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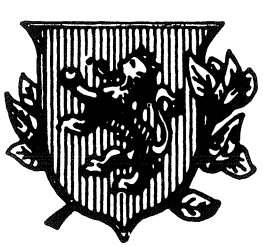
SEGUNDAS JORNADAS DE MECÁNICA CELESTE

Universidad de La Rioja



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SEGUNDAS JORNADAS DE MECÁNICA CELESTE

UNIVERSIDAD DE LA RIOJA

TABLA DE CONTENIDOS

Presentación

Esta Monografía de la Academia de Ciencias de Zaragoza, contiene artículos presentados en las Segundas Jornadas de Trabajo en Mecánica Celeste, que tuvieron lugar los días 17 y 18 de junio de 1999 en la Universidad de la Rioja. De este modo, se le ha dado continuidad a las primeras jornadas, iniciadas de un modo tan entusiasta por el Real Instituto y Observatorio de la Armada en 1998. El éxito de las primeras motivó la celebración de éstas y creemos, sin ninguna duda, que a la vista de las excelentes comunicaciones —algunas de las cuales se presentan en este volumen— del ambiente de trabajo y discusión científica encontrado en Logroño, será posible la continuidad de estas Jornadas.

El principal objetivo de estas jornadas era el tener un lugar de encuentro donde pudiéramos intercambiar con colegas nuestras líneas actuales de investigación, cuáles eran nuestros progresos científicos, sin tener que esperar a enterarnos por las publicaciones y, además, el poder trabajar conjuntamente, aportar diferentes puntos de vista y de técnicas a determinados problemas y el poder establecer los contactos que permitieran el trabajar conjuntamente grupos de investigación de diferentes centros. Creemos que todo eso ha sido cubierto, y nos consta que hay ya varios proyectos de investigación coordinados en marcha.

Queremos destacar desde estas líneas a los conferenciantes invitados, los doctores Alberto Abad, Sławek Breiter y Sebastián Ferrer, por el esfuerzo realizado en presentarnos en sus charlas el *estado del arte* en cuestiones como el Algebra computacional, Resonancias en la teoría del satélite artificial y la Integrabilidad de problemas keplerianos perturbados.

El éxito de estas jornadas se debe a la contribución de mucha gente. En primer lugar a los asistentes; estas jornadas han contado con 27 participantes de 8 instituciones diferentes y se presentaron 12 comunicaciones orales y tres conferencias invitadas. El Comité organizador local se preocupó en todo momento por hacernos agradable la estancia y programó unas actividades lúdicas a la par que informativas, como lo fue la visita a San Millán de la Cogolla, cuna del Castellano y donde tuvimos ocasión de contemplar e incluso hojear verdaderas maravillas de nuestros predecesores en la Ciencia, bajo la tutela de un guía, que estoy seguro tardaremos en encontrar otro de similar profesionalidad. El Departamento de Matemáticas y Computación de la Universidad de la Rioja y especialmente su Director, el Dr. Extremiana, puso a nuestra disposición sus instalaciones, por lo que le estamos reconocidos, así como al Vicerrectorado de Investigación de esta universidad

por su ayuda financiera.

Si estas actas ven la luz, se debe en gran parte al Profesor Rafael Cid Palacios, Académico editor de la Revista de la Academia de Ciencias de Zaragoza, y maestro directo de todos los que participamos en las Jornadas, que nos ha dado todo tipo de facilidades para que aparezcan como una monografía de la Academia.

Por último, mientras estábamos recopilando los artículos, recibimos la triste noticia del fallecimiento de nuestro colega Félix Mondéjar. Nada hacía presagiar tal desenlace, pues tan sólo un par de días antes nos remitía su comunicación revisada. En nuestro caso, hacía muy pocos años que lo conocíamos, pero en ellos, y a través de encuentros en congresos y un par de estancias de investigación que realizó en la Universidad de Zaragoza, reconocimos en él a un matemático brillante, con gran rigor en sus formulaciones y un trabajador incansable, al mismo tiempo que una gran persona. Lo echaremos en falta, pero nos queda su corta, pero excelente, labor científica y el recuerdo de su hombría de bien.

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Nota necrológica

Queremos recordar aquí a nuestro querido amigo y compañero Félix Mondéjar Alacid, desdichadamente fallecido el pasado dos de noviembre de 1999, que no sólo ha dejado desconsolados a sus padres, de los que era hijo único, sino a todos los que le conocíamos y apreciábamos.

Félix acababa de pasar de Profesor Ayudante a Profesor Asociado a tiempo completo del Departamento de Matemática Aplicada y Estadística en la recién creada Universidad Politécnica de Cartagena, con el fin resolver algunos problemas coyunturales del Departamento y para hacerse cargo de la docencia de nuevas asignaturas. Se hallaba entre nosotros desde octubre de 1995 y en este primer cuatrimestre del curso 1999-2000 estaba prevista la lectura de su tesis doctoral, titulada *Integrability and Reduction in Nonlinear Hamiltonian Mechanics*, que como codirectores teníamos sobre la mesa, para posibles correcciones, con el fin de que pudiese ser defendida a comienzos del año 2000.

Desde su entrada en el Departamento supimos de sus inquietudes científicas sobre los sistemas dinámicos hamiltonianos y sus aplicaciones, en particular a la Mecánica Celeste. Así, en relación con la línea de investigación sobre reducciones y equilibrios en problemas de Mecánica Celeste, aplicó los teroremas de Marsden y colaboradores a problemas de movimiento de uno, dos y tres giróstatos. Además, empezó a profundizar en la teoría de no integrabilidad de sistemas hamiltonianos, aplicando las técnicas más recientes, basadas en la teoría diferencial de Galois. En resumen, Félix era un buen matemático, con una sólida formación, que amaba las Matemáticas y la investigación.

Sobre sus extraordinarias dotes científicas, debemos señalar sus cualidades humanas de buen compañero, desprendido y generoso. Podemos decir, con palabras oídas a su madre, que *Félix era mucho Félix.*

Hasta siempre, amigo

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Lunisolar resonant effects on artificial satellites' orbits

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1. Introduction

Lunisolar perturbations are the most important mechanism delivering major Earth-orbiting objects into the upper atmosphere. Resonant lunisolar effects become the most interesting subject as far as the satellites lifetime is considered.

Resonances of the mean motion type are not possible at typical satellite orbits, but there exist a possibility of various commensurabilities between the secular perturbations of a satellite's node or apsis and the mean motion of the Sun or the Moon. These resonances, first identified by Musen (1960), can be of two types:

i) *eccentricity* or *apsidal* resonances, with critical arguments containing the argument of perigee g,

ii) *inclination* or *nodal* resonances, with critical arguments independent on g but containing the right ascension of the ascending node h.

The second group contains the well known heliosynchronous orbits and was studied extensively by many authors. The first one attracted less attention. Cook (1962) indicated 15 apsidal resonances and Hughes (1981) increased this number, but none of these authors answered the questions about the location of critical points, their stability and the width of libration zones. "Frozen orbits", i.e. five of apsidal resonances whose critical arguments do not depend on the perturbing bodies' longitudes are the only exception thanks to the works of Lidov (1961), Lorell (1965) and their followers. The remaining 10 apsidal resonances have been recently studied by the author (Breiter, 1999).

2. Formulation of the Problem

To achieve the first, general look at the family of 10 isolated single resonances, strong physical and mathematical assumptions have been imposed. The geopotential is restricted to the J_2 term and the Sun, as well as the Moon, is treated as a distant body (parallaxes neglected) moving on

circular orbits with constant inclinations. It is assumed that the Hamiltonian $\mathcal{H}$ of the problem has been normalized and it no longer depends on the satellite's mean anomaly.

Let us consider a single resonant periodic term with the argument

$$\varphi_{m,k} = g + \tfrac{1}{2} m\, h + k\, \lambda_p, \tag{1}$$

where λ_p is the mean longitude of the Sun or the Moon. If no other resonances appear, all remaining periodic terms can be removed from $\mathcal{H}$ by means of a properly defined Lie transformation. The new Hamiltonian $\mathcal{K}_{m,k}$ describes a one degree of freedom system with $\eta = \sqrt{1-e^2}$ as the momentum conjugate to $\varphi_{m,k}$. The general form is

$$\mathcal{K}_{m,k} = k\, n_p\, \eta + Z_1 + C_{m,k} \cos 2\, \varphi_{m,k}, \tag{2}$$

where Z_1 is the normalized J_2 part, $C_{m,k}$ comes from the Solar/Lunar perturbing function, and $n_p = \dot{\lambda}_p$. The terms Z_1 and $C_{m,k}$ depend on η and a as well as on a constant of motion α_m

$$\alpha_m = \begin{cases} \eta\,(\frac{1}{2}\, m - \cos I) & \text{for} \quad m > 0, \\ -\eta\,(\frac{1}{2}\, m - \cos I) & \text{for} \quad m \leq 0. \end{cases} \tag{3}$$

3. Location of critical points and their stability

Equations of motion in a single apsidal resonance take the form

$$\dot{\varphi}_{m,k} = \frac{\partial \mathcal{K}_{m,k}}{\partial \eta} = k\, n_p + Z_1' + C_{m,k}' \cos 2\, \varphi_{m,k}, \tag{4}$$

$$\dot{\eta} = -\frac{\partial \mathcal{K}_{m,k}}{\partial \varphi_{m,k}} = 2\, C_{m,k} \sin 2\, \varphi_{m,k}, \tag{5}$$

where the primes designate partial derivatives with respect to η. The critical points exist at

A) $\varphi_{m,k} = 0, \pi,$

B) $\varphi_{m,k} = \frac{1}{2}\,\pi, \frac{3}{2}\,\pi,$

The case of vanishing $C_{m,k}$ should not be considered in a cylindric $\varphi_{m,k}$, η parametrisation.

The values of η for the critical points A and B are given by the resonance conditions

$$k\, n_p + Z_1' + \sigma\, C_{m,k}' = 0. \tag{6}$$

Symbol σ selects a proper sign for a given critical point: $\sigma = 1$ for the points A, and $\sigma = -1$ for the points B. Equations (6) can be simplified if $C_{m,k}'$ is neglected, but even in this case we cannot solve them explicitly to obtain the critical values $\hat{\eta}$ as functions of α_m. Fortunately, Eqns. (6) are quadratic with respect to α_m and given the value of $\hat{\eta}$ we can obtain two roots

$$\alpha_{m,k,j} = \tfrac{1}{5}\,\hat{\eta} \left(2\,|m| + j \sqrt{5\,(1 - k\, z_p\, \hat{\eta}^4) + \tfrac{1}{4}\, m^2} \right), \quad j = \pm 1, \tag{7}$$

where

$$z_p = \frac{4}{3} \frac{n_p\, a^2}{J_2\, n\, R^2} \tag{8}$$

is an auxiliary dimensionless parameter. The symbols a, n, R stand for the satellite's semimajor axis, mean motion and Earth radius respectively.

For the stability of critical points we can use the eigenvalues ν of the associated variational equations

$$\nu^2 - 4\sigma \left[\dot{C}_{m,k}\, Z_1''\right]_{\eta=\hat{\eta}} = 0. \tag{9}$$

For sufficiently low orbits the approximated equation (9) provides qualitatively correct results.

The maximum amplitude of variations in η due to a resonance can be approximated as a half of the "resonance width," i.e. of the distance between two separatrices which encircle the libration zone (Garfinkel, 1966). Supposing, that the amplitude $\Delta_{m,k}$ is a small quantity, we obtain approximately

$$\Delta_{m,k} = 2\left[\sqrt{\frac{|C_{m,k}|}{|Z_1''|}}\right]_{\eta=\hat{\eta}}. \tag{10}$$

4. Brief summary of the results

The fact that α_m has the same definition regardless of k and certain symmetries in Z_1 or $C_{m,k}$ permit the division of 10 resonances in three families:

1. $m = 0$, with $\varphi_{0,k} = g + k\lambda_p$, and $k = \pm 1$,
2. $m = \pm 1$, with $\varphi_{\pm 1,k} = g \pm \frac{1}{2}h + k\lambda_p$, and $k = \pm 1$,
3. $m = \pm 2$, with $\varphi_{\pm 2,k} = g \pm h + k\lambda_p$, and $k = \pm 1$.

The resonances with $k = 1$ do not occur if the Moon is taken as a perturbing body. Depending on the values of α_m and z_p one can observe different numbers of critical points in $\varphi_{m,k}$, η plane. Speaking about the $0 \leq \varphi_{m,k} < \pi$ subdomain only, we can have:

1. up to two (A,B) pairs for $m = 0, \pm 1$,
2. up to three (A,B) pairs for $m = \pm 2$.

If there is more than one (A,B) pair, the stability of A points alternates along the line $\varphi_{m,k} = 0$. The same happens for the points B along $\varphi_{m,k} = \frac{1}{2}$.

Generally speaking, lunar resonances are much weaker than their solar counterparts. The maximum amplitude for a solar resonance is about 750 km in the perigee height for $m = -2, k = 1, j = -1$; next comes 340 km for $m = -2, k = 1$. For the Moon we have at most 15 km for $m = -2, k = 1$ and all other lunar resonances do not exceed few kilometers.

Suppressing some simplifications imposed in the presented model we discover a more complicated picture even for the single degree of freedom case. Tangent and pitchfork bifurcations occur as well as separatrix bifurcations, producing a large variety of the regimes of motion. It seems that lunisolar resonances constitute the problem which is not only important for space mission design but also mathematically attractive even in the first approximation.

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A solution to the artificial satellite problem in a focal formulation

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Abstract

We perform an analytical treatment of the *Main Problem* of Artificial Satellite Theory in terms of a set of *regular elements* attached to linearizing focal variables.

Focal–type variables reduce the Kepler problem to four uncoupled linear oscillators with the true anomaly as the independent variable, while perturbed Keplerian systems are generally brought into coupled and non–linearly forced oscillators.

Focal elements are constants of integration occurring in the general solution of the harmonic oscillator equations generated by the pure Kepler problem in focal variables, whereas for the perturbed problem they satisfy a system of first–order differential equations (with a true–like anomaly as the time parameter).

We give element equations corresponding to a focal formulation of perturbed two–body orbital motion, apply our developments to the J_2 problem of an artificial satellite, and approach the analytical integration of the resulting element equations by a Fourier–type series expansion method in terms of the said true–like anomaly.

1. Introduction

Within the framework of a *linear and regular* approach to Celestial Mechanics problems (Kustaanheimo & Stiefel 1965; Stiefel & Scheifele 1971; Deprit *et al.* 1994), and for application to *elliptic–type orbits*, Sharaf & Saad (1997) gave an analytical expansion of the Earth's gravitational *zonal potential* in Kustaanheimo–Stiefel (KS) *regular elements*.

Inspired in that KS–*regular–element* approach, and adapting the analytical treatment of Stiefel & Scheifele (1971), §19, we translate it into a *focal–method* version (Burdet 1969, §2; Ferrándiz 1988; Deprit *et al.* 1994, §4): we construct *element equations in a DEF–formulation* (differential equations for the variation, under perturbations, of the elements attached to the Kepler problem in DEF–variables), and then apply these developments to the *Main Problem* of Artificial Satellite Theory, once the J_2 harmonic is expressed in DEF–elements. We give a first approach to the element treatment of the J_2 Problem, studying the Hamiltonian system

in *focal variables* with a true–like anomaly as the pseudo–time. Our results are not limited to elliptic–type orbits: they are *valid* for any kind of orbit.

The (weakly) canonical extension of the point–transformation to focal coordinates proposed by Deprit, Elipe and Ferrer was designed to *exactly* linearize the equations of motion of the spatial Kepler problem, giving it the form of a *4–dimensional harmonic oscillator*; perturbed Keplerian systems are brought into perturbed harmonic equations.

The *DEF–elements of the motion* are integration constants in the general solution of the linear oscillator linked to the Kepler problem in DEF–variables. For the perturbed problem they satisfy a system of first–order equations: the equations of motion of the perturbed problem can be treated by the *method of variation of constants*, which leads to that system of differential equations for these quantities (with the true anomaly as the time parameter). These equations are *supplemented* by the equations for the variation of other quantities (law of variation of the angular momentum, variation of the energy) and by the equation for the variation of time. All these relations are the *element equations.*

Elements undergo slow variations under perturbations. Accordingly, their analytical and numerical behavior should be better than that of the coordinates of the DEF–set.

In our case, the canonical equations issued from the J_2–Hamiltonian are replaced by second–order equations (for the DEF–coordinates) with respect to the *true anomaly* as the pseudo–time. These quasi–linear equations govern a set of perturbed oscillators. To obtain a *solution with the DEF element equations*, we approach the integration of these equations by a Fourier series expansion method, by expressing the perturbing potential in terms of functions of the independent variable which are explicitly known by means of formulae constructed by harmonic analysis. The coefficients of that literal expressions will depend on the oscillator DEF–elements. Thus, the right-hand sides of the equations become functions of the independent variable given by true–anomaly expansions which are known by analytical formulae. An approximate analytical solution to the resulting equations is studied according to a procedure outlined by Stiefel & Scheifele (1971), §28.

