Delta_n)) \cdot h^2 \cdot \|g''\|_{L_\infty}.
\end{aligned}$$

We now substitute the function $g \in W_{2,\infty}[0,1]$ by $Z_h(f)$ from Žuk's paper [81], satisfying for $0 < h \leq \frac{1}{2}$ the inequalities

$$\begin{aligned}
\|f - Z_h(f)\| &\leq \frac{3}{4} \cdot \omega_2(f;h), \text{ and} \\
\|Z_h''(f)\|_{L_\infty} &\leq \frac{3}{2} \cdot \frac{1}{h^2} \cdot \omega_2(f;h).
\end{aligned}$$

Combining these estimates leads to

$$\begin{aligned}
\omega_2(S_{\Delta_n,k}f;\delta) &\leq 3\cdot\omega_2(f;\delta) + \frac{3}{4}\cdot\frac{k-1}{k}\cdot(1+2\cdot\rho(\Delta_n))\cdot\omega_2(f;\delta) \\
&= 3\cdot\left[1 + \frac{k-1}{4k}\cdot(1+2\cdot\rho(\Delta_n))\right]\cdot\omega_2(f;\delta),
\end{aligned}$$

which completes the proof. □

Since, for equidistant knots $x_j = \frac{j}{n}$, $0 \le j \le n$, one has $\rho(\Delta_n) = 1$, it follows immediately

Corollary 30 *For all $f \in C[0,1]$ and all $\delta \ge 0$, the variation–diminishing splines $S_{n,k}$ of degree $k \ge 3$ with $n \ge 2$ satisfy the following estimates:*

$$\begin{aligned} K_2(S_{n,k}f;\delta) &\le K_2\left(f;\frac{3}{2}\cdot\frac{k-1}{k}\cdot\delta\right), \text{ and} \\ \omega_2(S_{n,k}f;\delta) &\le \left(3+\frac{9}{4}\cdot\frac{k-1}{k}\right)\cdot\omega_2(f;\delta). \end{aligned}$$

6. Multivariate approaches

In the sequel we present statements on the degrees of approximation and simultaneous approximation for first and second derivatives in certain bivariate cases. We restrict ourselves to state inheritance principles (Theorems 31, 37) in terms of the classical second order modulus of smoothness, but similar statements can be formulated also for ω_2^{φ} and $\omega_2^{\varphi^\lambda}$ (see, e.g., [24, 16]).

All our results below should be compared with corresponding ones by Munteanu and Schumaker [62]. Due to the consequent use of ω_2 all our estimates will be of at least the orders given by Munteanu and Schumaker or improve them. Furthermore, again thanks to ω_2, we are able to better exploit smoothness properties of a given function f than they were able to do. This is true in particular in those cases in which f has two continuous partials in either x, y, or both. Note that we are also able to give quantitative information for more partials than they did.

Several results of this section will be given in terms of so–called partial moduli of smoothness of order r, given for the compact intervals $I, J \subset \mathbb{R}$, for $f \in C(I\times J)$, $r \in \mathbb{N}_0$ and $\delta \in \mathbb{R}_+$ by

$$\omega_r(f;\delta,0) := \sup\left\{\left|\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu}\cdot f(x+\nu h,y)\right| \,:\, (x,y),(x+rh,y)\in I\times J,\ |h|\le\delta\right\}$$

and symmetrically by

$$\omega_r(f;0,\delta) := \sup\left\{\left|\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu}\cdot f(x,y+\nu h)\right| \,:\, (x,y),(x,y+rh)\in I\times J,\ |h|\le\delta\right\}.$$

Occasionally we will use total moduli of smoothness of order r, defined by

$$\begin{aligned}\omega_r(f;\delta_1,\delta_2) := \sup\Bigg\{\left|\sum_{\nu=0}^{r}(-1)^{r-\nu}\binom{r}{\nu}\cdot f(x+\nu h_1,y+\nu h_2)\right| \,:\, \\ (x,y),(x+rh_1,y+rh_2)\in I\times J,\ |h_1|\le\delta_1,\ |h_2|\le\delta_2\Bigg\},\end{aligned}$$

for the compact intervals $I, J \subset \mathbb{R}$, for $f \in C(I \times J)$, $r \in I\!N_0$ and $\delta_1, \delta_2 \in \mathbb{R}_+$.

The third type of moduli figuring in this section will be the mixed moduli of smoothness, given for $r, s \in I\!N_0$ by

$$\omega_{r,s}(f;\delta_1,\delta_2) := \sup\left\{\left|\sum_{\nu=0}^{r}\sum_{\mu=0}^{s}(-1)^{r+s-\nu-\mu}\binom{r}{\nu}\binom{s}{\mu}\cdot f(x+\nu h_1, y+\mu h_2)\right| : \right.$$
$$\left. (x,y),(x+rh_1,y+sh_2) \in I \times J,\ |h_i| \leq \delta_i,\ i=1,2\right\}.$$

Several properties of these moduli can be found in Schumaker's book [73] and in [34].

6.1 Boolean sums

In order to cover Boolean sums of Schoenberg spline operators we will use the inheritance principle in the theorem below. The theorem is in analogy to two previous versions given in [35], [36], but it is adapted here to the situation we are dealing with.

In particular the operators L and M will be discretely defined, i.e., for finitely many, mutually distinct points $x_e,\ e \in E$ (E a suitable index set) of the compact interval I and fundamental functions A_e the operator L will be of the form

$$L(g;x) = \sum_{e\in E} g(x_e)\cdot A_e(x).$$

If $A_e \in C^p(I'),\ p \geq 0,\ I' \subseteq I$, then $L : C^p(I) \to C^p(I')$.

Likewise M will be of the form

$$M(h;y) = \sum_{f\in F} h(y_f)\cdot B_f(y)$$

and under analogous assumptions will map $C^q(J)$ into $C^q(J')$.

If L is of the form given above, then its parametric extension to $C^{p,q}(I \times J)$ is given by

$${}_xL(F;x,y) = L(F_y;x) = \sum_{e\in E} F_y(x_e)\cdot A_e(x) = \sum_{e\in E} F(x_e,y)\cdot A_e(x).$$

If we apply the partial differential operator $\frac{\partial^q}{\partial y^q} = D^{(0,q)}$ to this function we get

$$\begin{aligned}
\left(D^{(0,q)} \circ {}_xL\right)(F;x,y) &= \frac{\partial^q}{\partial y^q}\sum_{e\in E} F(x_e,y)A_e(x) \\
&= \sum_{e\in E}\frac{\partial^q}{\partial y^q}F(x_e,y)\cdot A_e(x) = \sum_{e\in E}\left(F^{(0,q)}\right)_y(x_e)\cdot A_e(x) \\
&= \left({}_xL \circ D^{(0,q)}\right)(F;x,y),
\end{aligned}$$

that is, $D^{(0,q)}$ and ${}_xL$ commute on $C^{p,q}(I \times J)$.

Analogously

$$D^{(p,0)} \circ {}_yM = {}_yM \circ D^{(p,0)}.$$

These commutativities will be crucial for the proof of the inheritance principle given in the theorem below.

One additional difference in comparison to earlier statements consists in our introducing certain intervals I', J', with $I' \subseteq I,\ J' \subseteq J$. This is due to the fact that, already for the second derivative, Schoenberg splines show a certain deficiency close to the endpoints. This was also observed by Marsden [55].