2. A Perturbed Keplerian System in DEF–Variables

Let $(\mathbf{x}, \mathbf{X}) = (x_1, x_2, x_3, X_1, X_2, X_3)$ be the *canonical Cartesian variables*, where $\mathbf{x}$ are the Cartesian coordinates with origin at the centre of mass of the primary, and the conjugate momenta are (X_1, X_2, X_3). The canonical set of (redundant dependent) *DEF–variables*, $(u_0, \mathbf{u}, U_0, \mathbf{U})$, is defined from the Cartesian ones by the *DEF–mapping*,

$$\mathbf{x} \;=\; u_0\mathbf{u}\,, \quad u_0 \in \mathbb{R}_+\,, \qquad \mathbf{X} \;=\; U_0\mathbf{u} + [\,(\mathbf{u}\times\mathbf{U})\times\mathbf{u}\,]/u_0\,, \quad U_0 \in \mathbb{R}\,, \tag{1}$$

with $r = \|\mathbf{x}\| = u_0\|\mathbf{u}\|$. Some useful notations are: $\|\mathbf{x}\times\mathbf{X}\|^2 = \|\mathbf{u}\|^4\,\|\mathbf{Q}\|^2 = \beta^4\,\|\mathbf{Q}\|^2 = \beta^4\,Q^2$, $\mathbf{Q} = \mathbf{u}\times\mathbf{U}$, $\beta = \|\mathbf{u}\|$, $Q = \|\mathbf{Q}\|$. Certain properties (e.g., *weak* canonicity) require the mapping to be restricted to the manifold $\beta = \|\mathbf{u}\| = 1$. Further details are analyzed by Deprit, Elipe & Ferrer (1994), §§4.1. The DEF–mapping converts the perturbed Keplerian Hamiltonian $\mathcal{H}$,

with the perturbing potential W depending on the position vector $\mathbf{x}$, into the transformed Hamiltonian $\mathcal{K}$:

$$\mathcal{H} \;=\; \frac{1}{2}\|\mathbf{X}\|^2 - \frac{\mu}{r} \;+\; W(\mathbf{x}) \;\longrightarrow\; \mathcal{K} = \frac{\beta^2}{2}\left(U_0^2 + \frac{Q^2}{u_0^2}\right) - \frac{\mu}{\beta u_0} + W(u_0, \mathbf{u})\,. \tag{2}$$

A pseudo–time f (true anomaly) is introduced by a *generalized Sundman time transformation*, and a *new dependent variable* σ is defined to replace the scalar variable u_0:

$$t \;\longrightarrow\; f: \;\; dt = \left(u_0^2/\beta^2 Q\right) df\,, \quad u_0 \;\longrightarrow\; \sigma: \;\; \sigma \;=\; Q^2/(\mu u_0)\,. \tag{3}$$

The DEF–linearization technique leads to the following system of *second–order differential equations* for the new coordinates $(\sigma, \mathbf{u})$, with f as the independent variable:

$$\mathbf{u}'' \;+\; \mathbf{u} \;=\; \left[\frac{1}{Q}\mathbf{Q}' - \frac{1}{Q^3}\left(\mathbf{Q}.\mathbf{Q}'\right)\mathbf{Q}\right] \times \mathbf{u}\,, \tag{4}$$

$$\sigma'' \;+\; \sigma \;=\; 1 \;+\; \left[\frac{2}{Q^2}\left\{\|\mathbf{Q}'\|^2 + \left(\mathbf{Q}.\mathbf{Q}''\right)\right\} - \frac{6}{Q^2}\left(Q'\right)^2\right]\sigma \tag{5}$$

$$+\frac{3Q'}{Q}\sigma' - \frac{Q^2}{\mu^2}\frac{\partial W}{\partial \sigma}\,, \tag{6}$$

$$\mathbf{Q}' \;=\; \frac{u_0^2}{Q}\left(\nabla_{\mathbf{u}} W \times \mathbf{u}\right) = \frac{Q^3}{\mu^2\sigma^2}\left(\nabla_{\mathbf{u}} W \times \mathbf{u}\right), \quad Q' \;=\; \frac{\left(\mathbf{Q}.\mathbf{Q}'\right)}{Q}\,, \tag{7}$$

$$t' \;=\; \left(Q^3/\mu^2\sigma^2\right)\,, \tag{8}$$

$\nabla_{\mathbf{u}} W$ being the gradient of the scalar function W with respect to the vector (u_1, u_2, u_3). A *solution to the Kepler problem* is really obtained starting from the above equations:

$$u_j \;=\; \alpha_j \cos f \;+\; \beta_j \sin f\,, \;\; j = 1,2,3\,, \quad \sigma \;=\; \alpha_0 \cos f \;+\; \beta_0 \sin f \;+\; 1\,. \tag{9}$$

The quantities α_k, β_k are *"elements of the motion"*. The equations of motion of the perturbed two–body problem can be treated by the method of *variation of constants*, which leads to a system of first–order differential equations for the functions $\alpha_k(f), \beta_k(f)$.

3. Towards a DEF–Treatment of the Main Problem

The *second zonal harmonic* of the geopotential, in Cartesian and DEF variables, reads:

$$V(\mathbf{x}) \;=\; \frac{\widetilde{\varepsilon}}{r^3}\left[\frac{1}{2}\left(3\left(\frac{x_3}{r}\right)^2 - 1\right)\right] \;\longrightarrow\; V(u_0, u_3) = \frac{\widetilde{\varepsilon}}{2\,u_0^3}\left[3u_3^2 - 1\right],\; \widetilde{\varepsilon} \;=\; \mu R^2 J_2\,. \tag{10}$$

Introducing a vector $\tilde{\mathbf{u}} = (u_2, -u_1, 0)$, the equations governing the *Main Problem* are

$$\mathbf{u}'' \;+\; \mathbf{u} \;=\; \frac{u_0^2}{\beta^2 Q^2}\frac{\partial V}{\partial u_3}\left[u_3'\mathbf{u}' - (\tilde{\mathbf{u}} \times \mathbf{u})\right], \tag{11}$$

$$\sigma'' \;+\; \sigma - 1 \;=\; 2\left[\frac{Q''}{Q} - 2\frac{(Q')^2}{Q^2}\right]\sigma + 3\frac{Q'}{Q}\sigma' + \frac{u_0^2}{\beta^2\mu}\frac{\partial V}{\partial u_0}\,. \tag{12}$$

As in Stiefel & Scheifele (1971, §28, §19), to develop an approximate analytical solution to this differential system, we start from the *reference solution* to the harmonic equations stemming from the pure Kepler problem, with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$:

$$\mathbf{u}(f) \;=\; \alpha(0)\cos f + \beta(0)\sin f\,, \quad \sigma(f) \;=\; \alpha_0(0)\cos f + \beta_0(0)\sin f + 1\,. \tag{13}$$

The above equations of motion are treated according to the method of variation of constants, which yields the first–order differential system of *element equations*

$$\alpha' \;=\; -\left[\frac{u_0^2}{\beta^2 Q^2}\frac{\partial V}{\partial u_3}\left[u_3'\mathbf{u}' - (\tilde{\mathbf{u}}\times\mathbf{u})\right]\right]\sin f\,, \tag{14}$$

$$\beta' \;=\; \left[\frac{u_0^2}{\beta^2 Q^2}\frac{\partial V}{\partial u_3}\left[u_3'\mathbf{u}' - (\tilde{\mathbf{u}}\times\mathbf{u})\right]\right]\cos f\,, \tag{15}$$

$$\alpha_0' \;=\; -\left[2\left(\frac{Q''}{Q} - 2\frac{(Q')^2}{Q^2}\right)\sigma + 3\frac{Q'}{Q}\sigma' + \frac{u_0^2}{\beta^2\mu}\frac{\partial V}{\partial u_0}\right]\sin f\,, \tag{16}$$

$$\beta_0' \;=\; \left[2\left(\frac{Q''}{Q} - 2\frac{(Q')^2}{Q^2}\right)\sigma + 3\frac{Q'}{Q}\sigma' + \frac{u_0^2}{\beta^2\mu}\frac{\partial V}{\partial u_0}\right]\cos f\,, \tag{17}$$

whereas the law of variation of the magnitude of the angular momentum vector is

$$Q' \;=\; -\left(u_0^2/Q\right)\left(\partial V/\partial u_3\right)u_3'\,. \tag{18}$$

Now the right–hand sides of these element differential equations are to be expressed as Fourier expansions in the true anomaly f, which allows us to undertake an approximate analytical solution of the element equations. A first–order solution, accounting for the first–order variations of the elements due to the main oblateness perturbation, is obtained after inserting the constant values for the elements of the Keplerian reference orbit into the right–hand sides of the resulting equations.

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Órbitas de satélites artificiales estrictamente disipativas

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Resumen

Estudiamos un sistema disipativo compuesto por un cuerpo rígido esférico y metálico que describe una órbita kepleriana circular en torno a un dipolo magnético; para ese sistema obtenemos el intervalo de inclinaciones en el que dichas órbitas corresponden a las soluciones de un sistema lagrangiano estrictamente disipativo y determinamos la función de Rayleigh correspondiente.

1. Sistema magneto-mecánico disipativo

En la descripción del movimiento de satélites artificiales metálicos se considera (véase [1]), entre otras, la acción magnética del cuerpo central que, debido a la rotación del satélite, induce corrientes eléctricas y a su vez éstas dan lugar a una interacción con el campo magnético exterior a través de fuerzas de fricción causantes del frenado de la rotación del satélite. Este sistema dinámico es de tipo lagrangiano y puede describirse en los siguientes términos: sea Q el espacio de configuración definido por el grupo euclidiano especial de los movimientos rígidos en $\mathbb{R}^3$, cuyos elementos representaremos por $(\mathbf{q}_1, \mathbf{q}_2)$ donde la primera componente denota las variables orbitales y la segunda las variable asociadas al cuerpo, y sea TQ el espacio de fases lagrangiano parametrizado por $(\mathbf{q}_1, \mathbf{q}_2, \dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2) \in TQ$. La función de Lagrange para este sistema mecánico es de la forma

$$L := TQ \to \mathbb{R}, \quad (\mathbf{q}, \dot{\mathbf{q}}) \mapsto \frac{1}{2} m \|\dot{\mathbf{q}}_1\|^2 + \frac{1}{2} \dot{\mathbf{q}}_2^T I \dot{\mathbf{q}}_2 + U(\mathbf{q}_1) \tag{1}$$

donde m representa la masa del satélite, I el tensor de inercia y $U(\mathbf{q}_1)$ designa a la energía potencial gravitatoria.

Supongamos que el satélite está formado por una esfera metálica que describe órbitas circulares con inclinación ι respecto al eje magnético $(O, \mathbf{u})$ —donde O es el punto del espacio ocupado por el centro de masas del cuerpo central (esférico y homogéneo) y $\mathbf{u}$ es un vector unitario— y admitamos que dicha órbita está contenida en una región de $\mathbb{R}^3 \backslash \{\mathbf{0}\}$ en la que está definido un campo magnético $\mathbf{H}$ de tipo dipolar, dado por

$$\mathbf{H} = \nabla \left(\|\mathbf{x}\|^{-3} (\mathcal{M} \cdot \mathbf{x}) \right), \tag{2}$$

donde $\mathcal{M}$ es el momento dipolar magnético del cuerpo central, que en el caso que vamos a considerar es un vector constante en la referencia del espacio $(O, \mathbf{e}_s)$, y $\mathbf{x}$ es el vector posición del satélite en dicha referencia. Denotemos por $\mathbf{m}$ el momento magnético inducido sobre el satélite:

$$\mathbf{m} = c_1 \mathbf{H} \times \omega, \qquad (c_1 = \text{cte}), \tag{3}$$

y por $\mathbf{N}$ el par de fuerzas correspondiente:

$$\mathbf{N} = \mathbf{m} \times \mathbf{H}. \tag{4}$$

Puesto que la perturbación de origen magnético produce variaciones pequeñas en la velocidad angular del satélite durante un periodo orbital, puede simplificarse el problema si se promedia $\mathbf{N}$ a lo largo de la órbita (cfr. [1]); se obtiene en este caso:

$$\langle \mathbf{N} \rangle = c_2 \, \mathbf{B}_s \omega, \tag{5}$$

donde $c_2 \, (> 0)$ es una constante y $\mathbf{B}_s$ es una matriz simétrica. Fijada una referencia en el espacio, $(O, \{\mathbf{e}_i\}_{i=1}^3)$ con $\mathbf{e}_3 = \mathbf{u}$, los elementos no nulos de la matriz $\mathbf{B}_s$ son funciones que dependen únicamente de la inclinación de la órbita y tienen la forma:

$$\begin{aligned} b_{11} &= \tfrac{1}{8}(27\cos^4 \iota - 39\cos^2 \iota + 20) & b_{22} &= \tfrac{1}{8}(-\cos^2 \iota + 11) \\ b_{33} &= \tfrac{3}{8}(-9\cos^4 \iota - 4\cos^2 \iota + 3) & b_{13} &= \tfrac{3}{8}(5 - \cos^2 \iota)\cos \iota \operatorname{sen} \iota. \end{aligned} \tag{6}$$

Puesto que conviene describir la dinámica del sólido rígido desde la referencia del cuerpo, $(O', \{\mathbf{e}'_i\}_{i=1}^3)$, con origen en el centro de masas del cuerpo y vectores unitarios en las direcciones principales de inercia, escribiremos en dicha referencia la igualdad (5) como

$$\mathbf{N}_c = -c_2 \mathbf{A} \mathbf{B}_s \mathbf{A}^T \omega_c, \tag{7}$$

donde $\mathbf{A}$ es la matriz ortogonal del cambio de base desde la referencia del espacio hasta la del cuerpo (la expresión de $\mathbf{A}$ en función de los ángulos de Euler puede verse en [2]), de modo que si se denotan por $\mathbf{R}$ las fuerzas generalizadas asociadas a (7) que actúan durante el movimiento, entonces —una vez efectuada la reducción al centro de masas del satélite— las ecuaciones lagrangianas del movimiento son de la forma (cfr. [4])

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_2}\right) - \frac{\partial L}{\partial \mathbf{q}_2} = \mathbf{R}. \tag{8}$$

2. Región de disipación estricta

En primer lugar vamos a probar que en el caso que venimos considerando *existe una forma cuadrática F, definida en el espacio al que pertenece* $\dot{\mathbf{q}}_2$, tal que $\mathbf{R} = -F'_{\dot{q}_2}$. Para ello representemos por $\mathbf{J}$ la matriz jacobiana de la transformación $\omega = \omega(\dot{\mathbf{q}})$; la relación entre el par de fuerzas (7) referido al cuerpo y la fuerza generalizada $\mathbf{R}$ puede escribirse en la forma siguiente

$$\mathbf{R} = -(\mathbf{J}^T \mathbf{A} \mathbf{B}_s \mathbf{A}^T \mathbf{J}) \dot{\mathbf{q}}_2 =: \mathbf{F}(\mathbf{q}_2, \iota) \, \dot{\mathbf{q}}_2 \tag{9}$$

donde $F(\mathbf{q}_2, \iota)$ es una matriz simétrica asociada a una forma cuadrática

$$F := \dot{\mathbf{q}}_2{}^T \mathbf{F} \dot{\mathbf{q}}_2. \tag{10}$$

Por consiguiente, el segundo miembro de la ecuación de Lagrange (8) puede escribirse como el gradiente $\mathbf{R} = -\partial F / \partial \dot{\mathbf{q}}_2$.

Veamos ahora que la forma cuadrática F es definida positiva , y por tanto *el sistema dinámico* (L, TQ) *es estrictamente disipativo* en el sentido de que la energía decrece monótonamente a lo largo de cualquier movimiento excepto los correspondientes a equilibrios relativos. La región $\mathcal{R} \subset Q_1$ en la que el sistema magneto-mecánico anterior es estrictamente disipativo queda determinada por el conjunto de puntos $(\theta, \varphi, \iota) \in \mathcal{R}$ donde los menores principales

$$\begin{aligned} \Delta_1 &:= b_{22} + (b_{11} - b_{22}) \cos^2(\varphi), \\ \Delta_2 &:= b_{22} b_{33} + (b_{11} b_{33} - b_{22} b_{33} - b_{13}^2) \cos^2 \varphi, \\ \Delta_3 &:= -b_{22} (b_{13}^2 - b_{11} b_{33}) \operatorname{sen}^2 \iota, \end{aligned} \tag{11}$$

de la forma cuadrática F sean positivos (cfr. [3]).

A partir de las expresiones (6) se deduce que la función $\Delta_1(\theta, \varphi, \iota)$ es positiva para cualquier elemento de $cqlR$.

En cuanto al segundo menor principal, $\Delta_2(\theta, \varphi, \iota)$, éste es positivo siempre que se cumpla la condición

$$(27\zeta^6 - 87\zeta^4 - 53\zeta^2 + 33) \cos^2 \varphi + 99\zeta^6 - 163\zeta^4 + 73\zeta^2 + 9 > 0,$$

donde $\zeta := \cos \iota$. La región máxima del plano (φ, ι) en la que se cumple esta condición para un valor fijo de ι y valores arbitrarios de φ es la contenida entre dos rectas $\iota = \text{cte}$ de dicho plano definidas por las ecuaciones

$$\begin{aligned} 27\zeta^6 - 87\zeta^4 - 53\zeta^2 + 33 &= 0 \\ 99\zeta^6 - 163\zeta^4 + 73\zeta^2 - 9 &= 0. \end{aligned} \tag{12}$$

La solución numérica de estas ecuaciones muestra que el menor principal Δ_2 es positivo para cualquier valor fijo de $\iota \in (0.9081, 2.2334)$ con (θ, φ) arbitrarios.

Por último $\Delta_3(\theta, \varphi, \iota)$, que sólo depende de las variables θ, ι, es positivo siempre que se cumpla la inecuación

$$135\zeta^6 - 201\zeta^4 + 136\zeta^2 - 30 > 0.$$

El conjunto solución de esta inecuación está formado valores de ι pertenecientes al intervalo $(0.9081, 2.2334)$. Por consiguiente, puede afirmarse que el sistema dinámico considerado es totalmente disipativo en la región $\mathcal{R} = [0, 2\pi) \times [0, 2\pi) \times (0.9081, 2.2334)$. En el caso de satélites artificiales como los LAGEOS I, su órbita está contenida en la región de disipación estricta anterior.

En la región $\mathcal{R}$ la función F definida en (10) es la función de Rayleigh para el sistema mecánico considerado.

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Órbitas periódicas alrededor de cuerpos alargados

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Resumen

Empleamos el campo gravitatorio de un segmento masivo en rotación pura y uniforme como aproximación de la atracción creada por cuerpos celestes alargados. Calculamos, en un referencial rotante solidario con el cuerpo, puntos estacionarios, familias de soluciones periódicas y determinamos su estabilidad lineal. Presentamos, además, una implementación diferente a la dada por Lara para el método de prolongación de familias naturales de órbitas periódicas dado por Deprit y Henrard.

1. Introducción

En este trabajo presentamos el problema de la descripción del movimiento de una partícula entorno a un segmento masivo en rotación pura. El estudio de este problema puede ayudar a comprender mejor la dinámica entorno a objetos celestes irregulares, especialmente los más alargados como los asteroides 433 Eros, 4179 Toutatis , 4769 Castalia ó Geógrafos, algunos de ellos objetivos de misiones de las agencias espaciales [2], [1]. La antigüedad de objetos celestes pequeños, como son los satélites menores de planetas o asteroides, hace suponer la estabilidad de su movimiento y por tanto podemos suponer su proximidad al estado de menor energía para un momento angular dado, es decir, en rotación pura alrededor del principal eje de inercia.

Para la prolongación de familias naturales de órbitas periódicas usamos el algoritmo de A. Deprit y J. Henrard [3] modificado por M. Lara et. al [4]. Además de utilizar la implementación habitual de la integración numérica mediante series recurrentes de potencias, hemos hecho uso, paralelamente, de métodos Runge-Kutta.

2. Formulación del problema

Supondremos un segmento masivo rotando uniformemente en torno a su eje principal de inercia, con respecto a un cierto sistema inercial. Tomaremos la varilla sobre el eje Ox y la rotación se efectúa entorno al eje Oz con vector velocidad angular constante, ω, de norma ω. La ecuación vectorial del movimiento de una partícula, cuyo vector de posición denotamos $r = (x, y, z)$, en

el sistema sinódico es

$$\ddot{\boldsymbol{r}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} = -\nabla_{\boldsymbol{r}} U(\boldsymbol{r}),$$

donde el potencial gravitatorio por unidad de masa creado por la varilla es,

$$U = -\frac{\mu}{2\ell} \log\left(\frac{s+2\ell}{s-2\ell}\right), \tag{1}$$

siendo s es la distancia de la partícula a los extremos del segmento y 2ℓ es la longitud de éste. Como es habitual en Mecánica, podemos definir un potencial efectivo $W(x,y,z)$ como

$$W(x,y,z) = U(x,y,z) - \frac{\omega^2}{2}(x^2+y^2),$$

con esto, la función lagrangiana de las ecuaciones de movimiento de este problema es

$$\mathcal{L} = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2) + \omega(x\dot{y} - y\dot{x}) - W(x,y,z). \tag{2}$$

Efectuamos el cambio de escala equivalente a elegir 2ℓ, la longitud del segmento, como la unidad de longitud y $P/2\pi$ como unidad de tiempo, con P el período de rotación del segmento. Tras esta operación, el lagrangiano queda

$$\mathcal{L} = \omega^2(2\ell)^2\left[\tfrac{1}{2}(\dot{x}^2+\dot{y}^2) + (x\dot{y} - y\dot{x}) + \tfrac{1}{2}(x^2+y^2) + k\,\log\left(\frac{s+1}{s-1}\right)\right],$$

donde $k = GM/(\omega^2(2\ell)^3)$. Este parámetro adimensional k es el cociente entre la aceleración gravitatoria y la aceleración centrífuga. Valores de k inferiores a la unidad indican rotación rápida mientras que valores por encima de la unidad indican rotación lenta. Finalmente, las ecuaciones del movimiento pueden escribirse por componentes como

$$\begin{gathered} \ddot{x} - 2\dot{y} = -W_x = x\left(1 - \frac{2k}{sp}\right), \qquad \ddot{y} + 2\dot{x} = -W_y = y\left(1 - \frac{2ks}{(s^2-1)p}\right), \\ \ddot{z} = -W_z = -\frac{2kzs}{(s^2-1)p} \end{gathered} \tag{3}$$

siendo $s = r_1 + r_2$ y $p = r_1 r_2$ las funciones auxiliares con

$$r_1 = \sqrt{y^2+z^2+(x-\tfrac{1}{2})^2}, \quad r_2 = \sqrt{y^2+z^2+(x+\tfrac{1}{2})^2}.$$

Se comprueba fácilmente que este sistema admite la integral primera denominada de Jacobi que, denotando T a la energía cinética, tiene como expresión

$$C = 2W(x,y,z) + 2T = 2U(x,y,z) - (x^2+y^2) + (\dot{x}^2+\dot{y}^2+\dot{z}^2). \tag{4}$$

3. Soluciones de equilibrio

Los equilibrios del sistema resultan de anular los segundos miembros de (3). La única solución de la tercera ecuación es $z = 0$, puesto que $s \neq 0$ al ser la suma de dos distancias no nulas al mismo tiempo. Se comprueba, además, que no existen soluciones de equilibrio con x e y

simultáneamente no nulos con lo que los equilibrios existentes se sitúan sobre los ejes Ox y Oy. El origen es, también, solución de equilibrio pero no la estudiamos al carecer de sentido físico.

Equilibrios sobre el eje Ox. Hallar los equilibrios sobre el semieje positivo de Ox se reduce a resolver la cúbica $4x^3 - x - 4k = 0$ que posee una única solución real positiva, x_0. Por simetría, hay otra solución en el semieje negativo y las coordenadas de los dos puntos de equilibrio son $(\pm x_0, 0)$ y los denominaremos puntos colineales.

Equilibrios sobre el eje Oy. Análogamente, la solución de la ecuación $4r^3 - r - 4k = 0$, con r, la distancia del extremo del segmento al punto de equilibrio, nos proporciona las coordenadas de las dos soluciones simétricas sobre el eje Oy, $(0, \pm\sqrt{r^2 - 1/4})$ que llamaremos puntos isósceles.

Para estudiar la estabilidad lineal de los puntos hallados no tenemos más que calcular la matriz de coeficientes de las ecuaciones variacionales asociadas a dichas soluciones de equilibrio. Como sobre el eje Oz el movimiento es armónico nos restringiremos a las variables x e y. Esta matriz se forma con las derivadas segundas del potencial efectivo W. Sin más que reemplazar los valores en la solución de equilibrio y estudiar el espectro de la matriz resultante podremos obtener la estabilidad lineal.

Para los colineales encontramos que el polinomio característico es $\lambda^4 + (1-b)\lambda^2 - (3+2b)b$ con $b = 1/(4\zeta(1+\zeta))$. Las raíces cuadradas de los ceros del polinomio son los valores propios que resultan ser distintos, con al menos uno de ellos siempre de parte real positiva para cualquier valor del parámetro k. Por tanto, los puntos son linealmente inestables.

Para los puntos isósceles el un polinomio característico de la forma $\lambda^4 + \lambda^2 + (3-a)a$, con $a = 1/(4r^2)$. Las distintas posibilidades para los valores propios dependiendo de a son: si $a_c = (3 - 2\sqrt{2})/2 < 1$, para $0 < a \leq a_c$ todos los valores propios distintos y tienen parte real nula y por lo tanto son linealmente estables, mientras que para $a_c < a < 1$ dos de los valores propios tienen parte real negativa y los otros dos positiva, y por consiguiente, estos puntos son linealmente inestables. Para el valor límite la matriz es no diagonalizable y tenemos inestabilidad lineal de nuevo.

4. Órbitas periódicas

En el sistema de referencia solidario con el segmento hemos encontrado soluciones de equilibrio, como es conocido, en las inmediaciones de estos puntos de equilibrio es posible encontrar órbitas periódicas (ver, por ejemplo, Verhulst [5]). Este hecho se deduce a partir de los términos cuadráticos del desarrollo de Taylor de W alrededor del punto de equilibrio. Si el punto de equilibrio es no singular y, en el desarrollo resultante, todos los coeficientes son positivos en un entorno del equilibrio, las superficies de nivel del flujo son difeomorfas a esferas. Las soluciones dentro de ese entorno tienen, por tanto, forma de pequeñas elipses, en primera aproximación.

Encontramos varias familias de órbitas periódicas en el plano Oxy usando el algoritmo formulado por M. Lara et. al. en [4] para la continuación numérica de soluciones periódicas dependientes de un parámetro. Empleando este algoritmo comenzamos con un conjunto de condiciones iniciales cercanas a una órbita periódica las corregimos hasta hallar verdaderas condiciones de una órbita periódica. Entonces, incrementando el valor del parámetro y calculando y refinando

una predicción tangente, obtenemos un nuevo conjunto de condiciones iniciales para una nueva solución periódica donde el parámetro elegido tiene un nuevo valor.

Los puntos colineales verifican las condiciones de existencia de órbitas periódicas en sus proximidades no siendo así para los puntos isósceles. Sin embargo, en las proximidades de ambos tipos de puntos, hemos hallado órbitas periódicas para variaciones de la energía. Ejemplos de estos resultados junto con el índice de estabilidad de las soluciones aparecen en la tabla 1.

Órbitas alrededor de los puntos colineales				**Órbitas alrededor de los puntos isósceles**			
$h - h_c$	x	T	κ	h	x	T	κ
0.335	1.79218281083	7.15575026737	4.2776	-1.4111	1.87854685860	19.3127912036	0.176
0.331	1.76592968828	6.97864350480	1.7248	-1.4117	1.86558684835	19.0184770839	3.329
0.325	1.73920838233	6.80622661684	2.2766	-1.4230	1.80881932997	17.9639771279	174.969
0.311	1.69584775454	6.54408046868	8.7342	-1.4410	1.76019773486	17.2065048860	306.521
0.281	1.63323125365	6.20958761222	16.0027	-1.4600	1.72095259938	16.6177984326	335.352
0.239	1.56988062670	5.93380221197	22.2754	-1.4800	1.68642219016	16.0690697749	297.563
0.149	1.46068199918	5.60960993063	35.4615	-1.5100	1.64521508749	15.2457205773	183.967
0.000	1.24370800804	5.33607314254	68.1469	-1.5400	1.62235353896	14.2847790224	76.138
				-1.5700	1.63848655345	13.0731550903	20.468
				-1.6200	1.78691029059	11.0683749386	0.813

Tabla 1.—Condiciones iniciales de algunas órbitas de familias cercanas a los puntos de equilibrio. El valor de la función energía de referencia para la familia colineal es $h_c = -1.550740055311294$.

El esquema de cálculo original del método empleado para la prolongación de las órbitas periódicas emplea el método de series recurrentes de potencias para la integración numérica de las ecuaciones de movimiento y las variaciones asociadas. Esta forma de integración numérica da, en general, buenos resultados en cuanto a precisión, estabilidad y velocidad de cálculo pero requiere hallar los coeficientes hasta el grado requerido de las series solución de las ecuaciones de movimiento y las variaciones asociadas. Esta tarea puede resultar sumamente costosa en muchos problemas reales. Como alternativa, hemos empleado un método Runge-Kutta continuo para la integración de las ecuaciones del movimiento. De este modo podemos evaluar ahora la matriz jacobiana del sistema variacional en la solución obtenida numéricamente y emplear otro método Runge-Kutta distinto para integrar las ecuaciones variacionales asociadas. Esta forma de proceder tiene como inconvenientes el costo computacional y la pérdida de precisión por emplear un valor aproximado de la matriz de coeficientes.

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Sobre el problema de Gyldén-Meshcherskii

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Abstract

In this paper we study the Gylden-Meshcherskii problem when the mass depends both on the time and the distance between two bodies.

We have fixed our attention on cases in which a certain transformation of the position vector and time converts the problem into another with constant mass and equations of motion arising from integrable potentials.

Tradicionalmente se denomina problema de Gylden-Meshcherskii al problema de dos cuerpos con variación isotrópica de masa. Es decir, aquél cuyas ecuaciones del movimiento son:

$$\ddot{\vec{r}} = -\frac{\mu(t)}{r^3}\vec{r} \tag{1}$$

En sintonía con este problema matemático, surgió en 1924 (Jeans 1924, Eddington 1924) la llamada ley de Eddington-Jeans:

$$\dot{m} = -\alpha m^n \tag{2}$$

que regulaba la pérdida de masa de las estrellas por radiación. Aquí α y n son dos números reales positivos, el primero próximo a cero y el segundo comprendido entre 0,4 y 4,4.

Las famosas soluciones exactas de Meshcherskii del problema (1) se corresponden con la ley (2) para $n = 2, 3$

Por otra parte, las estadísticas elaboradas en base a los elementos orbitales de las estrellas dobles visuales parecen indicar una tendencia de modo que a mayor período corresponde por término medio mayor excentricidad, e.

J. Dommanget (Dommanget 1963, 1964, 1981, 1982 y 1997) advirtió que quizá ello estuviera en relación con la pérdida de masa estelar, en el sentido de que al ir disminuyendo ésta, la excentricidad de la órbita fuese aumentando, ya que al mismo tiempo se produce un aumento del semieje y por ende del período orbital.

E.L.Martin(Martin 1934) y L.Chiara(Chiara 1957) demostraron que si la ley de pérdida de masa depende también de la distancia entre las dos estrellas, entonces la excentricidad crece

secularmente, mientras que la no dependencia de la distancia implica un comportamiento meramente periódico de dicho elemento.

Aplicando la ley de Martin:

$$\dot{m} = -\frac{\alpha m^n}{r^2} \tag{3}$$

(Docobo et al 1998, Docobo&Prieto 1998) se comprueba a su vez que el aumento de e es tanto mayor cuanto lo sea su valor inicial. Por ejemplo, con la misma pérdida de masa y en el mismo intervalo de tiempo, una excentricidad de 0,430 pasa a 0,678 en tanto que otra de 0,051 sólo aumenta a 0,052.

Recientemente han aparecido varios artículos (Moffar 1998, Walder 1998 y Katsova 1998) en los que, al menos en binarias cerradas, queda patente la relación entre la posición de las estrellas en su órbita y diversas alteraciones del viento estelar emitido por las mismas.

En fin, aunque se trate de un problema diferente, en el sistema solar tenemos en las órbitas cometarias un claro ejemplo de variación de masa en función de la distancia al Sol.

La presente comunicación va en la línea de estudiar el problema (1) pero considerando una ley de variación de masa más general: $m = m(t, r)$, haciendo especial hincapié en aquellos casos que dan lugar a soluciones exactas del problema.

En lo que sigue tomaremos $G = 1$ y por tanto $\mu = G(m_1 + m_2) = Gm = m$.

Recordemos que considerando un cambio de coordenadas y tiempo dado por:

$$\vec{R} = \frac{\vec{r}}{1+\alpha t}, \quad \tau = \frac{t}{1+\alpha t}, \quad \alpha \in R^+, \tag{4}$$

las ecuaciones (1) se transforman en

$$\vec{R}'' = -\mu(t)(1+\alpha t)\frac{\vec{R}}{R^3} \tag{5}$$

siendo $\vec{R}''$ la derivada segunda con respecto a τ

Meshcherskii (Meshcherskii 1893) tomó $\mu(t) = \frac{1}{1+\alpha t}$, con lo que (5) representa un movimiento kepleriano, de modo que deshaciendo la transformación (4), integró (1). Tal fue la primera solución exacta del problema de Gylden-Meshcherskii.

Es evidente que si elegimos funciones $\mu = \mu(t, R)$ tales que el tiempo sólo aparezca en el término $1 + \alpha t$ en el denominador, obtendremos casos integrables, pues (5) es entonces de la forma:

$$\vec{R}'' = -f(R)\frac{\vec{R}}{R^3} \tag{6}$$

dando lugar a un problema con un solo grado de libertad.

En particular vamos a fijarnos en tres casos: el primero, que podríamos llamar solución trivial, cuando $f(R) = R^3$, que conduce al problema del oscilador armónico en el plano $\vec{R}(X, Y)$, y otros dos que denominaremos la solución de Mestschersky perturbada, eligiendo, por ejemplo, $f(R) = 1 + \epsilon + \frac{\epsilon}{R}$ y $f(R) = 1 + \epsilon + \frac{\epsilon}{R^2}$, siendo ϵ un parámetro positivo adimensional.

CASO 1: $\mu(t,r) = (r/r_0)^3/(1+\alpha t)^4$

De acuerdo con la transformación (4), este caso se puede escribir como

$$\mu(t,R) = \frac{(R/R_0)^3}{1+\alpha t} = \frac{R^3}{1+\alpha t} \tag{7}$$

donde $R_0 = R(\tau = 0) = 1$

La trayectoria en el plano $\vec{R}$ es en general una elipse (que en casos particulares puede degenerar en una recta), lo que da lugar a una espiral en el plano $\vec{r}$.

CASO 2: $\mu(t,r) = \dfrac{1+\epsilon}{1+\alpha t} - \epsilon\dfrac{r_0}{r}$

Poniendo en esta función, $r = R(1+\alpha t)$, $r_0 = R_0$, obtenemos

$$\mu(t,R) = \frac{1+\epsilon-\epsilon R_0/R}{1+\alpha t} \tag{8}$$

que sustituida en (5) nos da:

$$\vec{R}'' = -(1+\epsilon)\frac{\vec{R}}{R^3} + \epsilon\frac{\vec{R}}{R^4} \tag{9}$$

es decir, son las ecuaciones del movimiento derivadas de un potencial

$$V = -\frac{1+\epsilon}{R} + \frac{\epsilon}{R^2} \tag{10}$$

que como es bien conocido, da lugar en el plano $\vec{R}$ a una elipse con movimiento de precesión de velocidad angular proporcional a ϵ.

En el plano $\vec{r}$ tendremos, por tanto, una espiral que también precesiona (ver figura 1).

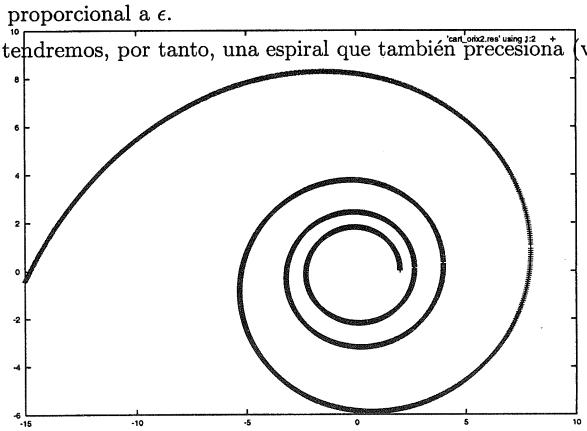

Figura 1.—Caso2. Movimiento para $\alpha = 0.25$, $\epsilon = 10^{-2}$

CASO 3: $\mu(t,r) = \dfrac{1+\epsilon}{1+\alpha t} - \epsilon(\frac{r_0}{r})^2(1+\alpha t)$

Esta función puede escribirse también en términos de R como

$$\mu(t,R) = \frac{1+\epsilon-\epsilon(\frac{R_0}{R})^2}{1+\alpha t} = \frac{1+\epsilon-\frac{\epsilon}{R^2}}{1+\alpha t} \tag{11}$$

y con esta ley de variación de masa, las ecuaciones (5) se transforman en:

$$\vec{R}'' = -(1+\epsilon)\frac{\vec{R}}{R^3} + \epsilon\frac{\vec{R}}{R^5} \tag{12}$$

por tanto, el potencial en este caso será:

$$V = -\frac{1+\epsilon}{R} + \frac{\epsilon}{R^3} \tag{13}$$

es decir, similar al que explica el avance relativista del perihelio en las órbitas planetarias y que por tanto conduce en el plano $\vec{r}$ a otra espiral con precesión (ver figura 2).

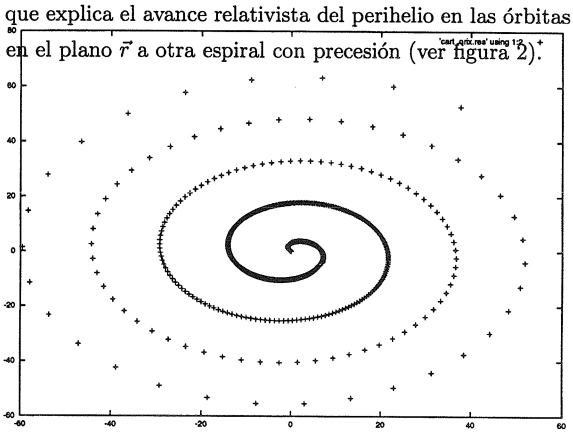

Figura 2.—Caso3. Movimiento para $\alpha = 0.25$, $\epsilon = 10^{-2}$

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Morales and Ramis non-integrability theory applied to some Keplerian Hamiltonian systems

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1. Introduction

Let us consider the family of Hamiltonian vector fields given by the meromorphic Hamiltonian function on $\mathbb{C}^{6+m}$

$$\mathcal{H} = \frac{1}{2}(X^2 + Y^2 + Z^2) - \frac{1}{r} + V(x, y, z, \beta), \tag{1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, V is a meromorphic function on $\mathbb{C}^{6+m}$ and $\beta \in \mathbb{C}^m$ is a parameter. We call those as Keplerian Hamiltonian systems. We adopt this name because when the components of β are small the systems defined by (1) may be treated as perturbed Keplerian systems, although we will not make use of perturbative techniques.