In other words, we will thus be able to give better estimates on $I' \times J'$ in the particular case where $I' \subsetneq I$ and $J' \subsetneq J$. Details will become clear in the applications.

Theorem 31 *Let I, I', J, J' be non–trivial compact intervals of the real axis $\mathbb{R}$, such that $I' \subseteq I$ and $J' \subseteq J$. For $(0,0) \leq (p',q') \leq (p,q)$ let discretely defined operators $L : C^p(I) \to C^{p'}(I')$ and $M : C^q(J) \to C^{q'}(J')$ be given such that for fixed $r, s \in I\!N_0$*

$$|(g - Lg)^{(p)}(x)| \leq \sum_{\rho=0}^{r} \Gamma_{\rho,p,L}(x) \cdot \omega_\rho(g^{(p)}; \Lambda_{\rho,p,L}(x)), \quad x \in I', g \in C^p(I), \tag{45}$$

and

$$|(h - Mh)^{(q)}(y)| \leq \sum_{\sigma=0}^{s} \Gamma_{\sigma,q,M}(y) \cdot \omega_\sigma(h^{(q)}; \Lambda_{\sigma,q,M}(y)), \quad y \in J', h \in C^q(J). \tag{46}$$

Here, Γ and Λ are positive, bounded functions.

Then we have for any $(x,y) \in I' \times J'$ and for all $f \in C^{p,q}(I \times J)$

$$\begin{aligned} &\left|(f - ({}_xL \oplus {}_yM)f)^{(p,q)}(x,y)\right| \\ &\qquad \leq \sum_{\rho=0}^{r} \sum_{\sigma=0}^{s} \Gamma_{\rho,p,L}(x) \cdot \Gamma_{\sigma,q,M}(y) \cdot \omega_{\rho,\sigma}\left(f^{(p,q)}; \Lambda_{\rho,p,L}(x), \Lambda_{\sigma,q,M}(y)\right)\right). \end{aligned}$$

Proof:

We want to estimate

$$\begin{aligned} &\left|D^{(p,q)} \circ [Id - ({}_xL \oplus {}_yM)]\,(f; x, y)\right| \\ &\quad = \left|D^{(p,0)} \circ D^{(0,q)} \circ [Id - ({}_xL \oplus {}_yM)]\,(f; x, y)\right| \\ &\quad = \left|D^{(p,0)}\left[D^{(0,q)}((Id - {}_yM)(f) - {}_xL \circ (Id - {}_yM)(f))\right](x,y)\right| \\ &\quad = \left|D^{(p,0)}\left[\left(D^{(0,q)} \circ (Id - {}_yM)\right)(f) - \left(D^{(0,q)} \circ {}_xL \circ (Id - {}_yM)\right)(f)\right](x,y)\right|. \end{aligned}$$

Using now the commutativity $D^{(0,q)} \circ {}_xL = {}_xL \circ D^{(0,q)}$, $x \in I'$, for the discretely defined operator L we get

$$\begin{aligned}&\left|D^{(p,0)}\left[\left(D^{(0,q)}\circ(Id-{}_yM)\right)(f)-\left(D^{(0,q)}\circ{}_xL\circ(Id-{}_yM)\right)(f)\right](x,y)\right|\\&\quad=\left|D^{(p,0)}\left[\left(D^{(0,q)}\circ(Id-{}_yM)\right)(f)-{}_xL\circ\left(D^{(0,q)}\circ(Id-{}_yM)\right)(f)\right](x,y)\right|.\end{aligned}$$

Now the assumption on the quantitative behaviour of the univariate operator L may be used since the function in [...] can also be written as a univariate function of x with parameter y, namely as

$$I' \ni x \mapsto \left[\left(D^{(0,q)}\circ(Id-{}_yM)\right)(f)\right]_y(x) - L\left(\left[\left(D^{(0,q)}\circ(Id-{}_yM)\right)(f)\right]_y; x\right) \in \mathbb{R}.$$

Applying $D^{(p,0)}$ to the function in [...] is the same as differentiating the latter univariate function with respect to x. Hence, by assumption (45), the quantity which we are interested in is bounded from above by

$$\sum_{\rho=0}^{r}\Gamma_{\rho,p,L}(x)\cdot\omega_\rho\left(\left(\frac{d}{dx}\right)^p\left[\left(D^{(0,q)}\circ(Id-{}_yM)\right)(f)\right]_y;\Lambda_{\rho,p,L}(x)\right).$$

The ρ–th modulus of smoothness can be replaced by

$$\left|{}_x\Delta^\rho_{\delta^*}\left[\left(\frac{d}{dx}\right)^p\left[\left(D^{(0,q)}\circ(Id-{}_yM)\right)(f)\right]_y\right](x^*)\right|$$

for some $x^* \in I'$ and $|\delta^*| \le \Lambda_{\rho,p,L}(x)$.

Next we investigate the latter quantity by using the information available on M. The absolute value of the ρ–th order difference is equal to

$$\left|\sum_{i=0}^{\rho}(-1)^i\binom{\rho}{i}\left[\left(\frac{d}{dx}\right)^p\left[\left(D^{(0,q)}\circ(Id-{}_yM)\right)(f)\right]_y\right](x^*+i\delta^*)\right|.$$

As in the above for L, we use now the commutativity for M, namely $D^{(p,0)} \circ {}_yM = {}_yM \circ D^{(p,0)}$, $y \in J'$. Since

$$\begin{aligned}\left(\frac{d}{dx}\right)^p\left[\left(D^{(0,q)}\circ(Id-{}_yM)\right)(f)\right]_y(x) &= \left(D^{(p,0)}\circ D^{(0,q)}\circ(Id-{}_yM)\right)(f;x,y)\\&= \left(D^{(0,q)}\circ D^{(p,0)}\circ(Id-{}_yM)\right)(f;x,y)\\&= \left(D^{(0,q)}\circ(Id-{}_yM)\circ D^{(p,0)}\right)(f;x,y),\end{aligned}$$

it follows that the ρ–th order difference can be written as

$$\begin{aligned}&\left|\sum_{i=0}^{\rho}(-1)^i\binom{\rho}{i}\left(D^{(0,q)}\circ(Id-{}_yM)\right)\left(D^{(p,0)}f;x^*+i\delta^*,y\right)\right|\\&\quad=\left|\sum_{i=0}^{\rho}(-1)^i\binom{\rho}{i}\left\{\left(\frac{d}{dy}\right)^q\left(D^{(p,0)}f\right)_{x^*+i\delta^*}(y)-\left(\frac{d}{dy}\right)^q\left({}_yM\circ D^{(p,0)}(f)\right)_{x^*+i\delta^*}(y)\right\}\right|\end{aligned}$$

$$= \left| \sum_{i=0}^{\rho} (-1)^i \binom{\rho}{i} \left\{ \left(\frac{d}{dy}\right)^q \left(D^{(p,0)} f\right)_{x^*+i\delta^*}(y) - \left(\frac{d}{dy}\right)^q M\left(\left(D^{(p,0)} f\right)_{x^*+i\delta^*}; y\right) \right\} \right|$$

$$= \left| \left[\left(\frac{d}{dy}\right)^q - \left(\frac{d}{dy}\right)^q \circ M \right] \left(\sum_{i=0}^{\rho} (-1)^i \binom{\rho}{i} \left(D^{(p,0)} f\right)_{x^*+i\delta^*}; y \right) \right| .$$