The question that we face is if the 3-DOF system defined by (1) is integrable or not. The issue of integrability has a long history (see [20],[32],[5] and references therein). As we know, two approaches may be taken in the studies of Hamiltonian systems: a) the search for integrals (separability, Darboux-Whittaker program, etc.) confirming the hints gathered from numerical experiments, and b) to prove that there are no integrals in some space of functions.

Before detailing our analysis, a few remarks about non-integrability results are in order. We say that a n-DOF Hamiltonian system is completely integrable in the extended Liouville-Arnold sense if there exist n meromorphic first integrals independent and in involution in a open dense subset of the complexified phase space. The criterion we use to study the integrability of our problem relies in the behaviour of the solutions in the complex domain. This kind of procedures began in the last century when in 1888 S. Kowalevski obtained a new integrable case in the rigid body problem with a fixed point, the Kowalevski case, showing then that the cases for which there exists a meromorphic solution were the Euler, Lagrange and Kowalevski cases. In 1894 Liapunov generalized the work of Kowalevski obtaining that the mentioned cases are the only

ones that have an univalued solution. Using Liapunov's idea of studying the variational equation associated to a Hamiltonian system, three years later Poincaré proved that if a Hamiltonian vector field $X_{\mathcal{H}}$ has k first integrals independent over a neighbourhood of a real periodic integral curve of $X_{\mathcal{H}}$, then k characteristic exponents of the monodromy matrix associated to the integral curve must be equal to 1. Later, at the beginning of this century, Painlevé (see [2] and references therein) settled the basic lines of the so called Painlevé analysis. This method has successfully been used in the search for integrable cases [1]. However, a Hamiltonian system may not satisfy the Painlevé property and, at the same time, it may be completely integrable [2].

An important progress was made by Ziglin in 1982 [35]. Ziglin proved a non-integrability result for complex analytic Hamiltonian systems based on some properties of the monodromy group of a normal variational equation along a complex integral curve.

Let M be a complex analytic manifold $2n$-dimensional. Let $\mathcal{H}$ be a holomorphic Hamiltonian function on M, and let us denote by $X_{\mathcal{H}}$ the associated Hamiltonian vector field. We take Γ the Riemann surface corresponding to an integral curve of $X_{\mathcal{H}}$: $z = z(t)$, and we compute the normal variational equation (NVE) along Γ

$$\dot{\xi} = A(t)\xi.$$

Then, the Ziglin's Theorem reads:

Theorem[Ziglin [35]] *Suppose that the Hamiltonian vector field $X_{\mathcal{H}}$ admits $n-k$ additional analytical first integrals, independent over a neighbourhood of Γ, but not necessarily on Γ itself. Furthermore, we assume that the monodromy group of the NVE contain a non-resonant element g. Then, any other element of the monodromy group of the NVE send eigendirections of g into eigendirections of g.*

An element g of the symplectic group over $\mathbb{C}$ ($Sp(m,\mathbb{C})$) is resonant if there exist $r_1,\ldots,r_n$ integer such that $\lambda_1^{r_1}\cdot\lambda_n^{r_n}=1$, being λ_i the eigenvalues of g. Ziglin himself, in [35], applied his result to the rigid body problem, to the Hénon-Heiles system and to a particular Yang-Mills field.

Apart from the proper Zigling applications of his theorem, the first satisfactory use of Ziglin's theorem is due to Ito in 1985 [15] who applied it to a generalization of the Hénon-Heiles system. From this work to the more recent works of Sansaturio *et al.* (see [30] and references therein), many papers have been published about this subject.

About ten years after the publication of Ziglin's Theorem, in separate researches carried out by Morales and Simó in [24] and by Churchill and Rod [4], they showed in the context of the differential Galois theory sufficient conditions in order to verify the hypothesis of Ziglin's theorem. Recently, Morales and Ramis have reached the core of the problem still within differential Galois theory [4, 5]. Under the hypothesis of complete integrability they have obtained that the identity component of the differential Galois group of the normal variational equation along a complex integral curve must be abelian. In particular, for 2-DOF Ziglin's theory appears as a corollary of Morales and Ramis theory. In the case of 2-DOF, other advantage to use this theory lies in the fact that we only need to check (e.g. by Kovacic's algorithm [19]) that the identity

component of the differential Galois group is not abelian, in contrast with further computations needed in the differential Galois approach of Ziglin's theory [4].

Therefore, in this paper we use Morales and Ramis theory to study the integrability problem of Keplerian Hamiltonian systems. We consider the non-integrability problem of two well known physical Hamiltonian systems that can be considered as Keplerian systems. Concretely, we consider the Zeeman-Stark Hamiltonian and the Generalized van der Waals Hamiltonian. Although they are similar, the reasoning for concluding is rather different for each of them. Thus, we illustrate different aspects of the application of the Non-Integrability theory of Morales and Ramis. For more details of the physics defining these models see [8, 9, 10, 23].

2. Non-integrability theory of Morales and Ramis

First, we begin by setting up the heuristic frame of the Morales-Ramis' theory. Let us consider a holomorphic vector field $X(z)$ $z \in \mathbb{C}^{2n}$, and let $\dot{z} = X(z)$ the associated nonlinear differential system. We suppose that in some adequate sense the vector field X is integrable. For instance, it possesses enough number of first integrals for the differential system to be integrated by quadratures. Then, the main idea of Morales-Ramis theory that comes from Liapunov and Poincaré is that the variational equation (VE) along an integral curve of X will be also integrable in the same sense. Concretely, if $z = z(t)$ is an integral curve of X, the VE

$$\dot{\xi} = X'(z(t)) \cdot \xi$$

will be integrable, where $X'(z(t))$ denotes de Jacobian matrix of the vector field X evaluated along the integral curve $z = z(t)$.

Therefore, the integrability problem of nonlinear differential systems reduces to the integrability problem of linear differential systems. This last problem has been thoroughly studied along this century by Kolchin, that continued the theory initiated by Picard and Vessiot at the end of the last century. We refer to the Differential Galois theory that Morales-Ramis' theory involves as an important part. We devote the next section to set up the main concepts and theorem of this theory.

An important remark about the heuristic frame that we have indicated is as follows. The planning made is based on the reduction of the integrability problem for nonlinear differential systems to linear differential systems. However, for general holomorphic differential systems it is not known a good definition of integrability. Thus, we find an obstacle to develop the theory. Then, Morales and Ramis restrict to Hamiltonian systems where there is a good definition of integrability.

2.1 Differential Galois Theory

As A. F. Magid points out in the preface of [22] "Differential Galois theory is the theory of solutions of differential equations over a differential base field, or rather, the nature of the differential field extension generated by the solutions, in much the same way that ordinary

Galois theory is the theory of field extensions generated by solutions of (one variable) polynomial equations, with the additional feature that the corresponding differential Galois groups."

For linear differential equations the Differential Galois theory is the so called Picard-Vessiot Theory. This theory provides a nice interpretation of the integrability of the linear differential equations: an equation is solvable if the solutions can be obtained by algebraic functions, quadratures and exponentiation of quadratures. As in Classical Galois Theory, Differential Galois Theory interprets the solvability of the linear differential equations as the solvability of the associated Differential Galois group.

There are three possible approaches to the Differential Galois Theory (see [5] and references therein). We will introduce the classical approach following the introductory lines given by Morales in [5]. However, a good introductory book to Differential algebra, that we have followed is [16]: "I have written this little book to make the subject more easily accessible to the mathematical community" (in the preface of [16]). For more advanced lectures see [18] and keep present the words of Kaplanski: "Differential algebra is easily described: it is (99 per cent or more) the work of Ritt and Kolchin."

2.1.1 Algebraic Groups

In this section we recover the basic concepts and properties on algebraic groups that we use along this paper. Two good references about algebraic groups are [14] and [31].

A linear algebraic group G over $\mathbb{C}$ is a subgroup of the Linear Group $GL(n, \mathbb{C})$ whose matrix coefficients satisfy algebraic equations over $\mathbb{C}$. We note that a linear algebraic group has compatible structures of group and of non singular variety. Moreover, in a linear algebraic group we have two topologies, the Zariski topology where the closed sets are the algebraic sets, and the usual topology inherited from $\mathbb{C}$.

Given a linear algebraic group G, the identity component of G, denoted by G° is the unique irreducible component that contains the identity element. We have the following proposition about the identity component.

Proposition 1 (Page 53 [14]) *Let G be a linear algebraic group.*

a) *G° is a normal subgroup of finite index in G, whose cosets are the connected as well as irreducible components of G.*

b) *Each closed subgroup of finite index in G contains G°.*

We recall that a Lie algebra over $\mathbb{C}$ is a subspace of an associative $\mathbb{C}$-algebra which is closed under the bracket operation $[x, y] = xy - yx$. Let G be a linear algebraic group, the space $\mathcal{L}(G)$ of the left invariant derivations of $\mathbb{C}[G]$ for a Lie algebra that we call the Lie algebra of G (see Chapter III of [14]). On the other hand, using the algebraic variety structure of G we can consider the tangent space TG, identified as the tangent space to G°, is a vector space over $\mathbb{C}$ of dimension equal to the dimension of G. We have that the spaces $\mathcal{L}(G)$ and TG are isomorphic as $\mathbb{C}$-vector spaces. Thus, we identify the Lie algebra $\mathcal{L}(G)$ with TG using the bracket operation

in TG inherited via the bracket in $\mathcal{L}(G)$ by the isomorphism. In the following we will denote TG as $\mathbf{g}$.

The characterization of the connected *solvable* linear algebraic groups is given by the Lie-Kolchin theorem.

Theorem 1 (Lie and Kolchin Theorem, [16] p. 30) *A connected linear algebraic group is solvable if and only if it is conjugated to a* triangular group.

The linear differential equations that arise from the examples presented in this memoir will be symplectic differential equations of second order. Therefore, we finish this section with the classification of the algebraic subgroups of $SL(2,\mathbb{C})$.

Proposition 2 ([16] p. 31) *Let V be an algebraic subgroup of $SL(2,\mathbb{C})$. Then, one of the following cases holds*

1. *V is triangulisable.*
2. *V is conjugate to a subgroup of*

$$U = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} : c \in \mathbb{C}\, c \neq 0 \right\} \bigcup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} : c \in \mathbb{C},\ c \neq 0 \right\}$$

   and case (1) does not hold.

3. *V is finite and cases (1) and (2) do not hold.*
4. *$V = SL(2,\mathbb{C})$.*

In the last case the identity component V° of V coincides with V.

2.1.2 Classical Approach

Let K be a differential field, i.e. a field endowed with a derivation $\delta = '$, and let C be the field of constants of K. Let us consider a linear homogeneous differential equation

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_0 y = 0,$$

with coefficients in K, or equivalently, a linear differential system of equation

$$\xi' = A\xi,$$

where A is $m \times m$ matrix with coefficients in K.

Let $u_1, \ldots, u_n$ be n solutions of $L(y) = 0$. We say that they are linearly independents over the C if the its wronskian not vanish.

Definition 1 (Picard-Vessiot Extensions,[16] p. 21) *Let $L(y) = 0$ be a linear homogeneous differential equation with coefficients in K. We say that a differential field M containing K is a Picard-Vessiot extension of K for $L(y) = 0$ if,*

1. $M = K < u_1, \ldots, u_n >$ *(quotients of differential polynomials in* $u_1, \ldots, u_n$ *and its derivatives with coefficients in* K*) where* $u_1, \ldots, u_n$ *are* n *solutions of* $L(y) = 0$ *linearly independent over* C.

2. *The field of constant of* M *is* C.

If the characteristic of K is zero and the field C is algebraically closed then for any linear homogeneous differential equation there exists an unique Picard-Vessiot extension. This result is due to Kolchin [17].

We define now a special type of extension that play an important role in the following theory.

Definition 2 (Liouvillian Extension, [16] p. 24) *An extension* $K \subset M$ *of differential field is called Liouvillian if there exists a chain of intermediate differential fields* $K = K_1 \subset K_2 \subset \ldots, \subset K_n = M$ *such that the each extension* $K_i \subset K_{i+1}$ *is given by the adjuntion of one element* a, $K_i \subset K_{i+1} = K_i < a, a', a'', \ldots >$ *such that* a *satisfies one of the following conditions:*

1. $a' \in K_i$, *i.e.* a *is a quadrature.*

2. $a' = ba$, $b \in K_i$, *i.e* a *is an exponential of a quadrature.*

3. a *is algebraic over* K_i.

We will say that the linear homogeneous differential equation (or the linear differential system of equations) $L(y) = 0$ is integrable if there exists a Picard-Vessiot extension of $L(y) = 0$ that is Liouvillian.

Definition 3 (The Galois Group, [16] p. 18) *Let* M *be a differential field and* K *a differential subfield of* M. *We define the Differential Galois group of* M/K, $Gal_K(M)$, *to be the group of all differential automorphisms of* M *living* K *elementwise fixed.*

When the extension M/K is a Picard-Vessiot extension the Differential Galois group $Gal_K(M)$ is an algebraic matrix group over the field of constants of K (Theorem 5.5 [16] p. 36). We say that an extension M/K is normal if any element in M invariant by $Gal_K(M)$ belongs to K. If M/K is a Picard-Vessiot extension, being K of characteristic zero with a field of constants algebraically closed, then the extension M/K is normal (Theorem 5.7 [16] p. 36). It is by this property why the correspondence between subgroups and subfield works well in the Differential Galois Theory. More precisely, we have the following theorem.

Theorem 2 (Kolchin. Theorem 5.9, [16] p. 38) *Let* K *be a differential field of characteristic zero with a field of constants* C *algebraically closed. Let* M/K *be a Picard-Vessiot extension associated to a linear homogeneous differential equation. Then, there is a one-to-one correspondence between the intermediary differential fields* $K \subset L \subset M$ *and the algebraic subgroups* $H \subset Gal_K(M)$, *such that* $H = Gal_L(M)$ *being the extension* M/L *a Picard-Vessiot extension. Furthermore, we have*

1. *The normal extensions* L/K *corresponds to normal subgroups* $H \subset Gal_K(M)$ *and* $Gal_K(M)/H = Gal_K(L)$.

2. *Let F be a subgroup of $Gal_K(M)$ and K_F the subfield of M given by the elements of M fixed by F. Then $H := Gal_{K_F}(M)$ is the Zariski closure (over the field of constants C) of F.*

In order to finish this section we set up the relation between the Differential Galois Theory and the integrability of the linear homogeneous differential equation.

Theorem 3 (Kolchin. Theorems 5.11 and 5.12 [16] p. 39) *Let K be a differential field of characteristic zero with a field of constants C algebraically closed. Let $L(y) = 0$ be a linear homogeneous differential equation over K. Then, $L(y) = 0$ is integrable, i.e. its unique Picard-Vessiot extension M/K is Liouvillian, if and only if the identity component $Gal_K(M)^\circ$ of the Differential Galois group $Gal_K(M)$ is solvable.*

2.1.3 Kovacic's Algorithm

The Kovacic's algorithm proportionate us a procedure to compute the Picard-Vessiot extension of a second order differential equation in $\mathbb{C}(x)$ provided the differential equation is solvable. Then, if the algorithm does not work the differential equation is non-integrable. We have used along this paper the original Kovacic's algorithm (see [19]). There are another more recent presentation of this algorithm (see [5] and [6]). We do not plan to include here the Kovacic's algorithm due to its extension. We limit ourselves to present two important Propositions that we have used in the applications.

Let us consider a second order differential equation in the invariant normal form

$$y'' = ry, \quad r \in \mathbb{C}(x), r \notin \mathbb{C}.$$

We will refer to this differential equation as "the DE".

It may happen four mutually exclusive cases for the solutions of the DE such that each case corresponds to the respective case of the Proposition 2:

Proposition 3 (Section 1.2, [19]) *There are precisely four cases that can occur.*

Case 1. *The DE has a solution of the form $e^{\int \omega}$ where $\omega \in \mathbb{C}(x)$.*

Case 2. *The DE has a solution of the form $e^{\int \omega}$ where ω is algebraic over $\mathbb{C}(x)$ of degree 2, and case 1 does not hold.*

Case 3. *All solutions of the DE are algebraic over $\mathbb{C}(x)$ and cases 1 and 2 do not hold.*

Case 4. *The DE ha no Liouvillian extension, i.e. the DE is non-integrable.*

If $r = s/t$ with $s, t \in \mathbb{C}(x)$ relatively prime, then the poles of r are the zeros of t and the order of the pole is the multiplicity of the zero of t. By the order of r at ∞ we shall mean the order of ∞ as a zero of r, thus the order of r at ∞ is $\deg(t) - \deg(s)$, where deg means the degree function. We give now the necessary conditions about the poles of r in order to verify the cases of Proposition 2.

Proposition 4 (Section 2.1 [19]) *Necessary conditions for the cases of Proposition 2 to hold are*

1. *Every pole of r must have even order or else have order 1, and the order of r at ∞ must be even or else be greater than 2 in order to the case (1) holds*

2. *r must have at least one pole that either has odd order greater than 2 or else has order 2 in order to the case (2) holds*

3. *The order of a pole of r can not exceed 2 and the order of r at ∞ must be at least 2. If the partial fraction expansion of r is*

$$r = \sum_i \frac{\alpha_i}{(x-c_i)^2} + \sum_j \frac{\beta_j}{x-d_j}$$

then $\sqrt{1+4\alpha_i} \in \mathbb{Q}$, for each i, $\sum_j \beta_j = 0$, and if $\gamma = \sum_i \alpha_i + \sum_j \beta_j$, then $\sqrt{1+4\gamma} \in \mathbb{Q}$. This condition is necessary for case (3) to holds.

2.2 Morales-Ramis' theorems

In this Section we present a short description of the main theorems of Morales-Ramis that we will use to prove the non-integrability results of the applications considered in the next Section.

Let us consider a 2n-dimensional complex analytic manifold M with a symplectic two form Ω and a holomorphic Hamiltonian vector field on M, X_H. Let $x = \phi(t)$ be a germ of a regular curve in M that is not an equilibrium point. We take $i(\Gamma)$ the maximal connected component analytically continued of the germ $x = \phi(t)$, and we consider Γ the abstract Riemann surface defined by $i(\Gamma)$. The inclusion $i : \Gamma \to i(\Gamma) \subset M$ is an immersion. Using the immersion i we define the fiber bundle $\pi : \Gamma \to \mathbf{T}\Gamma$ as the pull back of the fiber bundle $\mathbf{T}M$ restrict to $i(\Gamma)$. In the same way, we define a Hamiltonian vector field, denoted by X, on Γ by pull back of the X_H

Let us consider the holomorphic symplectic connection ∇ defined by pull back from the restriction to $i(\Gamma)$ of the Lie derivative with respect to the Hamiltonian vector field X_H

$$\nabla v = L_{X_H} Y|_\Gamma$$

where v is a section of the bundle $\mathbf{T}\Gamma$, Y is holomorphic vector field extension of the section v of the bundle $\mathbf{T}_{i(\Gamma)}M$. If we express the connection ∇ in a local trivialization of the bundle $\mathbf{T}\Gamma$ we obtain a linear differential system which is the variational equation (VE) along the integral curve defined by de germ $x = \phi(t)$. More precisely, we consider a coordinate system $\{x_1, \ldots, x_{2n}\}$ on M and the associated frame $\{e_1, \ldots, e_{2n}\}$ given by $e_i = \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, 2n$. Then, using the complex time t as local coordinate in Γ, by the definition of ∇ as the Lie derivative with respect to the vector field X_H we obtain the system

$$\frac{d\alpha}{dt} = A(t)\alpha,$$

where

$$A(t) = J\mathrm{Hess}H(x_1(t), \ldots, x_{2n}(t)),$$

being J the canonical symplectic matrix $\begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$, where $\mathbf{1}$ is the $n \times n$ identity matrix. The elements of $\mathrm{Ker}(\nabla)$ are the known horizontal sections that we have seen that expressed in a local trivialization of $\mathbf{T}\Gamma$ are the solutions of the VE.

In this situation, the first result about non-integrability of Morales and Ramis reads:

Theorem 4 (Theorem 7 [4]) *Assume that there are n first integrals of X_H which are meromorphic, in involution and independent in a neighborhood U of the curve $i(\Gamma)$ in M. Then the identity component of the Galois group of the VE is an abelian subgroup of the symplectic group.*

In some cases, if the vector field X_H has a finite set of equilibria that belong to the closure of $i(\Gamma)$ in M, we add to Γ this finite set of equilibria. We denote this new curve by $\underline{\Gamma}$. Then, we have $i(\Gamma) \subset \underline{\Gamma} \subset M$, where $\underline{\Gamma}$ is a closed analytic curve that in general will be singular in the equilibria points. By dessingularization of the curve $\underline{\Gamma}$ we consider $\overline{\Gamma}$ its corresponding connected Riemann surface, and $\Gamma \subset \overline{\Gamma}$ will be an open set of $\overline{\Gamma}$. We consider the restriction of the tangent bundle $\mathbf{T}M$ to $\underline{\Gamma}$. By pull back of $\mathbf{T}M|_{\underline{\Gamma}}$ by means of the immersion $i : \overline{\Gamma} \to \underline{\Gamma}$ we define an holomorphic vector bundle $\mathbf{T}\overline{\Gamma}$ over $\overline{\Gamma}$. Finally, as before, we define a meromorphic connection on $\mathbf{T}\overline{\Gamma}$ by means of the Lie derivative of the vector field X_H restrict to $\underline{\Gamma}$.

Remark 1 *In the applications we will not need to know a complete set of complex chart of the desingularized Riemann surface $\overline{\Gamma}$. We will use the fact that the curve $\underline{\Gamma}$ can be desingularized, because by applying the Theorem 6, in general, we will change the Riemann surface and the connection to the Riemann Sphere.*

In other cases we add to $\overline{\Gamma}$ (or Γ) a finite set of points corresponding to points at the infinity of $\underline{\Gamma}$ that correspond to poles of the parameterization of the curve. In these cases we suppose that the manifold M is contained in a connected manifold M', such that $M_\infty = M' - M$ is an *analytic hypersurface* in M', called hypersurface at infinity, and that the holomorphic symplectic 2-form Ω over M extends to a meromorphic symplectic 2-form Ω' over M'. Moreover, we consider the extension of the vector field X_H to M' that will have poles over M_∞.

Thus, we obtain a closed analytic curve $\underline{\Gamma}'$ in M' construct by adding to $\underline{\Gamma}$ the point at infinity. Then, by the desingularization process we consider $\overline{\Gamma}'$ the connected Riemann surface obtained from $\underline{\Gamma}'$. Then, we define a meromorphic connection over $\overline{\Gamma}'$ by the Lie derivative with respect to the extended vector field X_H restricted to $\underline{\Gamma}'$.

Remark 2 *In the applications M will be $\mathbb{C}^{2n}$ and we will consider $M' = (\mathbf{P}^1)^{2n}$. Then, the hypersurface at infinity is the set*

$$M_\infty = \bigcup_{i=1}^{n} \mathbf{P}^1 \times \cdots \times \overset{\hat{i}}{\{\infty\}} \times \cdots \times \mathbf{P}^1.$$

Let Ω the canonical two form in $\mathbb{C}^{2n}$. We extend the two form Ω to $\mathbf{P}^1$ in the following way. Let $x_1, \ldots, x_n$ coordinates at ∞ over n copies of $\mathbf{P}^1$ and $y_1, \ldots, y_n$ the same for the rest copies.

Then, at the point of M_∞, Ω extends as

$$\Omega = \sum_{i=1}^{n} \frac{dx_i \wedge dy_i}{x_i^2 y_i^2}.$$

We observe that the extended two form Ω is degenerated at the point of M_∞.

In this situation Morales and Ramis proved the following non-integrability result.

Theorem 5 (Theorem 9 [4]) *Assume that there is a finite set of equilibrium points and points at infinity. Assume that there are n first integrals of X_H which are meromorphic, in involution and independent in a neighborhood U of the curve Γ' in M'. Then the identity component of the Galois group $\overline{G}'$ of the VE over the differential field of the meromorphic functions on $\overline{\Gamma}'$ is an abelian subgroup of the symplectic group.*

In general $G \subset \overline{G} \subset \overline{G}'$ with strict inclusion. However, when the extended connection over of the variational equation over $\overline{\Gamma}$ (resp. $\overline{\Gamma}'$) is Fuchsian (i.e. the singular points are regular singular points) we have $G = \overline{G}$ (resp. $G = \overline{G}'$).

In the applications is useful to be able to change the Riemann surface and the connection that we are dealing with in order to simplify the procedure of computing the Galois group. However, not all changes are allowed. Morales and Ramis proved a result involving a vast class of changes that makes invariant the component of the identity of the Galois group. This theorem will be applied some times along the next Section. The Theorem reads.

Theorem 6 (Theorem 2.5 [5], [4]) *Let X be a connected Riemann surface. Let (X', f, X) be a finite ramified covering of X by a connected Riemann surface X'. Let ∇ be a meromorphic connection over X. We set $\nabla' = f^*\nabla$. Then, we have a natural injective homomorphism*

$$Gal(\nabla') \longrightarrow Gal(\nabla)$$

of differential Galois groups which induces an isomorphism between their Lie algebras.

In terms of differential Galois groups this theorem means that the identity component of the differential Galois group is invariant by the covering.

In the applications we will not compute the differential Galois group of the VE. We will obtain the differential Galois group of a system obtained from the VE by a reduction process. This system is the so called normal variational equation NVE that come from Ziglin's papers. Below, we give a short description of the reduction process generalized by Morales and Ramis in the context of meromorphic bundles.

Let V be a symplectic (meromorphic) vector bundle of rank $2n$ over $\overline{\Gamma}$ ($\overline{\Gamma}'$) with a symplectic connection ∇, and Ω the two form that defines the symplectic structure. Let $v_1, \ldots, v_k$ be k global horizontal meromorphic sections of V (holomorphic in Γ) linearly independent over Γ and in involution, i.e. $\Omega(v_i, v_j) = 0$, $i, j = 1, \ldots, k$. Let F be the subbundle of V generated by $v_1, \ldots, v_k$ (here the section are identified with their images), and define $F^\perp$ the subbundle orthogonal to F with respect to Ω, i.e. $w \in F^\perp$ if and only if $\Omega(w, v) = 0$ for all $v \in F$. Clearly

$F \subset F^{\perp}$ by the involutivity of the sections. Then, we can define $N = F^{\perp}/F$ that is a symplectic subbundle of V of rank $2(n-k)$. Moreover, we take the connections ∇_F and $\nabla_{F^{\perp}}$ by restriction of ∇ to F and $F^{\perp}$ respectively. Thus, we can define the normal connection ∇_N on N given by the action of ∇ over the representatives of the classes of N. Furthermore, Morales and Ramis show in Proposition 4.1 of [5] that the connection ∇_N is symplectic. The local differential system defined by ∇_N is called the normal system. When, the connection is defined from a Hamiltonian vector field we will call to this system the normal variational equation (NVE).

Morales and Ramis supplied a general method to obtained the normal variational equation (see p. 79 [5]). We not plan recover this method of reduction because we have not needed to use this method in our applications. The common situation in our examples is the existence of an invariant symplectic hyperplane, and in that special case the normal variational equation is found without difficulty as we will show in the next Section.

The relation between the Galois groups of the above connections is given in the following proposition.

Proposition 5 (Proposition 4.2 [5]) *Let $\alpha_1, \ldots, \alpha_k$ be an involutive linearly independent set of global horizontal meromorphic section of (∇, V, Ω). Let ∇_N be the normal connection defined by the above set. Then, we have*

i) *The linear differential equation corresponding to the connection ∇ is integrable if and only if the normal equation corresponding to ∇_N is integrable.*

ii) *If the identity component of the differential Galois group of ∇ is abelian then the identity component of the differential Galois group of ∇_N is also abelian.*

We observe that from this proposition, by the Theorems 4 and 5, we only need to study the identity component of the differential Galois group of the normal variational equation when we are investigating the integrability of a problem.

Finally, after the theoretical frame introduced along this Section, we can formulate the algorithm's Morales-Ramis in order to study the non-integrability of a Hamiltonian vector field X_H:

1. Select a particular integral curve of X_H. From the experience of the author, this integral curve ought to be some kind of special curves as homoclinic or heteroclinic curves, or rectilinear solutions containing in its closure singular points of X_H.

2. Compute the VE and the NVE.

3. Compute the differential Galois group of the NVE. This part of the algorithm may be a cumbersome task even in the case of applying known algorithms as Kovacic's algorithm.

4. If the identity of the differential Galois group of the NVE is not abelian then X_H is not integrable by meromorphic functions, else we can not conclude either integrability or non-integrability.

2.2.1 Application to Homogeneous Potentials

In a third paper (see [26]), Morales and Ramis have proved an extension of a known criterion of non-integrability of Yoshida [34] over Hamiltonian systems with homogeneous potential. We recover this result here because we will use it to prove the non-integrability of the Generalized van der Waals Hamiltonian system.

We sketch below the criterion contained in [26] (see also Section 5.1 of [5]).

We consider a natural Hamiltonian function with homogeneous potential, i.e. of the type

$$H(x,y) = \frac{1}{2}(y_1^2 + \ldots + y_n^2) + W(x_1, \ldots, x_n), \tag{2}$$

where W is a homogeneous function of integer degree k. Let us suppose $n \geq 2$, and $k \neq 0$. First, we select a solution $c = (c_1, \ldots, c_n)$ of the equation

$$\frac{\partial}{\partial(x,y)} W(c) = c, \tag{3}$$

where $\frac{\partial}{\partial(x,y)}$ denote the gradient operator. Then, we compute the eigenvalues of the Hessian matrix of W at c. We denote them by λ_i for $i = 1, \ldots, n$. Then, we have the following theorem.

Theorem 7 (Theorem 3 in [26]) *If the Hamiltonian system with Hamiltonian (2) is completely integrable with holomorphic (or meromorphic) first integrals, then each pair (k, λ_i) belongs to one of the following lists*

1. $(-2, \alpha)$, $\alpha \in \mathbb{C}$
2. $(2, \alpha)$, $\alpha \in \mathbb{C}$,
3. $\left(-3, \frac{25}{24} - \frac{1}{6}(1+3p)^2\right)$,
4. $\left(3, -\frac{1}{24} + \frac{1}{6}(1+3p)^2\right)$,
5. $\left(-3, \frac{25}{24} - \frac{3}{8}(\frac{1}{2}+2p)^2\right)$,
6. $\left(3, -\frac{1}{24} + \frac{3}{8}(\frac{1}{2}+2p)^2\right)$,
7. $\left(-3, \frac{25}{24} - \frac{3}{2}(\frac{2}{5}+p)^2\right)$,
8. $\left(3, -\frac{1}{24} + \frac{3}{2}(\frac{2}{5}+p)^2\right)$,
9. $\left(-3, \frac{25}{24} - \frac{3}{2}(\frac{1}{5}+p)^2\right)$,
10. $\left(3, -\frac{1}{24} + \frac{3}{2}(\frac{2}{5}+p)^2\right)$,
11. $\left(-4, \frac{9}{8} - 2(\frac{1}{3}+p)^2\right)$,
12. $\left(4, -\frac{1}{8} + 2(\frac{1}{3}+p)^2\right)$,
13. $\left(-5, \frac{49}{40} - \frac{5}{2}(\frac{1}{3}+p)^2\right)$,
14. $\left(5, -\frac{9}{40} + \frac{5}{2}(\frac{1}{3}+p)^2\right)$,
15. $\left(-5, \frac{49}{40} - \frac{1}{10}(2+5p)^2\right)$,
16. $\left(5, -\frac{9}{40} + \frac{1}{10}(2+5p)^2\right)$,
17. $(k, p + p(p-1)k/2)$,
18. $\left(k, \frac{1}{2}(\frac{k-1}{k} + p(p+1)k)\right)$,

where p is an arbitrary integer.

3. Non-integrability of some Keplerian systems

The present Section is dedicated to illustrate the Morales and Ramis' Theory by the study of the integrability of some known Keplerian systems.

First we abord the non-integrability in the Liouvillian sense of the Stark-Zeeman Hamiltonian. In particular, we generalize the result of Kummer and Saenz about the non-integrability of the pure Zeeman Hamiltonian.

The second example is dévoted to prove that, except for the three known cases, the uniparametric family of Hamiltonian systems defined by the generalized van der Waals potential is non integrable in the Liouville-Arnold sense.

3.1 *On the Non-integrability of the Stark-Zeeman Hamiltonian System*

In the class of perturbed Coulomb systems, the one most thoroughly investigated is the quadratic Zeeman effect (see [11] and references therein). One of the variants of this problem is obtained by introducing an electric field parallel to the magnetic field: the Stark-Zeeman effect [3].

In the case of the quadratic Zeeman effect there is a numerical evidence for the occurrence of chaos. This was taken as a first hint of non-integrability; the same happens in the Stark-Zeeman effect [29]. A rigorous mathematical study of the non-integrability in the system defined by the Zeeman effect is due to Kummer and Saenz [21] using Ziglin's theorem. A common limitation for applying Ziglin's theorem to prove non-integrability is the restriction to fuchsian variational equations (their singularities must be regular singular). This is the case in the known particular solutions of the Stark-Zeeman effect: we cannot apply Ziglin's theorem. Nevertheless, using the recent theorems of Morales and Ramis we prove the non-integrability of the Zeeman-Stark effect by meromorphic functions in a sense to be specified later.

We consider the problem of the dynamics of an electron of reduced mass μ in an atom of infinite massive nucleus under the effect of a magnetic and electric parallel fields. Choosing the z axis as the direction of the fields, the Hamiltonian function takes the form

$$\mathcal{H} = \frac{1}{2}(X^2+Y^2+Z^2) - \frac{1}{\sqrt{x^2+y^2+z^2}} + Fz + \frac{1}{8}(x^2+y^2), \tag{4}$$

where F is a non-negative adimensional parameter. The phase space is the six-dimensional real manifold

$$M = \{(U,V) \in \mathbb{R}^6 : \ U=(x,y,z),\ V=(X,Y,Z),\ x^2+y^2+z^2>0\}.$$

In order to apply Morales and Ramis (MR) theory we consider the Hamiltonian (4) as a holomorphic function on the six-dimensional complexification of the manifold M

$$\widehat{M} = \{(U,V) \in \mathbb{C}^6 : \ U=(x,y,z),\ V=(X,Y,Z),\ x^2+y^2+z^2 \neq 0\},$$

equipped with the non-degenerated two-form $d\Theta$, where Θ is the canonical one-form $\Theta = V \cdot dU$. We regard $\widehat{M}$ as an open subset of the six-dimensional complex connected manifold $\widehat{M}' = (\mathbf{P^1})^6$. The holomorphic two-form $d\Theta$ extends uniquely to a meromorphic two-form over $\widehat{M}'$ (see Section 2.2).

The Hamiltonian vector field $X_{\mathcal{H}}$ associated to $\mathcal{H}$ on $\widehat{M} \subset \widehat{M}'$ is

$$\begin{aligned} \dot{x} &= X, & \dot{X} &= -x\Big(\frac{1}{r^3}+\frac{1}{4}\Big), \\ \dot{y} &= Y, & \dot{Y} &= -y\Big(\frac{1}{r^3}+\frac{1}{4}\Big), \\ \dot{z} &= Z, & \dot{Z} &= -\frac{z}{r^3} - F, \end{aligned} \tag{5}$$

where $r = \sqrt{x^2+y^2+z^2}$. This vector field is tangent to the submanifold $x=y=X=Y=0$. We take $\widetilde{M} = 0\times 0\times \mathbb{C}\times 0\times 0\times \mathbb{C} \bigcap \widehat{M}$ and define the symplectic form by $d\Theta|_{\widetilde{M}} = dz \wedge dZ$. Then, the vector field (5) becomes the Hamiltonian vector field on $\widetilde{M}$ associated to the Hamiltonian meromorphic function

$$\widetilde{\mathcal{H}} = \frac{1}{2}Z^2 + Fz - \frac{1}{z}. \tag{6}$$

For the non-equilibrium solutions needed in MR theorems we use the curve $\varphi = \varphi(t) = (0, 0, \varphi_1(t), 0, 0, \varphi_2(t))$, where $\varphi = (\varphi_1, \varphi_2)$ is a maximally continued integral curve of (6) in the zero level energy, value that we have taken for simplicity of our computations; and we denote $i(\Gamma)$ the image of φ in $\widehat{M}'$. The projection of $i(\Gamma)$ over $\mathbb{C} \times \mathbb{C}$ is the analytic set given by (6) at the zero energy value.