This difference may now be evaluated using assumption (46) on M. Hence, its absolute value is less than or equal to

$$\sum_{\sigma=0}^{s} \Gamma_{\sigma,q,M}(y) \cdot \omega_\sigma \left(\left(\frac{d}{dy}\right)^q \sum_{i=0}^{\rho} (-1)^i \binom{\rho}{i} \left(D^{(p,0)} f\right)_{x^*+i\delta^*}; \Lambda_{\sigma,q,M}(y) \right) .$$

The σ–th order modulus can be written as

$$\left| {}_y\Delta^{\sigma}_{\eta^*} \left[\left(\frac{d}{dy}\right)^q \sum_{i=0}^{\rho} (-1)^i \binom{\rho}{i} \left(D^{(p,0)} f\right)_{x^*+i\delta^*} \right] (y^*) \right|$$

for some $y^* \in J'$ and a suitable η^* such that $|\eta^*| \le \Lambda_{\sigma,q,M}(y)$. More explicitly, the latter quantity is equal to

$$\begin{aligned}
&\left| {}_y\Delta^{\sigma}_{\eta^*} \sum_{i=0}^{\rho} (-1)^i \binom{\rho}{i} \left(\frac{d}{dy}\right)^q \left(D^{(p,0)} f\right)_{x^*+i\delta^*} (y^*) \right| \\
&\quad = \left| {}_y\Delta^{\sigma}_{\eta^*} \sum_{i=0}^{\rho} (-1)^i \binom{\rho}{i} \left(D^{(p,q)} f\right)_{x^*+i\delta^*} (y^*) \right| \\
&\quad = \left| \sum_{j=0}^{\sigma} (-1)^j \binom{\sigma}{j} \sum_{i=0}^{\rho} (-1)^i \binom{\rho}{i} \left(D^{(p,q)} f\right) (x^* + i\delta^*, y^* + j\eta^*) \right| \\
&\quad = \left| \sum_{j=0}^{\sigma} \sum_{i=0}^{\rho} (-1)^{j+i} \binom{\sigma}{j} \binom{\rho}{i} \left(D^{(p,q)} f\right) (x^* + i\delta^*, y^* + j\eta^*) \right| \\
&\quad \le \omega_{\rho,\sigma} \left(f^{(p,q)};\ \Lambda_{\rho,p,L}(x),\ \Lambda_{\sigma,q,M}(y) \right) .
\end{aligned}$$

Combining the latter inequality with the observations made earlier in this proof shows the validity of the statement of Theorem 31. □

We will now give a number of applications for Boolean sums of Schoenberg splines. In doing so we will not strive to be as general as possible, but restrict ourselves to some cases of special interest, namely to estimates involving only the mesh gauge of the splines, but other direct inequalities from Section 3 can be used as well. The results given here should be also compared to the early paper [14] by Coman.

Theorem 32 *We consider the operators $S_{\Delta_n,k} : C[0,1] \to C^{k-1}[0,1]$ and $S_{\Delta_m,l} : C[0,1] \to C^{l-1}[0,1]$ for $n, m \ge 1$ and $k, l \ge 1$. For their Boolean sums we have*

$$\|f - ({}_xS_{\Delta_n,k} \oplus {}_yS_{\Delta_m,l})f\|_\infty \le \left(1 + \frac{k+1}{24}\right) \cdot \left(1 + \frac{l+1}{24}\right) \cdot \omega_{2,2}\left(f; \|\Delta_n\|, \|\Delta_m\|\right) .$$

The proof is immediate: put $r = s = 2$ and $\Gamma_{0,p,L} = \Gamma_{1,p,L} = \Gamma_{0,q,M} = \Gamma_{1,q,M} = 0$ in the general theorem for Boolean sums and use (20) twice.

We now turn to statements concerning also the approximation of derivatives. However, the upper bounds which are derived from Theorem 31 are quite complex. We thus focus in the following results on certain smooth functions to derive more instructive upper bounds. Nonetheless the following proof will provide the reader with an idea of what can be stated for less smooth functions.

Theorem 33 *For the operators $S_{\Delta_n,k} : C^1[0,1] \to C^{k-1}[0,1]$ and $S_{\Delta_m,l} : C^1[0,1] \to C^{l-1}[0,1]$ for $n, m \geq 1$ and $k, l \geq 2$ the following inequalities hold for any function $f \in C^{2,2}[0,1]^2$:*

$$\begin{aligned} &(i) \quad \|f - ({}_xS_{\Delta_n,k} \oplus {}_yS_{\Delta_m,l})f\|_\infty = \mathcal{O}\left(\|\Delta_n\|^2 \cdot \|\Delta_m\|^2\right); \\ &(ii) \quad \|(f - ({}_xS_{\Delta_n,k} \oplus {}_yS_{\Delta_m,l})f)^{(1,0)}\|_\infty = \mathcal{O}\left(\|\Delta_n\| \cdot \|\Delta_m\|^2\right); \\ &(iii) \quad \|(f - ({}_xS_{\Delta_n,k} \oplus {}_yS_{\Delta_m,l})f)^{(0,1)}\|_\infty = \mathcal{O}\left(\|\Delta_n\|^2 \cdot \|\Delta_m\|\right); \\ &(iv) \quad \|(f - ({}_xS_{\Delta_n,k} \oplus {}_yS_{\Delta_m,l})f)^{(1,1)}\|_\infty = \mathcal{O}\left(\|\Delta_n\| \cdot \|\Delta_m\|\right). \end{aligned}$$

In all four cases $\mathcal{O}$ depends on k and l.

Proof:

(i) is an immediate consequence of Theorem 32. It is only necessary to observe that

$$\omega_{2,2}\left(f; \|\Delta_n\|, \|\Delta_m\|\right) \leq \|\Delta_n\|^2 \cdot \|\Delta_m\|^2 \cdot \|f^{(2,2)}\|_\infty.$$

(ii) We apply Theorem 31 (for $p = 1$, $q = 0$) with $r = s = 2$, $\Gamma_{0,0,S_{\Delta_n,k}} = \Gamma_{0,1,S_{\Delta_n,k}} = 0$ and collect the others Γ's and the Λ's from the univariate case, that is

$$\begin{aligned} &\Gamma_{1,0,S_{\Delta_n,k}}(x) = 0, && \\ &\Gamma_{2,0,S_{\Delta_n,k}}(x) = 1 + \tfrac{k+1}{24}, && \Lambda_{2,0,S_{\Delta_n,k}}(x) = \|\Delta_n\|; \\ &\Gamma_{1,1,S_{\Delta_n,k}}(x) = 1, && \Lambda_{1,1,S_{\Delta_n,k}}(x) = \|\Delta_n\|; \\ &\Gamma_{2,1,S_{\Delta_n,k}}(x) = \tfrac{3}{2}\left(1 + \sqrt{\tfrac{k}{12}}\right)^2, && \Lambda_{2,1,S_{\Delta_n,k}}(x) = \|\Delta_n\|. \end{aligned}$$

The Γ's and the Λ's with respect to $S_{\Delta_m,l}$ are to be chosen analogously.