The vector field associated with the Hamiltonian $\widetilde{\mathcal{H}}$ has two equilibrium points in an energy level different from zero. Then, there are not equilibrium points in the closure of $i(\Gamma)$. Thus, we take Γ the abstract Riemann surface defined by $i(\Gamma)$. Because $\varphi_1(t)$ is an elliptic function we have that Γ is a complex torus without two points (the poles of the elliptic function).

We consider now the curve $\underline{\Gamma}'$ in $\widehat{M}'$ which is the curve $i(\Gamma)$ adding two points of its closure in $\widehat{M}'$ that correspond to the poles and zeros of the parameterization of $i(\Gamma)$ by the elliptic function. This points are $(0, 0, \infty, 0, 0\infty)$ and $(0, 0, 0, 0, 0, \infty)$. We take the abstract Riemann surface $\overline{\Gamma}'$ defined by $\underline{\Gamma}'$.

The variational equation along $\overline{\Gamma}'$ is the differential system

$$\begin{aligned} \frac{d\xi}{dt} &= \widetilde{A}(t)\,\xi, \\ \widetilde{A}(t) &= J\,\mathrm{Hess}\mathcal{H}(\varphi(t)) \end{aligned}$$

where J is the standard symplectic matrix.

The normal variational equation along $\overline{\Gamma}'$ is composed of two uncoupled equations

$$\ddot{\xi}_i - \Big(\frac{1}{\varphi_1(t)^3} - \frac{1}{4}\Big)\xi_i = 0, \quad i = 1, 2. \tag{7}$$

We denote by G_i $(i = 1, 2)$ the differential Galois group of each equation of (7) over the field of meromorphic functions over $\overline{\Gamma}'$, and by G the differential Galois group of the normal variational equation (7). A representation of G is $G_1 \times G_2$. Then, the identity component of G is not abelian if the identity component of G_1 or G_2 is not abelian. Then, in what follows we will consider the normal variational equation

$$\ddot{\xi} - \Big(\frac{1}{\varphi_1(t)^3} - \frac{1}{4}\Big)\xi = 0$$

over $\overline{\Gamma}'$, and we will compute the differential Galois group of this equation over the field of meromorphic functions over $\overline{\Gamma}'$. We denote this group by G_3.

First, we carry out the change of variables $t \leftrightarrow z$, $z = \varphi_1(t)$. Then, we obtain $\overline{\Gamma}' \simeq \mathbf{P}^1$, and the algebraic expression of the normal variational equation (ANVE) on $\mathbf{P}^1$ reads

$$\ddot{\eta} - \frac{1 + Fz^2}{2z(1 - Fz^2)}\dot{\eta} - \frac{4 - z^3}{8z^2(1 - Fz^2)}\eta = 0. \tag{8}$$

We observe that the poles $z = 0$ and $z = \infty$ correspond to the two points at infinity of $\overline{\Gamma}'$, and the poles $z = \pm\frac{1}{\sqrt{F}}$ are ramification points of the finite covering $\overline{\Gamma}' \simeq \mathbf{P}^1$.

We suppose first $F \neq 0$. Then, by a second change of variables $z \leftrightarrow u$, $u = \sqrt{F}z$ on $\mathbf{P}^1$, we obtain that equation (8)

$$\ddot{\eta} - \frac{1 + u^2}{2u(1 - u^2)}\dot{\eta} - \frac{1 - \delta u^3}{2u^2(1 - u^2)}\eta = 0, \tag{9}$$

where $\delta = \frac{1}{4F^{(3/2)}}$. Let us denote by G_B the differential Galois group of the equation (9) over the differential field of meromorphic functions on $\mathbf{P}^1$. By Theorem 6 we have that the identity components of G_3 and G_B coincide. Then, we will compute G_B.

Transforming the ANVE (9) to its normal invariant form is done by means of the usual change $\chi = \exp(-\frac{1}{2}\int p)\eta$, where $p = -\frac{1+u^2}{2u(1-u^2)}$. We obtain

$$\ddot{\chi} = r\chi \tag{10}$$

where $r(z) = -q(z) + \frac{1}{4}p(z)^2 + \frac{1}{2}p'(z)$, with

$$r = \frac{\frac{13}{16}}{u^2} + \frac{\frac{-1}{16} + \frac{\delta}{4}}{u-1} + \frac{\frac{-3}{16}}{(u-1)^2} + \frac{\frac{5}{16} + \frac{\delta}{4}}{u+1} + \frac{\frac{-3}{16}}{(u+1)^2}. \tag{11}$$

We note that the singular points $u = 0, \pm 1$ are regular and $u = \infty$ is a irregular singular point.

The solvability of the equation (9) is equivalent to the solvability of the ANVE. Then, we determine the differential Galois group of the equation (10) over the field of meromorphic functions over $\mathbf{P}^1$. We denote this group by G_4 (in general the groups G_4 and G_B do not coincide).

In order to obtain the group G_4, we apply the original Kovacic's algorithm (see 2.1.3). G_4 is an algebraic subgroup of $SL(2,\mathbb{C})$. Then, applying Proposition 4 to equation (10) only cases (2) or (4) of Proposition 2 can be possible. Thus, we only need to compute the second step of the Kovacic's algorithm. Then, we obtain $G_4 = G_4^\circ = SL(2,\mathbb{C})$. As final conclusion the group G_4 has a not abelian identity component, and so, the identity component of the group G is not abelian. Proceeding analogously in the case $F = 0$ we also obtain that the identity component of the group G is not abelian.

Summarizing the results obtained for $F \geq 0$, and using Theorem 5, we trivially obtain:

Theorem 8 *Let $U \subset (\mathbf{P^1})^6$ be an arbitrary open neighborhood of $\underline{\Gamma}'$. Then the Stark-Zeeman Hamiltonian does not admit three independent meromorphic integrals in involution defined on U.*

Then, in terms of the original Hamiltonian vector field on M, we have the following result:

Theorem 9 *The Stark-Zeeman Hamiltonian does not admit three independent globally defined analytic integrals in involution which extend meromorphically to $(\mathbf{P^1})^6$.*

As a consequence of the above theorems we have the following result:

Theorem 10 *The Stark-Zeeman Hamiltonian system is not completely integrable by rational functions on M.*

Readers should note the possibility of the existence of three independent analytic first integrals in involution for the Stark-Zeeman system which can be extended meromorphically to $\widehat{M}$ but not meromorphically to $(\mathbf{P^1})^6$; this has already been noted by Morales and Ramis in [5]. Finally, we note that our work includes an alternate proof of the non-integrability of the Zeeman Hamiltonian obtained by Kummer and Saenz [21]. However, the non-integrability result obtained by

them is different from our result, because in their paper it is proved that the reduced Zeeman Hamiltonian system by the S^1 symmetry is not integrable by meromorphic functions defined in the reduced manifold.

Remark.- We have studied also the integrability of the system defined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(X^2 + Y^2 + Z^2) - \frac{1}{\sqrt{x^2 + y^2 + z^2}} + Fx + \frac{1}{8}(x^2 + y^2), \tag{12}$$

corresponding to the 3-D Hydrogen atom under motional Stark effect or circularly polarized microwave combined with magnetic fields. Restricted to the invariant manifold $z = Z = 0$, Apostolakis *et al.* have shown that the problem separates in elliptical coordinates and Raković *et al.* have completed the analysis providing the second integral (see [28]). We have proved the non-integrability of the system defined by (12) in 3 dimensions. Although rather similar to the Stark-Zeeman it presents some peculiar features in the application of Morales and Ramis theory. For details see [9].

3.2 On the Non-integrability of the Generalized van der Waals Hamiltonian

The generalized van der Waals Hamiltonian in cylindrical coordinates is

$$\mathcal{H}_\beta = \frac{1}{2}(P^2 + \frac{\Lambda^2}{\rho^2} + Z^2) - \frac{1}{r} + \frac{\gamma}{2}(\rho^2 + \beta^2 z^2), \tag{13}$$

in which $r = \sqrt{\rho^2 + z^2}$, and P, Λ and Z are the canonical momenta conjugate to the coordinates ρ, λ and z respectively, γ is square of a frequency, and the parameter β, which is dimensionless. For details see [7] and references therein.

The problem of the integrability of (13) has been considered in different ways. One such approach is the Painlevé analysis. Ganesan and Lakshmanan [12] showed that the Hamiltonian vector field derived from (13), when $\Lambda = 0$, has the Painlevé property when $\beta = \frac{1}{2}, 1, 2$, and they obtained these integrals. The next advance was due to Howard and Farrelly [13] obtaining the third integral in three dimensions for the values $\beta = \frac{1}{2}, 2$ valid except for the z-axis. Here we show that, except for the three known cases, the generalized van der Waals Hamiltonian is non-integrable in the Liouville-Arnold sense (for more details see [10]).

We will fix one value of the energy so that, in suitable coordinates, the resulting Hamiltonian function is a natural Hamiltonian with a homogeneous potential. In this way, we have taken into account a recent extension of a known criterion of non-integrability by Yoshida [33]. This extended criterion is the one obtained by Morales and Ramis presented in Section 2.2.1 (see [26]).

Then, in order to apply Theorem 7 we convert the Hamiltonian (13) into a natural Hamiltonian with homogeneous potential. We begin by writing the Hamiltonian (13) in Cartesian coordinates. Then, by rescaling we obtain

$$\mathcal{H}_\beta = \frac{1}{2}(X^2 + Y^2 + Z^2) - \frac{1}{r} + \frac{1}{2}(x^2 + y^2 + \beta^2 z^2). \tag{14}$$

Using the Kustaanheimo-Stiefel transformation of coordinates the Hamiltonian (14) takes the form

$$\begin{aligned}\mathcal{H}_\beta &= \frac{1}{8\boldsymbol{u}^2}\boldsymbol{P}^2 - \frac{1}{\boldsymbol{u}^2} + \frac{1}{2}\Big(4(u_1u_3+u_2u_4)^2 \\ &\quad +4(u_1u_2-u_3u_4)^2 + \beta^2(u_1^2-u_2^2-u_3^2+u_4^2)^2\Big). \end{aligned} \tag{15}$$

This change does not introduce essential singularities; any meromorphic integral in cartesian coordinates becomes a meromorphic integral in Kustaanheimo-Stiefel coordinates. We convert the Hamiltonian (15) into a system of four coupled anharmonic oscillators making a transformation $t \longrightarrow \tau$ of the independent variable $dt/d\tau = 4r = 4\boldsymbol{u}^2$. Multiplying (15) by $4r$ leads to a four anharmonic oscillators system, defined by

$$\begin{aligned} 4 &= \frac{1}{2}(\boldsymbol{P}^2+\omega^2\boldsymbol{u}^2) + 2\boldsymbol{u}^2\Big(4(u_1u_3+u_2u_4)^2 \\ &\quad +4(u_1u_2-u_3u_4)^2 + \beta^2(u_1^2-u_2^2-u_3^2+u_4^2)^2\Big), \end{aligned} \tag{16}$$

where $\omega^2 = -8h$, h being the energy of the Hamiltonian (15).

In order to prove the non integrability of the Hamiltonian system defined by (16) it is sufficient to show that for a specific value of the energy, the Hamiltonian is not completely integrable. Thus, we fix the energy to be zero. By denoting

$$V_\beta(\boldsymbol{u}) = 2\boldsymbol{u}^2(4(u_1u_3+u_2u_4)^2 + 4(u_1u_2-u_3u_4)^2 + \beta^2(u_1^2-u_2^2-u_3^2+u_4^2)^2\Big),$$

we obtain the following natural Hamiltonian H with homogeneous potential V of degree 6

$$4 = H = \frac{1}{2}\boldsymbol{P}^2 + V_\beta(\boldsymbol{u}). \tag{17}$$

Thus, we can apply the indicated procedure of Morales and Ramis to show non integrability of the system defined by (17). Because the exponent of the parameter β in the Hamiltonian is even we will analyze the case $\beta > 0$. The case $\beta = 0$ is considered later.

In our problem, from the system of equations (3) $\frac{\partial}{\partial(x,y)}V(c) = c$, in order to apply Theorem 7 we have identified some of the solutions which are sufficient for our purposes. Those solutions are

$$\{u_3,u_4\} = \{\frac{\Omega}{2}(\frac{1}{3})^{\frac{1}{4}}, \frac{-\Omega}{2}(\frac{1}{3})^{\frac{1}{4}}\}, \tag{18}$$

$$\{u_3,u_4\} = \{\chi\frac{\Omega}{\sqrt{2}}(\frac{1}{3\beta^2})^{\frac{1}{4}}, (1-\chi)\frac{\Omega}{\sqrt{2}}(\frac{1}{3\beta^2})^{\frac{1}{4}}\}, \tag{19}$$

$$\{u_3,u_4\} = \{\pm\mathrm{i}(4(4-\beta^2))^{-\frac{1}{4}}, \pm\mathrm{i}(4(4-\beta^2))^{-\frac{1}{4}}\} \tag{20}$$

where $\chi = 0, 1$, and Ω belongs to the set of the four complex roots of unity. In what follows, in order to prove the non integrability, we will only need to make use of the first two sets of solutions.

Indeed, in the second step of the procedure we compute the eigenvalues of the Hessian matrix of the potential V valued in the above sets of particular solutions. Concretely, any family of solutions in set (18) leads to the following eigenvalues:

$$\{\frac{-1+4\beta^2}{3}, 1, 1, 5\},$$

and any family of solutions in the set (19) leads to the following eigenvalues

$$\{\frac{4-\beta^2}{3\beta^2}, \frac{4-\beta^2}{3\beta^2}, 1, 5\}. \tag{21}$$

We will consider first the eigenvalue $\frac{-1+4\beta^2}{3}$. Taking into account Theorem 7, the necessary condition for the Hamiltonian (17) to be integrable is that the $(6, \frac{-1+4\beta^2}{3})$ pair belongs to either the first set or to the last set in the list of Theorem 7. Thus, two cases can be considered.
a) The $(6, \frac{-1+4\beta^2}{3})$ pair belongs to the first set in the Theorem 7. Then, parameter β must be in set

$$A = \{\frac{3p-1}{2} : p \geq 1, \text{ integer}\} \cup \{\frac{1-3p}{2} : p \leq 0, \text{ integer}\}.$$

b) The $(6, \frac{-1+4\beta^2}{3})$ pair belongs to the last set in the Theorem 7. Then, parameter β must be in set

$$B = \{\frac{6p+3}{4} : p \geq 0, \text{ integer}\} \cup \{-\frac{6p+3}{4} : p \leq -1, \text{ integer}\}.$$

In the case of eigenvalues 1 and 5 it is easy to check that the pairs $(6,1)$ and $(6,5)$ belong to the first list in Theorem 7.

From the above analysis we conclude that if β does not belong to $A \cup B$ the Hamiltonian system defined by (17) is not completely integrable in the Liouville-Arnold sense by the Theorem 7. Now, in order to finish the proof we will consider the second set of eigenvalues (21) and proceed as with the previous set. However, we only need to consider those values of $\beta \in A \cup B$.

Let us suppose first that $\beta = 2$. Then, the set of eigenvalues is $\{0, 0, 1, 5\}$. Immediately we find that the $(6,0)$ pair belongs to the list in Theorem 7. Now, suppose $\beta \neq 2$, and that $\frac{4-\beta^2}{3\beta^2}$ belongs to the first set of conditions in Theorem 7. Then, there must exist an integer p solution to the equation

$$p + 3p(p-1) = \frac{4-\beta^2}{3\beta^2}.$$

By solving the equation we obtain that $L_1 = \frac{\beta+2}{3\beta}$ or $L_2 = \frac{\beta-2}{3\beta}$ must be integer. Thus, it is easy to show that if $\beta \in A \cup B$, only $\beta = 1$ makes L_1 integer and $\beta = \frac{1}{2}$ makes L_2 integer.

Finally, let us suppose that $\frac{4-\beta^2}{3\beta^2}$ belongs to the last list of Theorem 7. Then, it must exist an integer p solution to the equation:

$$\frac{5}{12} + 3p(p+1) = \frac{4-\beta^2}{3\beta^2}.$$

By solving this equation we obtain that $L_3 = \frac{-3\beta+4}{6\beta}$ or $L_4 = -\frac{3\beta+4}{6\beta}$ must be integer. Now, taking $\beta \in A \cup B$ it is immediate to show that L_3 and L_4 are not integers.

The value $\beta = 0$ remains to be considered. In this case, by imposing $u_1 = u_2 = 0$ we easily obtain a solution to the system of equations (18)-(20)

$$\{u_3, u_4\} = \{12^{-\frac{1}{4}}, 12^{-\frac{1}{4}}\}.$$

The eigenvalues of the Hessian matrix of V_0 valued in the above solution are

$$\{-\frac{1}{3}, 1, 1, 5\}.$$

It is straightforward to check that the eigenvalue $-\frac{1}{3}$ does not belong to the list in Theorem 7. Then, the Hamiltonian system defined by (17) is not completely integrable if $\beta = 0$. Thus, we have obtained the result of Kummer and Saenz about the non integrability of the Zeeman effect by a shorter way.

In conclusion, by Theorem 7, we have proved the following theorem.

Theorem 11 *The Generalized van der Waals Hamiltonian given by (13) is non integrable if* $\beta \notin \{1, 2, \frac{1}{2}\}$.

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On the non-integrability of parametric Hamiltonian systems by differential Galois theory

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Abstract

In this note[1] we consider a family of Hamiltonian systems $\mathcal{H}_\epsilon$ depending continually of a real parameter ϵ. We prove that, under certain conditions on the singular points, if for a given value ϵ_0 of the parameter the corresponding Hamiltonian system $\mathcal{H}_{\epsilon_0}$ is not completely integrable, then for values in a neigborghood of ϵ_0 the Hamiltonian system will not be completely integrable.

1. Introduction

Let us consider a non-integrable Hamiltonian system. A question that we face is if we will have non-integrability for Hamiltonian systems in an small neigborhood of the non-integrable Hamiltonian for an appropiate topology in the space of Hamiltonian systems. This general question is not close to be answered. Thus, in a first step of approximation to the core of the problem, we consider a parametric family of Hamiltonian systems. We impose the non-integrability for a given value of the parameter and we ask us about the non-integrability for parametric values close enough of the given value of non-integrability.

Before detailing our analysis we set up the concept of integrability for Hamiltonian systems that we will use. We say that a n-DOF Hamiltonian system is completely integrable in the extended Liouville-Arnold sense if there exist n meromorphic first integrals independent and in involution in a open dense subset of the complexified phase space. Thus, the esential tool used to prove the main result of this note is the non-integrability theory Morales and Ramis [4, 5] based on Differential Galois theory.