For brevity we write in the sequel $\Gamma_{1,1}(x)$ instead of $\Gamma_{1,1,S_{\Delta_n,k}}(x)$, etc. The upper bound which is derived from Theorem 31 is then as follows:

$$\begin{aligned} &\|(f - ({}_xS_{\Delta_n,k} \oplus {}_yS_{\Delta_m,l})f)^{(1,0)}\|_\infty \\ &\quad \leq \Gamma_{1,1}(x) \cdot [\Gamma_{1,0}(y) \cdot \omega_{1,1}(f^{(1,0)}; \Lambda_{1,1}(x), \Lambda_{1,0}(y)) + \Gamma_{2,0}(y) \cdot \omega_{1,2}(f^{(1,0)}; \Lambda_{1,1}(x), \Lambda_{2,0}(y))] \end{aligned}$$

$$+\Gamma_{2,1}(x)\cdot[\Gamma_{1,0}(y)\cdot\omega_{2,1}(f^{(1,0)};\Lambda_{2,1}(x),\Lambda_{1,0}(y))+\Gamma_{2,0}(y)\cdot\omega_{2,2}(f^{(1,0)};\Lambda_{2,1}(x),\Lambda_{2,0}(y))]$$

$$=\left(1+\frac{l+1}{24}\right)\cdot\omega_{1,2}\left(f^{(1,0)};\|\Delta_n\|,\|\Delta_m\|\right)$$

$$+\frac{3}{2}\left(1+\sqrt{\frac{k}{12}}\right)^2\cdot\left(1+\frac{l+1}{24}\right)\cdot\omega_{2,2}\left(f^{(1,0)};\|\Delta_n\|,\|\Delta_m\|\right)$$

$$=\mathcal{O}\left(\omega_{1,2}\left(f^{(1,0)};\|\Delta_n\|,\|\Delta_m\|\right)+\omega_{2,2}\left(f^{(1,0)};\|\Delta_n\|,\|\Delta_m\|\right)\right)$$

$$=\mathcal{O}\left(\|\Delta_n\|\cdot\|\Delta_m\|^2\cdot\|f^{(2,2)}\|_\infty\right)$$

$$=\mathcal{O}\left(\|\Delta_n\|\cdot\|\Delta_m\|^2\right)\ \text{for } f\in C^{2,2}[0,1]^2.$$

(iii) This is analogous to case (ii).

(iv) The functions Γ and Λ are the same as in case (ii); they just appear in different combinations now:

$$\|(f-({}_xS_{\Delta_n,k}\oplus{}_yS_{\Delta_m,l})f)^{(1,1)}\|_\infty$$

$$\leq\Gamma_{1,1}(x)\cdot[\Gamma_{1,1}(y)\cdot\omega_{1,1}(f^{(1,1)};\Lambda_{1,1}(x),\Lambda_{1,1}(y))+\Gamma_{2,1}(y)\cdot\omega_{1,2}(f^{(1,1)};\Lambda_{1,1}(x),\Lambda_{2,1}(y))]$$

$$+\Gamma_{2,1}(x)\cdot[\Gamma_{1,1}(y)\cdot\omega_{2,1}(f^{(1,1)};\Lambda_{2,1}(x),\Lambda_{1,1}(y))+\Gamma_{2,1}(y)\cdot\omega_{2,2}(f^{(1,1)};\Lambda_{2,1}(x),\Lambda_{2,1}(y))]$$

$$=\omega_{1,1}\left(f^{(1,1)};\|\Delta_n\|,\|\Delta_m\|\right)+\frac{3}{2}\left(1+\sqrt{\frac{l}{12}}\right)^2\cdot\omega_{1,2}\left(f^{(1,1)};\|\Delta_n\|,\|\Delta_m\|\right)$$

$$+\frac{3}{2}\left(1+\sqrt{\frac{k}{12}}\right)^2\cdot\left[\,\omega_{2,1}\left(f^{(1,1)};\|\Delta_n\|,\|\Delta_m\|\right)\right.$$

$$\left.+\frac{3}{2}\left(1+\sqrt{\frac{l}{12}}\right)^2\cdot\omega_{2,2}\left(f^{(1,1)};\|\Delta_n\|,\|\Delta_m\|\right)\,\right]$$

$$=\mathcal{O}\left(\omega_{1,1}\left(f^{(1,1)};\|\Delta_n\|,\|\Delta_m\|\right)+\omega_{1,2}\left(f^{(1,1)};\|\Delta_n\|,\|\Delta_m\|\right)\right.$$

$$\left.+\,\omega_{2,1}\left(f^{(1,1)};\|\Delta_n\|,\|\Delta_m\|\right)+\omega_{2,2}\left(f^{(1,1)};\|\Delta_n\|,\|\Delta_m\|\right)\right)$$

$$=\mathcal{O}\left(\|\Delta_n\|\cdot\|\Delta_m\|\cdot\|f^{(2,2)}\|_\infty\right)$$

$$=\mathcal{O}\left(\|\Delta_n\|\cdot\|\Delta_m\|\right)\ \text{for } f\in C^{2,2}[0,1]^2.\ \square$$

Remark 34 *If in Theorem 33 we take the* sup *norms over* $\left[\frac{k-1}{n},1-\frac{k-1}{n}\right]\times\left[\frac{l-1}{n},1-\frac{l-1}{n}\right]$ *only and* $f\in C^{3,3}[0,1]^2$*, we get* $\mathcal{O}\left(\|\Delta_n\|^2\cdot\|\Delta_m\|^2\right)$ *as an upper bound for all the quantities from (i) to (iv) there.*

In order to give inequalities for the partial derivatives of orders up to $(2,2)$ which are not covered by the previous theorem we restrict our attention to $\left[\frac{k-1}{n},1-\frac{k-1}{n}\right]\times\left[\frac{l-1}{n},1-\frac{l-1}{n}\right]$ and to the case of equidistant knots.

Theorem 35 *For $S_{n,k} : C^2[0,1] \to C^{k-1}[0,1]$ and $S_{m,l} : C^2[0,1] \to C^{l-1}[0,1]$, $3 \le k \le \frac{n}{2}+1$, $3 \le l \le \frac{m}{2}+1$, the following inequalities are true for $f \in C^{3,3}[0,1]^2$:*

$$(i) \quad \|(f - ({}_xS_{n,k} \oplus {}_yS_{m,l})f)^{(2,0)}\|_\infty = \mathcal{O}\left(\frac{1}{n} \cdot \frac{1}{m^2}\right);$$

$$(ii) \quad \|(f - ({}_xS_{n,k} \oplus {}_yS_{m,l})f)^{(2,1)}\|_\infty = \mathcal{O}\left(\frac{1}{n} \cdot \frac{1}{m^2}\right);$$

$$(iii) \quad \|(f - ({}_xS_{n,k} \oplus {}_yS_{m,l})f)^{(2,2)}\|_\infty = \mathcal{O}\left(\frac{1}{n} \cdot \frac{1}{m}\right).$$

Analogous statements hold for the partial derivatives of orders $(0,2)$ and $(1,2)$. The $\mathcal{O}$'s depend on k and l and the sup *norms are to be taken over $\left[\frac{k-1}{n}, 1-\frac{k-1}{n}\right] \times \left[\frac{l-1}{n}, 1-\frac{l-1}{n}\right]$.*

Proof:

The functions needed now are $\Gamma_{0,0} = \Gamma_{0,1} = \Gamma_{0,2} = 0$ and

$$\begin{aligned}
&\Gamma_{1,0}(x) = 0, && \\
&\Gamma_{2,0}(x) = 1 + \tfrac{k+1}{24}, && \Lambda_{2,0}(x) = \tfrac{1}{n}; \\
&\Gamma_{1,1}(x) = 0, && \\
&\Gamma_{2,1}(x) = \tfrac{3}{2}\left(1 + \sqrt{\tfrac{k}{12}}\right)^2, && \Lambda_{2,1}(x) = \tfrac{1}{n}; \\
&\Gamma_{1,2}(x) = 1, && \Lambda_{1,2}(x) = \tfrac{1}{n}; \\
&\Gamma_{2,2}(x) = \tfrac{3}{2}\left(1 + \sqrt{\tfrac{k-1}{12}}\right)^2, && \Lambda_{2,2}(x) = \tfrac{1}{n}.
\end{aligned}$$

Again, the Γ's and the Λ's with respect to $S_{m,l}$ are analogous.

(i) From the general theorem we obtain

$$\begin{aligned}
&\|(f - ({}_xS_{n,k} \oplus {}_yS_{m,l})f)^{(2,0)}\|_\infty \\
&\le \Gamma_{1,2}(x) \cdot \left[\Gamma_{1,0}(y) \cdot \omega_{1,1}\left(f^{(2,0)}; \frac{1}{n}, \frac{1}{m}\right) + \Gamma_{2,0}(y) \cdot \omega_{1,2}\left(f^{(2,0)}; \frac{1}{n}, \frac{1}{m}\right)\right] \\
&\quad + \Gamma_{2,2}(x) \cdot \left[\Gamma_{1,0}(y) \cdot \omega_{2,1}\left(f^{(2,0)}; \frac{1}{n}, \frac{1}{m}\right) + \Gamma_{2,0}(y) \cdot \omega_{2,2}\left(f^{(2,0)}; \frac{1}{n}, \frac{1}{m}\right)\right] \\
&= \left(1 + \frac{l+1}{24}\right) \cdot \omega_{1,2}\left(f^{(2,0)}; \frac{1}{n}, \frac{1}{m}\right) \\
&\quad + \frac{3}{2}\left(1 + \sqrt{\frac{k-1}{12}}\right)^2 \cdot \left(1 + \frac{l+1}{24}\right) \cdot \omega_{2,2}\left(f^{(2,0)}; \frac{1}{n}, \frac{1}{m}\right) \\
&= \mathcal{O}\left(\omega_{1,2}\left(f^{(2,0)}; \frac{1}{n}, \frac{1}{m}\right) + \omega_{2,2}\left(f^{(2,0)}; \frac{1}{n}, \frac{1}{m}\right)\right) \\
&= \mathcal{O}\left(\frac{1}{n} \cdot \frac{1}{m^2}\right) \text{ for } f \in C^{3,3}[0,1]^2.
\end{aligned}$$

(ii) Now

$$\begin{aligned}
&\|(f-({}_xS_{n,k}\oplus {}_yS_{m,l})f)^{(2,1)}\|_\infty \\
&\le \Gamma_{1,2}(x)\cdot\left[\Gamma_{1,1}(y)\cdot\omega_{1,1}\left(f^{(2,1)};\frac{1}{n},\frac{1}{m}\right)+\Gamma_{2,1}(y)\cdot\omega_{1,2}\left(f^{(2,1)};\frac{1}{n},\frac{1}{m}\right)\right] \\
&\quad +\Gamma_{2,2}(x)\cdot\left[\Gamma_{1,1}(y)\cdot\omega_{2,1}\left(f^{(2,1)};\frac{1}{n},\frac{1}{m}\right)+\Gamma_{2,1}(y)\cdot\omega_{2,2}\left(f^{(2,1)};\frac{1}{n},\frac{1}{m}\right)\right] \\
&=\frac{3}{2}\left(1+\sqrt{\frac{l}{12}}\right)^2\cdot\omega_{1,2}\left(f^{(2,1)};\frac{1}{n},\frac{1}{m}\right) \\
&\quad +\frac{3}{2}\left(1+\sqrt{\frac{k-1}{12}}\right)^2\cdot\frac{3}{2}\left(1+\sqrt{\frac{l}{12}}\right)^2\cdot\omega_{2,2}\left(f^{(2,1)};\frac{1}{n},\frac{1}{m}\right) \\
&=\mathcal{O}\left(\omega_{1,2}\left(f^{(2,1)};\frac{1}{n},\frac{1}{m}\right)+\omega_{2,2}\left(f^{(2,1)};\frac{1}{n},\frac{1}{m}\right)\right) \\
&=\mathcal{O}\left(\frac{1}{n}\cdot\frac{1}{m^2}\right)\ \text{ for } f\in C^{3,3}[0,1]^2.
\end{aligned}$$

(iii) In this case

$$\begin{aligned}
&\|(f-({}_xS_{n,k}\oplus {}_yS_{m,l})f)^{(2,2)}\|_\infty \\
&\le \Gamma_{1,2}(x)\cdot\left[\Gamma_{1,2}(y)\cdot\omega_{1,1}\left(f^{(2,2)};\frac{1}{n},\frac{1}{m}\right)+\Gamma_{2,2}(y)\cdot\omega_{1,2}\left(f^{(2,2)};\frac{1}{n},\frac{1}{m}\right)\right] \\
&\quad +\Gamma_{2,2}(x)\cdot\left[\Gamma_{1,2}(y)\cdot\omega_{2,1}\left(f^{(2,2)};\frac{1}{n},\frac{1}{m}\right)+\Gamma_{2,2}(y)\cdot\omega_{2,2}\left(f^{(2,2)};\frac{1}{n},\frac{1}{m}\right)\right] \\
&=\omega_{1,1}\left(f^{(2,2)};\frac{1}{n},\frac{1}{m}\right)+\frac{3}{2}\left(1+\sqrt{\frac{l-1}{12}}\right)^2\cdot\omega_{1,2}\left(f^{(2,2)};\frac{1}{n},\frac{1}{m}\right) \\
&\quad +\frac{3}{2}\left(1+\sqrt{\frac{k-1}{12}}\right)^2\cdot\omega_{2,1}\left(f^{(2,2)};\frac{1}{n},\frac{1}{m}\right) \\
&\quad +\frac{3}{2}\left(1+\sqrt{\frac{k-1}{12}}\right)^2\cdot\frac{3}{2}\left(1+\sqrt{\frac{l-1}{12}}\right)^2\cdot\omega_{2,2}\left(f^{(2,2)};\frac{1}{n},\frac{1}{m}\right) \\
&=\mathcal{O}\left(\frac{1}{n}\cdot\frac{1}{m}+\frac{1}{n}\cdot\frac{1}{m}+\frac{1}{n}\cdot\frac{1}{m}+\frac{1}{n}\cdot\frac{1}{m}\right)=\mathcal{O}\left(\frac{1}{n}\cdot\frac{1}{m}\right)\ \text{ for } f\in C^{3,3}[0,1]^2.\square
\end{aligned}$$

Remark 36 *The fact that the partials of orders* $(2,0)$ *and* $(2,1)$ *are approximated with the same order* $\mathcal{O}\left(\frac{1}{n}\cdot\frac{1}{m^2}\right)$ *is due to the fact that on the small interval* $\left[\frac{l-1}{n},1-\frac{l-1}{n}\right]$ *for the univariate operator* $S_{m,l}$ *both a function* $f\in C^3[0,1]$ *and its derivative are approximated with the same order. When it comes to the second derivative a power of one is lost in the univariate case.*

6.2 Tensor products

For the tensor product case the situation is the same as for Boolean sums: the reader should compare our results with those of Munteanu and Schumaker in order to confirm that the consequent use of ω_2 provides more insight. Further papers on tensor products were provided by Coman and Frenţiu [15] and Felicia Stancu [77], for example.