Concretely, in this note, under suitable conditions, we show that the non-solvability of the Differential Galois group of a Riemann surface associated to the non-integrable Hamiltonian system, for a given value of the parameter, implies the non-abelianess of the Differential Galois

[1]This note is part of a longer paper submitted for publication to *Comm. Math. Phys.*

group of the Riemann surface corresponding to the parameter values close enough of the given value of non-integrability.

2. Terminology and basic theorems

In this Section we present a short description of the theorems of Morales-Ramis that we will use to prove the results of the note.

Let us consider a 2n-dimensional complex analytic manifold M with a symplectic two form Ω and a holomorphic Hamiltonian vector field on M, X_H. Let $x = \phi(t)$ be a germ of a regular curve in M that is not an equilibrium point. We take $i(\Gamma)$ the maximal connected component analytically continued of the germ $x = \phi(t)$, and we consider Γ the abstract Riemann surface defined by $i(\Gamma)$. The inclusion $i : \Gamma \to i(\Gamma) \subset M$ is an immersion. Using the immersion i we define the fiber bundle $\pi : \Gamma \to \mathbf{T}\Gamma$ as the pull back of the fiber bundle $\mathbf{T}M$ restrict to $i(\Gamma)$. In the same way, we define a Hamiltonian vector field, denoted by X, on Γ by pull back of the X_H

Let us consider the holomorphic symplectic connection ∇ defined by pull back from the restriction to $i(\Gamma)$ of the Lie derivative with respect to the Hamiltonian vector field X_H

$$\nabla v = L_{X_H} Y|_\Gamma$$

where v is a section of the bundle $\mathbf{T}_\Gamma$ Y is holomorphic vector field extension of the section v of the bundle $\mathbf{T}_{i(\Gamma)}M$. If we express the conexion ∇ in a local trivialization of the bundle $\mathbf{T}\Gamma$ we obtain a linear differential system which is the variational equation (VE) along the integral curve defined by de germ $x = \phi(t)$ (see [4, 5]). The elements of $\mathrm{Ker}(\nabla)$ are called horizontal sections. The horizontal sections expressed in a local trivialization of $\mathbf{T}\Gamma$ are the solutions of the VE.

Using the previous definition we will define the monodromy group of ∇ denoted by $Mon(\nabla)$ (see [1]). Let $x_0 \in \Gamma$ be a fixed point. Let us denote by $\pi_1(\Gamma, x_0)$ the fundamental group of Γ, and if $\mathbf{T}\Gamma|_{x_0}$ is the fiber over x_0, $Aut(\mathbf{T}\Gamma|_{x_0})$ denotes de group of automorphism of $\mathbf{T}\Gamma|_{x_0}$. We take an homotopy class $[\gamma] \in \pi_1(\Gamma, x_0)$ and we construct the inverse $\gamma^{-1} : [0,1] \to \Gamma$. Let $v_1, \ldots, v_{2n} \in \mathbf{T}\Gamma|_{x_0}$ $2n$ linearly independent vectors, by the existence theorem for ordinary differential equations applied locally in neighbourhood of $\gamma^{-1}(0)$ in Γ we take $s_1, \ldots, s_{2n}$ linearly independent horizontal sections of ∇ defined in a neighbourhood of $\gamma^{-1}(0)$ with initial conditions $v_1, \ldots, v_{2n}$ respectively. Then, we continue holomorphicaly the horizontal section $s_1, \ldots, s_{2n}$ along γ^{-1} terminating in a new set of linearly independent horizontal section $s_1^1, \ldots, s_{2n}^1$ in neighbourhood of $\gamma^{-1}(0)$. The map $I_{\gamma^{-1}} : \{s_1(x_0), \ldots, s_{2n}(x_0)\} \mapsto \{s_1^1(x_0), \ldots, s_{2n}^1(x_0)\}$ is an automorphism of $\mathbf{T}\Gamma|_{x_0}$ that only depends on the homotopy class of γ^{-1}. Finally, we define the representation of $\pi_1(\Gamma, x_0)$

$$\mathrm{Mon} : \pi_1(\Gamma, x_0) \longrightarrow Aut(\mathbf{T}\Gamma|_{x_0}) \tag{1}$$

as $\mathrm{Mon}([\gamma]) = I_{\gamma^{-1}}$. Then, the monodromy group, denoted by $\mathrm{Mon}(\nabla)$, is the image of the fundamental group by this representation. The representation and the monodromy group are independent of x_0 up to equivalence.

In this situation it is proved in [4], (Theorem 7)

Theorem 12 *Assume that there are n first integrals of X_H which are meromorphic, in involution and independent in a neighborhood U of the curve $i(\Gamma)$ in M. Then the identity component of the Galois group of the VE is an abelian subgroup of the symplectic group.*

In some cases, if the vector field X_H has a finite set of equilibria that belong to the closure of $i(\Gamma)$ in M, we add to Γ this finite set of equilibria. We denote this new curve by $\underline{\Gamma}$. Then, we have $i(\Gamma) \subset \underline{\Gamma} \subset M$, where $\underline{\Gamma}$ is a closed analytic curve and $\overline{\Gamma}$ its corresponding connected Riemann surface, $\Gamma \subset \overline{\Gamma}$.

In other cases we add to $\overline{\Gamma}$ (or Γ) a finite set of points corresponding to points at infinity of $\underline{\Gamma}$. In these cases we suppose that we suppose that the manifold M is contained in a connected manifold M', being $M_\infty = M' - M$ is an analytic hypersurface in M', called hypersurface at infinity. The holomorphic symplectic 2-form Ω over M extends to a meromorphic symplectic 2-form Ω' over M' (see [4]). Then, we obtain $\underline{\Gamma} \subset \underline{\Gamma}' \subset M'$ and $\overline{\Gamma} \subset \overline{\Gamma}'$ where $\underline{\Gamma}'$ is a closed analytic curve in M', and $\overline{\Gamma}'$ is the corresponding connected Riemann surface. After that, the meromorphic connection over $\overline{\Gamma}$ extends to a meromorphic connection over $\overline{\Gamma}'$. Finally, we compute the differential Galois group $\overline{G}$ (resp. $\overline{G}'$) of the VE relatively to the differential field of meromorphic functions over $\overline{\Gamma}$ (resp. $\overline{\Gamma}'$). Let us remember that the above differential Galois group is isomorphic to a linear algebraic group over $\mathbb{C}$. A linear algebraic group is a subgroup of $GL(m, \mathbb{C})$ whose matrix coefficients satisfy polynomial equations over $\mathbb{C}$ (see [3]).

In this situation it is proved in [4], (Theorem 9)

Theorem 13 *Assume that there is a finite set of equilibrium points and points at infinity. Assume that there are n first integrals of X_H which are meromorphic, in involution and independent in a neighborhood U of the curve $\underline{\Gamma}'$ in M'. Then the identity component of the Galois group $\overline{G}'$ of the VE over the differential field of the meromorphic functions over $\overline{\Gamma}'$ is an abelian subgroup of the symplectic group.*

In general we have $G \subset \overline{G} \subset \overline{G}'$ with strict inclusion. However, when the extended connection over of the variational equation over $\overline{\Gamma}$ (resp. $\overline{\Gamma}'$) is Fuchsian (i.e. the singular points are regular singular points) we have $G = \overline{G}$ (resp. $G = \overline{G}'$).

Morales and Ramis considered the relation between the Galois groups of finite covering of a Riemann surface and of the proper Riemann surface. This relation is given by the following theorem, Theorem 5 of [4].

Theorem 14 *Let X be a connected Riemann surface. Let (X', f, X) be a finite ramified covering of X by a connected Riemann surface X'. Let ∇ be a meromorphic connection over X. We set $\nabla' = f^*\nabla$. Then, we have a natural injective homomorphism*

$$Gal(\nabla') \longrightarrow Gal(\nabla)$$

of differential Galois groups which induces an isomorphism between their Lie algebras.

In terms of differential Galois groups this theorem means that the identity component of the differential Galois group is invariant by the covering.

3. The main result

We will state first a technical result needed in the proof of the main theorem of the note.

Let M be a complex analytic manifold and U an open neigbourhood of $0 \in \mathbb{C}^m$. Let us consider a continuous function

$$\mathcal{H} : M \times U \longrightarrow \mathbb{C}, \tag{2}$$

where for fixed $\epsilon \in U$, $\mathcal{H}_\epsilon(z) = \mathcal{H}(z, \epsilon)$ $z \in M$, is a holomorphic Hamiltonian function. We suppose additionally that the map

$$X_{\mathcal{H}} : M \times U \longrightarrow TM,$$

defined by $X_{\mathcal{H}}(z, \epsilon) = X_{\mathcal{H}_\epsilon}(z)$ is continuous.

Let us consider, for each $\epsilon \in U$, $x = \phi_\epsilon(t)$ a germ of a regular curve in M such that $\phi_\epsilon(0) = x_0 \in M$, for a fixed point $x_0 \in M$. We take $i(\Gamma_\epsilon)$ the maximal connected component analytically continued of the germ $x = \phi_\epsilon(t)$ (we suppose that this is not an equilibrium point), and we consider Γ_ϵ the abstract Riemann surface defined by $i(\Gamma_\epsilon)$. The inclusion $i : \Gamma_\epsilon \to i(\Gamma_\epsilon) \subset M$ is an immersion. Using the immersion i we obtain by pull back of the $X_{\mathcal{H}_\epsilon}$ a Hamiltonian vector field on Γ_ϵ that we denote by X_ϵ.

We take into account two different hypothesis:

hypothesis *1* For each $\epsilon \in U$ we need not to add to the surfaces Γ_ϵ equilibria points of $X_{\mathcal{H}_\epsilon}$ or point at infinity.

hypothesis *2* For $\epsilon = 0$ we have to add to the surface Γ_0, as it was described in the previous section, equilibrium points $u_1^0, \ldots, u_m^0$ and/or the points at infinity $v_1^0, \ldots, v_r^0$. We denote the new Riemann surface by $\overline{\Gamma}_0$ (resp. $\overline{\Gamma}_0'$). Moreover, in a small neighbourhood of 0 $V \subset U$, for each $\epsilon \in V$ we have to add to Γ_ϵ the equilibrium points and/or the points at infinity

$$u_1^\epsilon, \ldots, u_m^\epsilon, \quad v_1^\epsilon, \ldots, v_r^\epsilon$$

respectively. We denote the new Riemann surface by $\overline{\Gamma}_\epsilon$ (resp. $\overline{\Gamma}_\epsilon'$).

Let us consider the holomorphic symplectic connection ∇_ϵ, defined by pull back from the restriction to $i(\Gamma_\epsilon)$ of the Lie derivative with respect to the Hamiltonian vector field $X_{\mathcal{H}_\epsilon}$

$$\nabla_\epsilon v = L_{X_{\mathcal{H}_\epsilon}} Y|_{\Gamma_\epsilon},$$

where v is a section of the bundle $\mathbf{T}_{\Gamma_\epsilon)}$ Y is holomorphic vector field extension of the section v of the bundle $\mathbf{T}_{i(\Gamma_\epsilon)}M$.

In the case of the **hypothesis** *2*, we extend the connection ∇_ϵ to a meromorphic symplectic connection $\overline{\nabla}_\epsilon$ (resp. $\overline{\nabla}_\epsilon'$) on $\overline{\Gamma}_\epsilon$ (resp. $\overline{\Gamma}_\epsilon'$) (see [4]). The singularities of $\overline{\Gamma}_\epsilon$ (resp. $\overline{\Gamma}_\epsilon'$) are the points $u_1^\epsilon, \ldots, u_m^\epsilon$ (resp. $u_1^\epsilon, \ldots, u_m^\epsilon$ and $v_1^\epsilon, \ldots, v_r^\epsilon$).

We remember now some results that we will use in the proof of the technical result of this section.

Theorem 15 *(Tits, Theorems 1,3 in [6])*

1. *Over a field of characteristic 0, a linear group either has a non-abelian free subgroup or possesses a solvable subgroup of finite index.*
2. *Let G be a nontrivial semisimple algebraic group defined over a field k of characteristic 0 and let P be a (Zariski) dense subgroup of G. Then G has a countable free subset F such that every element of F generates a connected subgroup of G and that every pair of elements of F generates a dense non-abelian free subgroup of G.*

Lemma 1 *(Tits, Lemma 4.2 in [6])*

Let U be a k-vector space and let H be a finitely generated subgroup of $GL(U)$. Then, there exists $m \in \mathbb{N}^*$ such that, for every $h \in H$, the group generated by h^m is connected.

Finally, we can set up the main result of this Section.

Theorem 16 *Let $\mathbf{P}^1$ be the Riemann sphere. Let $V_1 \subset V$ be an open neighbourhood of $0 \in \mathbb{C}^m$. We suppose that for each $\epsilon \in V_1$ there exists a finite ramified covering $f_\epsilon : \mathbf{P}^1 \longrightarrow \Gamma_\epsilon$ (resp. over $\overline{\Gamma}_\epsilon$, $\overline{\Gamma}'_\epsilon$). Let $\nabla^*_\epsilon = f^*_\epsilon \nabla_\epsilon$ (resp. $\overline{\nabla}^*_\epsilon$ and $(\overline{\nabla}')^*_\epsilon$) be the corresponding meromorphic connection on Γ by the covering f_ϵ. Let us consider the following hypothesis.*

i) We assume that for $\epsilon = 0$ the singularities of ∇^*_0 (resp. $\overline{\nabla}^*_0$, $(\overline{\nabla}')^*_0$) are the points $x_1, \ldots, x_m$, all regular singular points.

ii) We suppose that for each $\epsilon \in V_1 \setminus 0$, the singularities of ∇^*_ϵ are the points $x^\epsilon_1, \ldots, x^\epsilon_m$. Moreover, for each neighbourhood N of an arbitrary point x_j on $\mathbf{P}^1$, there exists a positive number δ and a polyadisk $P(\delta)$ such that, for each $\epsilon \in P(\delta)$ the point x^ϵ_j belongs to N.

iii) The vector field $X_{\mathcal{H}_0}$ is non-integrable, being the identity component of the differential Galois group of ∇_0 (resp. $\overline{\nabla}_0$, $\overline{\nabla}'_0$) over the meromorphic functions on Γ_0 (resp. $\overline{\Gamma}_0$, $\overline{\Gamma}'_0$) not solvable.

Then, there exists a neighbourhood $V_2 \subset V_1$ such that, for each $\epsilon \in V_2$ the identity component of the differential Galois group of ∇_ϵ (resp. $\overline{\nabla}_\epsilon$, $\overline{\nabla}'_\epsilon$) over the meromorphic functions on Γ_ϵ (resp. $\overline{\Gamma}_\epsilon$, $\overline{\Gamma}'_\epsilon$) is not abelian. Thus, the vector field $X_{\mathcal{H}_\epsilon}$ is non-integrable.

Proof.-

Let us denote by $\mathrm{Gal}(\nabla_\epsilon)$ (resp. $\mathrm{Gal}(\nabla^*_\epsilon)$) the differential Galois group of ∇_ϵ (resp. ∇^*_ϵ) over the meromorphic functions on Γ_ϵ (resp. $\mathbf{P}^1$). By using Theorem 14, we have that $\mathrm{Gal}(\nabla^*_\epsilon)^\circ$ is not solvable, because $\mathrm{Gal}(\nabla_\epsilon)^\circ$ is not solvable. Then $\mathrm{Gal}(\nabla^*_\epsilon)^\circ/R$ is a semisimple group, where R is the semisimple radical of $\mathrm{Gal}(\nabla^*_\epsilon)^\circ$.

Let us denote the monodromy group of ∇^*_ϵ for each $\epsilon \in V_1$ over $\mathbf{P}^1$ by $\mathrm{Mon}(\nabla^*_\epsilon)$, and by $\mathrm{Mon}(\nabla^*_\epsilon)^\circ = \mathrm{Gal}(\nabla^*_\epsilon)^\circ \bigcap \mathrm{Mon}(\nabla^*_\epsilon)$. We have that $\mathrm{Mon}(\nabla^*_\epsilon)^\circ$ is dense in $\mathrm{Gal}(\nabla^*_\epsilon)^\circ$, because the singular points of ∇^*_0 are regular singular points . By using Theorem 15 we have that $\mathrm{Mon}(\nabla^*_0)^\circ/R$ has a non abelian free subgroup since $\mathrm{Gal}(\nabla^*_\epsilon)^\circ/R$ is a semisimple group.

Thus, we select two non commuting elements $A_1, A_2 \in \mathrm{Mon}(\nabla_0^*)^\circ$. Then, there exist paths γ_1 and γ_2 in $\mathbf{P}^1$ with base point x_0 such that $\mathrm{Mon}([\gamma_i]) = A_i$ for $i = 1, 2$, where Mon is the monodromy map defined by (1). Let us suppose that the path γ_1 encircles the points $x_{i_1}, \ldots, x_{i_k}$ with $k \geq 0$. By hypothesis **ii)** of Theorem 16 for each neighbourhood N of x_{i_j} there exists a positive number $\delta = \delta(x_{i_j}, N)$ and a polyadisk $P(\delta)$ contained in V_1 such that, for each $\epsilon \in P(\delta)$ the point $x_{i_j}^\epsilon$ belongs to N. Then, for each $x_{i_1}, \ldots, x_{i_k}$ we select neighbourhood $N_{i_1}, \ldots, N_{i_k}$ contained in V_1 verifying $N_{i_j} \bigcap \mathrm{Im}\gamma_1 = \emptyset$. Thus, for each $\epsilon \in P(\delta_1)$ the point $x_{i_j}^\epsilon \in N_{i_j}$, where $\delta_1 = \min\{\delta(x_{i_j}, N_{i_j})\}_{j=1,\ldots,k}$. Then, the curve γ_1 encircles the points $x_{i_j}^{(\epsilon,s)}$ for $j = 1, \ldots, k$, $s = 1, \ldots, i_k$ and $\epsilon \in P(\delta_1)$.

Given a set $\{s_1^0, \ldots, s_{2n}^0\}$ linearly independent of horizontal section of ∇_0^* in W_0, by definition of $\mathrm{Mon}([\gamma_1])$, there exists a partition $0 = t_0 < \ldots < t_q = 1$ of $[0, 1]$ and open neighbourhoods $W_i \subset \mathbf{P}^1$ with $\gamma_1^{-1}([t_{i-1}, t_i]) \subset W_i$ for $i = 1, \ldots, q$, such that there exist a family of set $\{s_1^j, \ldots, s_{2n}^j\}$ of linearly independent horizontal sections of ∇_0^* in W_j for $j = 2, \ldots, q$ verifying $s_i^j|_{W_j} = s_i^{j+1}|_{W_{j+1}}$ for $j = 0, \ldots, q-1$. Then $\mathrm{Mon}([\gamma_1])$ is the authomorfism of $\mathbf{T}_{x_0}\mathbf{P}^1$ which transforms the set $\{s_1^0(0), \ldots, s_{2n}^0(0)\}$ into the set $\{s_1^q(0), \ldots, s_{2n}^q(0)\}$.