Also in the tensor product case we will use an inheritance principle. It is stated in a form which exactly suits our purposes and makes the same assumptions concerning the univariate building blocks as in the Boolean sum case.

Theorem 37 *Let $I' \subseteq I, J' \subseteq J$ be non–trivial compact intervals of the real axis $\mathbb{R}$. For $p, q \in \mathbb{N}_0$ let $L : C^p(I) \to C^p(I')$ and $M : C^q(J) \to C^q(J')$ be discretely defined operators as given above and such that for fixed $r, s \in \mathbb{N}_0$*

$$\left|(g - Lg)^{(p)}(x)\right| \leq \sum_{\rho=0}^{r} \Gamma_{\rho,p,L}(x) \cdot \omega_\rho(g^{(p)}; \Lambda_{\rho,p,L}(x)), \quad x \in I', g \in C^p(I),$$

and

$$\left|(h - Mh)^{(q)}(y)\right| \leq \sum_{\sigma=0}^{s} \Gamma_{\sigma,q,M}(y) \cdot \omega_\sigma(h^{(q)}; \Lambda_{\sigma,q,M}(y)), \quad y \in J', h \in C^q(J).$$

Here, Γ and Λ are bounded functions.

(i) Then for $(x, y) \in I' \times J'$ and $f \in C^{p,q}(I \times J)$ the following hold:

$$\begin{aligned}\left|\left[f - ({}_xL \circ {}_yM)f\right]^{(p,q)}(x, y)\right| &\leq \sum_{\rho=0}^{r} \Gamma_{\rho,p,L}(x) \cdot \omega_\rho(f^{(p,q)}; \Lambda_{\rho,p,L}(x), 0) \\ &+ \|D^p \circ L\|^* \cdot \sum_{\sigma=0}^{s} \Gamma_{\sigma,q,M}(y) \cdot \omega_\sigma(f^{(p,q)}; 0, \Lambda_{\sigma,q,M}(y)).\end{aligned}$$

Here

$$\|D^p \circ L\|^* := \inf\{c : \|(D^p \circ L)g\|_{\infty,I'} \leq c \cdot \|g^{(p)}\|_{\infty,I}, \ \forall \ g \in C^p(I)\}.$$

(ii) A symmetric upper bound is given by

$$\sum_{\sigma=0}^{s} \Gamma_{\sigma,q,M}(y) \cdot \omega_\sigma(f^{(p,q)}; 0, \Lambda_{\sigma,q,M}(y)) + \|D^q \circ M\|^* \cdot \sum_{\rho=0}^{r} \Gamma_{\rho,p,L}(x) \cdot \omega_\rho(f^{(p,q)}; \Lambda_{\rho,p,L}(x), 0).$$

Proof:

Recall first that

$$D^{(0,q)} \circ {}_xL = {}_xL \circ D^{(0,q)} \text{ on } C^{(p,q)}(I \times J).$$

Then

$$
\begin{aligned}
&\left|[f-({}_xL\circ{}_yM)f]^{(p,q)}(x,y)\right|\\
&=\left|D^{(p,0)}\circ D^{(0,q)}\circ(Id-{}_xL\circ{}_yM)(f;x,y)\right|\\
&=\left|[D^{(p,0)}\circ D^{(0,q)}\circ(Id-{}_xL)+D^{(p,0)}\circ D^{(0,q)}\circ{}_xL\circ(Id-{}_yM)](f;x,y)\right|\\
&=\left|[D^{(p,0)}\circ(Id-{}_xL)\circ D^{(0,q)}+D^{(p,0)}\circ{}_xL\circ D^{(0,q)}\circ(Id-{}_yM)](f;x,y)\right|\\
&\le\left|D^{(p,0)}\circ(Id-{}_xL)\circ D^{(0,q)}(f;x,y)\right|+\left|D^{(p,0)}\circ{}_xL\circ D^{(0,q)}\circ(Id-{}_yM)(f;x,y)\right|\\
&=:E_1(x,y)+E_2(x,y).
\end{aligned}
$$

Now, for $x\in I'$,

$$
\begin{aligned}
E_1(x,y) &= \left|D^{(p,0)}\circ(Id-L)((f^{(0,q)})_y;x)\right|\le\sum_{\rho=0}^{r}\Gamma_{\rho,p,L}(x)\cdot\omega_\rho((f^{(p,q)})_y;\Lambda_{\rho,p,L}(x))\\
&\le \sum_{\rho=0}^{r}\Gamma_{\rho,p,L}(x)\cdot\omega_\rho(f^{(p,q)};\Lambda_{\rho,p,L}(x),0).
\end{aligned}
$$

Furthermore, with $F:=D^{(0,q)}\circ(Id-{}_yM)f$, we have

$$E_2(x,y)=\left|\left(D^{(p,0)}\circ{}_xL\right)(F;x)\right|=|(D^p\circ L)(F_y;x)|\le\|(D^p\circ L)(F_y)\|_{\infty,I'}\,.$$

Here again $F_y\in C^p(I)$ for all $y\in J'$. By our assumption on L we have for any $g\in C^p(I)$ that

$$\|(D^p\circ L)g\|_{\infty,I'}\le\left(1+\sum_{\rho=0}^{r}2^\rho\cdot\|\Gamma_{\rho,p,L}\|_{\infty,I'}\right)\cdot\|g^{(p)}\|_{\infty,I}.$$

Hence

$$\|D^p\circ L\|^*:=\inf\{c:\|D^p\circ L)g\|_{\infty,I'}\le c\cdot\|g^{(p)}\|_{\infty,I}\quad\forall\, g\in C^p(I)\}<\infty.$$

In our present situation we have

$$
\begin{aligned}
\|F_y^{(p)}\|_{\infty,I} &= \left\|\frac{d^p}{dx^p}\left[D^{(0,q)}\circ(Id-{}_yM)f\right]_y(x)\right\|_{\infty,I}=\left\|D^{(p,0)}\circ D^{(0,q)}\circ(Id-{}_yM)f(\cdot,y)\right\|_{\infty,I}\\
&= \left\|D^{(0,q)}\circ(Id-{}_yM)f^{(p,0)}(\cdot,y)\right\|_{\infty,I}=\left\|\frac{d^q}{dy^q}\circ(Id-{}_yM)\left(f^{(p,0)}\right)_x(y)\right\|_{\infty,I}\\
&\le \left\|\sum_{\sigma=0}^{s}\Gamma_{\sigma,q,M}(y)\cdot\omega_\sigma\left(\frac{d^q}{dy^q}\left(f^{(p,0)}\right)_x;\ \Lambda_{\sigma,q,M}(y)\right)\right\|_{\infty,I}\\
&\le \sum_{\sigma=0}^{s}\Gamma_{\sigma,q,M}(y)\cdot\sup_{x\in I}\omega_\sigma\left(\frac{d^q}{dy^q}\left(f^{(p,0)}\right)_x;\Lambda_{\sigma,q,M}(y)\right)\\
&= \sum_{\sigma=0}^{s}\Gamma_{\sigma,q,M}(y)\cdot\omega_\sigma\left(f^{(p,q)};0,\Lambda_{\sigma,q,M}(y)\right).
\end{aligned}
$$

Hence

$$E_1(x,y)+E_2(x,y) \leq \sum_{\rho=0}^{r}\Gamma_{\rho,p,L}(x)\cdot\omega_\rho\left(f^{(p,q)};\Lambda_{\rho,p,L}(x),0\right)$$

$$+ \quad \|D^p\circ L\|^*\cdot\sum_{\sigma=0}^{s}\Gamma_{\sigma,q,M}(y)\cdot\omega_\sigma\left(f^{(p,q)};0,\Lambda_{\sigma,q,M}(y)\right).$$

The second upper bound can be obtained in an analogous fashion. □

For the tensor product of two Schoenberg spline operators we first state

Theorem 38 *For $n,m\geq 1$ and $k,l\geq 1$ we have*

$$\begin{aligned}&\|f-({}_xS_{\Delta_{n,k}}\circ{}_yS_{\Delta_{m,k}})f\|_{\infty,I\times J}\\ &\leq\left(1+\frac{k+1}{24}\right)\cdot\omega_2(f;\|\Delta_n\|,0)+\left(1+\frac{l+1}{24}\right)\cdot\omega_2(f;0,\|\Delta_m\|)\\ &\leq\left(2+\frac{k+l+2}{24}\right)\cdot\omega_2(f;\|\Delta_n\|,\|\Delta_m\|).\end{aligned}$$

Proof:

This is the case $p=q=0$, $r=s=2$. With $\Gamma_{0,0}(x)=\Gamma_{1,0}(x)=0$, $\Gamma_{2,0}(x)=1+\frac{k+1}{24}$, $\Lambda_{2,0}(x)=\|\Delta_n\|$ and analogous choices with respect to the variable y we arrive at the above upper bound, also observing that $\|D^0\circ S_{\Delta_{n,k}}\|^*=1$. □

For the partial derivatives up to order $(1,1)$ we arrive at

Theorem 39 *For $n,m\geq 1$ and $k,l\geq 2$ we have the following inequalities for any $f\in C^{2,2}[0,1]^2$.*

(i) $\|f-({}_xS_{\Delta_{n,k}}\circ{}_yS_{\Delta_{m,l}})f\|_\infty=\mathcal{O}(\|\Delta_n\|^2+\|\Delta_m\|^2)$;

(ii) $\|(f-({}_xS_{\Delta_{n,k}}\circ{}_yS_{\Delta_{m,l}})^{(1,0)}\|_\infty=\mathcal{O}(\|\Delta_n\|+\|\Delta_m\|^2)$;

(iii) $\|(f-({}_xS_{\Delta_{n,k}}\circ{}_yS_{\Delta_{m,l}})^{(0,1)}\|_\infty=\mathcal{O}(\|\Delta_n\|^2+\|\Delta_m\|)$;

(iv) $\|(f-({}_xS_{\Delta_{n,k}}\circ{}_yS_{\Delta_{m,l}})^{(1,1)}\|_\infty=\mathcal{O}(\|\Delta_n\|+\|\Delta_m\|)$.

In all four cases $\mathcal{O}$ depends in k and l, and the sup *norms are those over* $[0,1]^2$.

Proof:

The Γ's and the Λ's are again the same as in the Boolean sum case (see the proof of Theorem 33).

(i) This is an immediate consequence from Theorem 38.

(ii) With $r=s=2,\ p=1,\ q=0$, in the general theorem we have

$$\begin{aligned}
&\|(f-({}_xS_{\Delta_{n,k}}\circ {}_yS_{\Delta_{m,l}})f)^{(1,0)}\|_{\infty,I\times J}\\
&\le 1\cdot\omega_1\left(f^{(1,0)};\|\Delta_n\|,0\right)+\frac{3}{2}\left(1+\sqrt{\frac{k}{12}}\right)^2\cdot\omega_2\left(f^{(1,0)};\|\Delta_n\|,0\right)\\
&\qquad+\|D^1\circ S_{\Delta_{n,k}}\|^*\cdot\left(1+\frac{l+1}{24}\right)\cdot\omega_2\left(f^{(1,0)};0,\|\Delta_m\|\right).
\end{aligned}$$

From the representation of $(D^1\circ S_{\Delta_{n,k}})g$ for $g\in C^1[0,1]$ it follows that $\|D^1\circ S_{\Delta_{n,k}}\|^*\le 1$.

Thus we obtain an upper bound of order

$$\begin{aligned}
&\mathcal{O}\left(\omega_1(f^{(1,0)};\|\Delta_n\|,0)\ +\ \omega_2(f^{(1,0)};\|\Delta_n\|,0)\ +\ \omega_2(f^{(1,0)};0,\|\Delta_m\|)\right)\\
&=\mathcal{O}\left(\|\Delta_n\|+\|\Delta_m\|^2\right)\ \text{ for } f\in C^{2,2}[0,1]^2.
\end{aligned}$$

(iii) The proof for the partial of order $(0,1)$ is 'symmetric' to that for $(1,0)$.

(iv) Again with $r=s=2,\ p=q=1$, we have

$$\begin{aligned}
&\left\|(f-({}_xS_{\Delta_{n,k}}\circ {}_yS_{\Delta_{m,l}})f)^{(1,1)}\right\|_{\infty,I\times J}\\
&\le 1\cdot\omega_1\left(f^{(1,1)};\|\Delta_n\|,0\right)+\frac{3}{2}\left(1+\sqrt{\frac{k}{12}}\right)^2\cdot\omega_2\left(f^{(1,1)};\|\Delta_n\|,0\right)\\
&\qquad+1\cdot\left(\omega_1\left(f^{(1,1)};0,\|\Delta_m\|\right)+\frac{3}{2}\left(1+\sqrt{\frac{l}{12}}\right)^2\cdot\omega_2\left(f^{(1,1)};0,\|\Delta_m\|\right)\right)\\
&=\mathcal{O}\left(\|\Delta_n\|+\|\Delta_m\|\right)\ \text{ for } f\in C^{2,2}[0,1]^2.\ \square
\end{aligned}$$

For the remaining partials up to order $(2,2)$ again we consider only the case of equidistant knots and the smaller intervals $\left[\frac{k-1}{n},1-\frac{k-1}{n}\right]\times\left[\frac{l-1}{m},1-\frac{l-1}{m}\right]$. We now have

Theorem 40 *For* $3\le k\le\frac{n}{2}+1,\ 3\le l\le\frac{m}{2}+1$ *the following are true for* $f\in C^{3,3}[0,1]^2$*:*

(i) $\left\|(f-({}_xS_{n,k}\circ {}_yS_{m,l})f)^{(2,0)}\right\|_\infty=\mathcal{O}\left(\frac{1}{n}+\frac{1}{m^2}\right)$;

(ii) $\left\|(f-({}_xS_{n,k}\circ {}_yS_{m,l})f)^{(2,1)}\right\|_\infty=\mathcal{O}\left(\frac{1}{n}+\frac{1}{m^2}\right)$;

(iii) $\left\|(f-({}_xS_{n,k}\circ {}_yS_{m,l})f)^{(2,2)}\right\|_\infty=\mathcal{O}\left(\frac{1}{n}+\frac{1}{m}\right)$.

Analogous statements hold for the partials of orders $(0,2)$ *and* $(1,2)$*;* $\mathcal{O}$ *depends on* k *and* l *in all cases and the* sup *norms are those over the smaller subinterval given above.*

Proof:

The Γ's and the Λ's are again the same as in the Boolean sum case (see the proof of Theorem 35).

(i) From the general statement we obtain with $p=2, q=0, r=s=2$,

$$\left\|(f-({}_xS_{n,k}\circ {}_yS_{m,l})f)^{(2,0)}\right\|_\infty$$
$$\leq \Gamma_{1,2}(x)\cdot\omega_1\left(f^{(2,0)};\Lambda_{1,2}(x),0\right)+\Gamma_{2,2}(x)\cdot\omega_2\left(f^{(2,0)};\Lambda_{2,2}(x),0\right)$$
$$+\left\|D^2\circ S_{n,k}\right\|^*\cdot\Gamma_{2,0}(y)\cdot\omega_2\left(f^{(2,0)};0,\Lambda_{2,0}(y)\right)$$
$$\leq \omega_1\left(f^{(2,0)};\frac{1}{n},0\right)+\frac{3}{2}\left(1+\sqrt{\frac{k-1}{12}}\right)^2\cdot\omega_2\left(f^{(2,0)};\frac{1}{n},0\right)$$
$$+\frac{3}{2}\cdot\frac{k-1}{k}\cdot\left(1+\frac{l+1}{24}\right)\cdot\omega_2\left(f^{(2,0)};0,\frac{1}{m}\right)$$
$$=\mathcal{O}\left(\frac{1}{n}+\frac{1}{m^2}\right)\ \text{ for } f\in C^{3,2}[0,1]^2.$$

Note that $\|D^2\circ S_{n,k}\|^*\leq\frac{3}{2}\cdot\frac{k-1}{k}$ was shown in the section on global smoothness preservation.

(ii) For $p=2,\ q=1,\ r=s=2$ we have

$$\left\|(f-({}_xS_{n,k}\circ {}_yS_{m,l})f)^{(2,1)}\right\|_\infty$$
$$\leq \Gamma_{1,2}(x)\cdot\omega_1\left(f^{(2,1)};\Lambda_{1,2}(x),0\right)+\Gamma_{2,2}(x)\cdot\omega_2\left(f^{(2,1)};\Lambda_{2,2}(x),0\right)$$
$$+\left\|D^2\circ S_{n,k}\right\|^*\cdot\left\{\Gamma_{1,1}(y)\cdot\omega_1\left(f^{(2,1)};0,\Lambda_{1,1}(y)\right)+\Gamma_{2,1}(y)\cdot\omega_2\left(f^{(2,1)};0,\Lambda_{2,1}(y)\right)\right\}$$
$$\leq \omega_1\left(f^{(2,1)};\frac{1}{n},0\right)+\frac{3}{2}\left(1+\sqrt{\frac{k-1}{12}}\right)^2\cdot\omega_2\left(f^{(2,1)};\frac{1}{n},0\right)$$
$$+\frac{3}{2}\cdot\frac{k-1}{k}\cdot\frac{3}{2}\left(1+\sqrt{\frac{l}{12}}\right)^2\cdot\omega_2\left(f^{(2,1)};0,\frac{1}{m}\right)$$
$$=\begin{cases}\mathcal{O}\left(\frac{1}{n}+\frac{1}{m}\right) & \text{for } f\in C^{3,2},\\ \mathcal{O}\left(\frac{1}{n}+\frac{1}{m^2}\right) & \text{for } f\in C^{3,3}.\end{cases}$$

(iii) Now $p=q=2,\ r=s=2$. Thus

$$\left\|(f-({}_xS_{n,k}\circ {}_yS_{m,l})f)^{(2,2)}\right\|_\infty$$
$$\leq \omega_1\left(f^{(2,2)};\frac{1}{n},0\right)+\frac{3}{2}\left(1+\sqrt{\frac{k-1}{12}}\right)^2\cdot\omega_2\left(f^{(2,2)},\frac{1}{n},0\right)$$

$$+\frac{3}{2}\cdot\frac{k-1}{k}\cdot\left[\omega_1\left(f^{(2,2)};0,\frac{1}{m}\right)+\frac{3}{2}\left(1+\sqrt{\frac{l-1}{12}}\right)^2\cdot\omega_2\left(f^{(2,2)};0,\frac{1}{m}\right)\right]$$

$$=\mathcal{O}\left(\omega_1\left(f^{(2,2)},\frac{1}{n},0\right)+\omega_2\left(f^{(2,2)};\frac{1}{n},0\right)+\omega_1\left(f^{(2,2)};0,\frac{1}{m}\right)+\omega_2\left(f^{(2,2)};0,\frac{1}{m}\right)\right)$$

$$=\mathcal{O}\left(\frac{1}{n}+\frac{1}{m}\right)\ \text{for } f\in C^{3,3}[0,1]^2.\ \square$$

7. Concluding remarks and open problems

1. For the case of equidistant knots we were able to show

$$(S_{n,k}(e_1-x)^2)(x)\le 1\cdot\frac{\min\left\{2x(1-x),\frac{k}{n}\right\}}{n+k-1}.$$

For $n=1,\ k\geq 1$ and $n\geq 2,\ k=1$ the constant 1 can be replaced by $\frac{1}{2}$. It should be clearified if 1 is globally optimal.

It would likewise be desirable to have an analogous inequality for general knot sequences.

Instructive exact representations (and thus lower bounds) for $(S_{n,k}(e_1-x)^2)(x)$ are only known in a few exceptional cases. It would thus be of interest to find such representations or lower bounds for more general combinations of n and k.

2. Strong converse inequalities also seem to be known only in very special cases. For piecewise linear interpolation at equidistant knots see Ditzian and Ivanov [23], for Bernstein and related operators consult the papers by Zhou, Totik, Knoop, Ivanov and Ditzian [51, 79, 23]. It seems as if there is no strong converse inequality even for the case of quadratic Schoenberg splines, rather a popular tool (cf. their use in packages such as MacDraw, for example).

3. Likewise it would be desirable to prove at least inverse and saturation results including such for derivatives.

4. We have only given estimates for the approximation of derivatives up to order 2. We are not aware of corresponding estimates for derivatives of order $l\geq 3$.

5. The preservation of global smoothness is well understood for Bernstein operators and their derivatives (see [17]). For the Schoenberg operator this appears to be much more difficult since derivatives of order $l\geq 3$ would have to be also represented appropriately in order to come up with inequalities such as

$$\begin{aligned}\omega_2(D^1S_{n,k}f;\delta)&\le\ \ldots\omega_2(f';\ldots),\ \text{ or}\\ \omega_2(D^2S_{n,k}f;\delta)&\le\ \ldots\omega_2(f'';\ldots).\end{aligned}$$

Acknowledgments

The present article was significantly improved and finalized while all four authors were RiP–fellows at the Mathematical Research Institute in Oberwolfach. They gratefully acknowledge the support of the Volkswagen–Stiftung for this opportunity. The fourth author's contribution was also made possible by a DAAD–fellowship for visiting the University of Duisburg in the fall of 2001.

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