Without lost of generality we can suppose, by taking smaller neighbourhoods $\overline{W}_i \subset W_i$, that $W_i \bigcap N_{x_{i_j}} = \emptyset$ for all i, j. We take for each $\epsilon \in P(\delta_1)$ a linearly independent set of horizontal section $\{s_1^{(\epsilon,0)}, \ldots, s_{2n}^{(\epsilon,0)}\}$ of ∇_ϵ^* on W_0, and we continue holomorphicaly this set along γ_1^{-1}. Thus, we construct the monodromy matrices $A_1(\epsilon)$ associated to the the authomorphism that transforms the set $\{s_1^{(\epsilon,0)}(0), \ldots, s_{2n}^{(\epsilon,0)}(0)\}$ into the set $\{s_1^{(\epsilon,q)}(0), \ldots, s_{2n}^{(\epsilon,q)}(0)\}$.

From the definition of ∇_ϵ^*, it is not hard to show that the VE that ∇_ϵ^* defines locally is a differential system depending continuously on ϵ. By the continuity dependence of the solution of the differential equations with respect to parameters we have that the functions $s_i^j(t, \epsilon) = s_i^{(\epsilon,j)}(t)$ are continuous in $W_j \times P(\delta_1)$, for $j = 0, \ldots, q$. Then, the matrices $A_1(\epsilon)$ define a continuous function on $P(\delta_1)$ such that $A_1(0) = A_1$. By arguing analogously with the monodromy matrix A_2, we obtain a continuous function $A_2(\epsilon)$ defined in some polyadisk $P(\delta_2)$, such that $A_2(\epsilon)$ are the monodromy matrices of ∇_ϵ^* associated to γ_2 and $A_2(0) = A_2$.

We note that the non-commutativity relationship between A_1 and A_2 can be expressed in terms of a finite set of strict inequalities involving only the coefficients of the matrices. By this and the continuity of the functions $s_i^j(t, \epsilon)$ we deduce the existence of a positive number $\delta_3 < min\{\delta_1, \delta_2\}$ such that, for each $\epsilon \in P(\delta_3)$ the matrices $A_1(\epsilon)$ and $A_2(\epsilon)$ do not commute.

For each fixed $\epsilon \in P(\delta_3)$, we denote the non abelian subgroup of $\mathrm{Gal}(\nabla_\epsilon^*)$ generated by $A_1(\epsilon)$ and $A_2(\epsilon)$ by $H(\epsilon)$. By applying the Lemma 1, we have that there exists a natural number d such that for every $h \in H(\epsilon)$ the subgroup generated by h^d is (Zariski) connected. In particular, if we take the elements $A_i^{-d}(\epsilon)$ $(i = 1, 2)$ we have that the subgroups generated by $A_1(\epsilon)$ and by $A_2(\epsilon)$, $< A_1(\epsilon) >$ and $< A_2(\epsilon) >$ respectively, are connected. Then, these subgroups must be contained in the identity component $\mathrm{Gal}(\nabla_\epsilon^*)^\circ$. Thus, we obtain that $H(\epsilon) \subset \mathrm{Gal}(\nabla_\epsilon^*)^\circ$. Therefore, $\mathrm{Gal}(\nabla_\epsilon^*)^\circ$ is not abelian. Then, by Theorem 14, the identity component $\mathrm{Gal}(\nabla_\epsilon)^\circ$ is not abelian. Finally, by applying the Theorem 13, we deduce that for each fixed $\epsilon \in P(\delta_3)$, the vector field $X_{\mathcal{H}_\epsilon}$ is non integrable. ∎

We finish this Section with two remarks.

Remark 3 *In [2] it is considered the family of vector field $X_{\mathcal{H}_\epsilon}$ where the function $\mathcal{H}$ defined in (2) is meromorphic in the variables and in the parameters. It is proved there that, if the vector field $X_{\mathcal{H}_0}$ is non integrable by meromorphic functions the vector fields $X_{\mathcal{H}_\epsilon}$ will not be integrable by meromorphic integrals in the variables and in the parameters.*

By considering a continuous family of vector fields we relax the hypothesis of the mentioned result in [2], and we obtain non-integrability by meromorphic integrals in the variables.

Remark 4 *We conjecture that the hypothesis* **ii)** *of Theorem 16 it is unnecessary. It is just only necessary the hypothesis of the regularities of the singular points of the Riemann surface $\overline{\Gamma}_0$ (resp. $\overline{\Gamma}'_0$). A more general result with this hypothesis it is now being investigated by the authors.*

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A gyrostat in the three-body problem: Reductions

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Abstract

The problem of three bodies when one of them is a gyrostat is considered. Using the symmetries of the system we carry out two reductions, giving in each step the Poisson structure of the reduced space and the Hamiltonian of the problem.

1. Introduction

In the last years new research about the problem of roto-translational motion of celestial bodies has appeared; some papers within differential geometry frame, others still with the classical approach. In particular, they show a new interest in the study of configurations of relative equilibria in different models. In the problem of three rigid bodies Vidiakin [1] and Duboshine [2] proved the existence of Euler and Lagrange configurations of equilibria when the bodies possess symmetries (for a recent review see [3]). More recently Wang [4] considers the problem of a rigid body in a central Newtonian field and Maciejewski [5] takes into account the problem of two rigid bodies in mutual Newtonian attraction. In the same way, these problems have been generalized to the case when the rigid bodies are gyrostats [6], [7], [8].

In order to study the configurations of equilibria of the general problem of three rigid bodies from a global geometrical point of view it is natural to consider first the problem when two bodies have spherical distribution of mass. Fanny and Badaoui [9] study of the configuration of the equilibria in terms of the global variables in the unreduced problem. There, simplifications such as considering spherical or axisymmetric bodies are made in order to get specific results. It is clear, as the papers of [5] and [8] show, that to work in the reduced system (if the problem has symmetries) produces natural simplifications in the conditions of the equilibria, and then more general results can be obtained. This is the approach we will follow in this paper.

In the way just mentioned above, the problem of three rigid bodies when two are spherical and the other is a gyrostat is considered. Using the symmetries of the translational and rotational group possessed by the system, we perform a reduction process in two steps, giving explicitly at each step the Poisson structure of the reduced system. We note also that the reduction procedure presented here applies immediately to rigid body case when we take the girostatic momentum be zero.

2. Configuration and phase space

Let us denote by S_0 a gyrostat of mass m_0, by S_1 and S_2 two rigid bodies with spherical symmetry of masses m_1 and m_2 respectively. We remember that a gyrostat is a mechanical system $\mathcal{G}$ composed of a rigid body and other bodies (deformable or rigid) connected to it such that their relative motion do not change the distribution of mass of $\mathcal{G}$. Let us consider an inertial reference frame $\mathcal{I} = \{O, u_1, u_2, u_3\}$ and a body frame $\mathcal{B} = \{C_0, b_1, b_2, b_3\}$ fixed at the center of mass C_0 of S_0. A particle in the body S_0 with coordinates Q in $\mathcal{B}$ is represented in the inertial frame $\mathcal{I}$ by the vector $q = R_0 + BQ$, where $B \in SO(3)$ and R_0 is the vector position of the center of mass of S_0 in $\mathcal{I}$. Let us denote by R_1 and R_2 the vector position of center of mass of the bodies S_1 and S_2 respectively in $\mathcal{I}$. Then, at any instant, the configuration of the system is uniquely determined by $((B, R_0), R_1, R_2)$. The configuration space of the problem is the Lie group $\boldsymbol{Q} = SE(3) \times \mathbb{R}^3 \times \mathbb{R}^3$, where $SE(3)$ is the known semidirect product of $SO(3)$ and $\boldsymbol{R}^3$.

The Kinetic energy of the system is $\mathcal{T} = 1/2 \int_{S_0} |\dot{q}|^2 dm(Q) + m_1|\dot{R}_1|^2/2 + m_2|\dot{R}_2|^2/2$, where $dm(.)$ denotes the mass measure of S_0, and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$. The expression of the Kinetic energy simplifies (see [6]) to

$$\mathcal{T} = \frac{1}{2}\Omega \cdot \boldsymbol{I}\Omega + L_r \cdot \Omega + \frac{m_0}{2}|\dot{R}_0|^2 + \frac{m_1}{2}|\dot{R}_1|^2 + \frac{m_2}{2}|\dot{R}_2|^2 + \mathcal{T}_r,$$

where $\boldsymbol{I}$ is the tensor of inertia of S_0 in the body frame, Ω is the angular velocity of S_0 defined by $\dot{B} = B\widehat{\Omega}$, and L_r and $\mathcal{T}_r$ are the momentum and the Kinetic energy of the moving part of the gyrostat respectively. Here, $\widehat{\Omega}$ is the image by the standard isomorphism between the Lie algebras $so(3)$ and $\mathbb{R}^3$, i.e., for $X = (X_1, X_2, X_3) \in \mathbb{R}^3$,

$$\widehat{X} = \begin{pmatrix} 0 & -X_3 & X_2 \\ X_3 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{pmatrix}.$$

In what follows we assume that $\mathcal{T}_r$ is a known function of the time and L_r is constant.

The gravitational potential energy is the function $\mathcal{V} : \boldsymbol{Q} \longrightarrow \mathbb{R}$

$$\mathcal{V} = -Gm_1m_2\frac{1}{|R_2 - R_1|} - Gm_1\int_{S_0}\frac{dm(Q)}{|BQ + R_0 - R_1|} - Gm_2\int_{S_0}\frac{dm(Q)}{|BQ + R_0 - R_2|}.$$

Then, the Lagrangian of the problem is $\mathcal{L} : \boldsymbol{TQ} \to \mathbb{R}$

$$\mathcal{L} = \mathcal{T} - \mathcal{V} \circ \tau \tag{1}$$

where $\tau : TQ \longrightarrow Q$ is the canonical projection.

The phase space is the cotangent bundle T^*Q. By left trivialization we identify T^*Q with $Q \times Q^*$. We denote by $\Xi = ((B, R_0), R_1, R_2, (B\widehat{\Pi}, P_0), P_1, P_2)$ the elements of T^*Q, where $\Pi = I\Omega + L_r$ is the total angular momentum of the gyrostat in the body frame $\mathcal{B}$ and $P_i = m_i\dot{R}_i$, $i = 0, 1, 2$ are the linear momentum of the bodies in the fixed frame $\mathcal{I}$.

T^*Q carries a canonical symplectic structure ω defined as $\omega = \omega^{SE} + \omega^{\mathbb{R}^3} + \omega^{\mathbb{R}^3}$, where ω^{SE} denotes the symplectic 2-form in $SE(3) \times se(3)^*$ by left trivialization of the canonical 2-form on $T^*SE(3)$, and $\omega^{\mathbb{R}^3}$ denotes de canonical 2-form in $T^*\mathbb{R}^3$. Associated to the symplectic structure on T^*Q given by ω we have a Poisson structure where the Poisson bracket is obtained from ω. The Poisson brackect takes the form

$$\begin{aligned}\{f,g\}_{T^*Q}(\Xi) &= \Big(< D_B f, \frac{\partial g}{\partial B\widehat{\Pi}} > - < D_B g, \frac{\partial f}{\partial B\widehat{\Pi}} > + \frac{\partial f}{\partial R_0}\frac{\partial g}{\partial P_0} - \frac{\partial g}{\partial R_0}\frac{\partial f}{\partial P_0} \\ &\quad + \frac{\partial f}{\partial R_1}\frac{\partial g}{\partial P_1} - \frac{\partial g}{\partial R_1}\frac{\partial f}{\partial P_1} + \frac{\partial f}{\partial R_2}\frac{\partial g}{\partial P_2} - \frac{\partial g}{\partial R_2}\frac{\partial f}{\partial P_2}\Big)(\Xi), \end{aligned} \tag{2}$$

where $< \cdot, \cdot >$ denotes the natural pairing between $T^*SO(3)$ and $TSO(3)$ and $\frac{\partial f}{\partial R_i}\frac{\partial g}{\partial P_i}$ denotes the Euclidean inner product on $\mathbb{R}^3$.

The Hamiltonian of the problem is the function $\mathcal{H} : T^*Q \to \mathbb{R}$

$$\mathcal{H} = \frac{1}{2}\Pi \cdot I^{-1}\Pi - L_r \cdot I^{-1}\Pi + \frac{|P_0|^2}{2m_0} + \frac{|P_1|^2}{2m_1} + \frac{|P_2|^2}{2m_2} + \mathcal{V}.$$

3. Symmetries and reduction

The problem can be reduced by the action of the group $SE(3)$. Thus, we might reduce using the semidirect product reduction theorem (see [10] and [11]). However, in the case of semidirect products we can proceed by stages [13]: we will use of the symplectic reduction procedure by the action of the translation group $\mathbb{R}^3$ in a first stage, and a Poisson reduction procedure by the group $SO(3)$ in a second stage.

3.1 Reduction by the translation group

Consider the action of the translation group $\mathbb{R}^3$

$$\begin{aligned}\Phi_1 &: \mathbb{R}^3 \times Q \longrightarrow Q \\ (u, z) &\mapsto ((B, R_0 + u), R_1 + u, R_2 + u),\end{aligned}$$

where z denotes the point $((B, R_0), R_1, R_2) \in Q$. This action lifts to a free and proper action on T^*Q,

$$\begin{aligned}\Phi_1^{T^*} &: \mathbb{R}^3 \times T^*Q \longrightarrow T^*Q \\ (u, z) &\mapsto ((B, R_0 + u), R_1 + u, R_2 + u, (B\widehat{\Pi}, P_0), P_1, P_2),\end{aligned}$$

with an Ad*-equivariant momentum map $j : T^*Q \to (\mathbb{R}^3)^* \cong \mathbb{R}^3$, $j(z) = P_0 + P_1 + P_2$. We will use the regular reduction Theorem (see [12]). Let us take κ a regular value for j, and we

consider then the submanifold $j^{-1}(\kappa)$. Because the translation group is abelian, its isotropy subgroup $(\mathbb{R}^3)_\kappa$ under the co-adjoint action is the whole group of translation. Then, by the regular reduction Theorem, the reduced space is the symplectic manifold $\boldsymbol{T}^*\boldsymbol{Q}_\kappa = j^{-1}(\kappa)/\mathbb{R}^3$ with symplectic form $\pi_\kappa^*\omega_\kappa = i_\kappa^*\omega$, where $\pi_\kappa : j^{-1}(\kappa) \to \boldsymbol{T}^*\boldsymbol{Q}_\kappa$ is the canonical projection and $i_\kappa : j^1(\kappa) \to \boldsymbol{T}^*\boldsymbol{Q}$ is the inclusion.

We will obtain a model for $\boldsymbol{T}^*\boldsymbol{Q}_\kappa$. We define by $M_1 = SO(3) \times so(3)^* \times \boldsymbol{T}^*\mathbb{R}^3 \times \boldsymbol{T}^*\mathbb{R}^3$. Let us consider the symplectic manifold (M_1, ω_1) where ω_1 is defined by $\omega_1 = \omega^{SO} + \omega^{\boldsymbol{R}^3} + \omega^{\boldsymbol{R}^3}$, where ω^{SO} is the symplectic form in $SO(3) \times so(3)^*$ obtained by left trivialization from the canonical 2-form on $\boldsymbol{T}^*SO(3)$. Now, we look for a symplectic diffeomorphism

$$\begin{aligned}&\Psi_1 : (\boldsymbol{T}^*\boldsymbol{Q}_\kappa, \omega_\kappa) \longrightarrow (M_1, \omega_1)\\ &[((B, R_0), R_1, R_2, (B\widehat{\Pi}, P_0), P_1, P_2)] \mapsto (B, B\widehat{\Pi}, r, \widetilde{r}, s, \widetilde{s}),\end{aligned}$$

where $[\cdot]$ denotes the class of an element of $\boldsymbol{T}^*\boldsymbol{Q}$, and we take $r = R_2 - R_1$, $s = R_0 - (m_1R_1 + m_2R_2)/\overline{M}$. Then, imposing that $\Psi_1^*\omega_1 = \omega_\kappa$, and taking into account the relation $\pi_\kappa^*\omega_\kappa = i_\kappa^*\omega$, it is not hard to show that we can take as momenta variables $\widetilde{r} = (m_1P_2 - m_2P_1)/\overline{M}$ and $\widetilde{s} = M_t(P_0 - m_0(P_1 + P_2))/\overline{M}$, where $M_t = m_0 + m_1 + m_2$. Thus, Ψ_1 becomes a symplectic diffemorphism. Then, we adopt as model for $(\boldsymbol{T}^*\boldsymbol{Q}_\kappa, \omega_\kappa)$ the symplectic manifold (M_1, ω_1).

In order to get a Poisson structure on M_1, we compute the Poisson bracket $\{\cdot,\cdot\}_I$ associated to the symplectic form ω_1. It is not hard to show that the Poisson brackect $\{\cdot,\cdot\}_I$ is given by

$$\begin{aligned}\{f,g\}_I(z) \;=\; &\Big(< D_B f, \frac{\partial g}{\partial B\widehat{\Pi}} > - < D_B g, \frac{\partial f}{\partial B\widehat{\Pi}} >\\ &+ \frac{\partial f}{\partial r}\frac{\partial g}{\partial \widetilde{r}} - \frac{\partial g}{\partial r}\frac{\partial f}{\partial \widetilde{r}} + \frac{\partial f}{\partial s}\frac{\partial g}{\partial \widetilde{s}} - \frac{\partial f}{\partial s}\frac{\partial g}{\partial \widetilde{s}}\Big)(z),\end{aligned} \tag{3}$$

for any $f, g \in C^\infty(M_1)$.

The Hamiltonian $\mathcal{H}$ is $\Phi_1^{T^*}$-invariant. Then, the projection π_κ induces a Hamiltonian function on $\boldsymbol{T}^*\boldsymbol{Q}_\kappa$, $\mathcal{H}_\kappa(\pi_\kappa(z)) = \mathcal{H}(i_\kappa(z))$, and by the diffeomorphism Ψ_1 we obtain the Hamiltonian

$$\mathcal{H}_I(z) = \mathcal{H}_\kappa(\Psi_1^{-1}(z)) \tag{4}$$

for $v \in M_1$. The reduced dynamics is $X_{\mathcal{H}_I}(z) = \{Id_{M_1}, \mathcal{H}_I\}_I(z)$. The reduced Hamiltonian (2) on M_1 is the function

$$\mathcal{H}_I(z) = \frac{|\widetilde{s}|^2}{2M} + \frac{|\widetilde{r}|^2}{2\widetilde{M}} + \frac{1}{2}\Pi \cdot \boldsymbol{I}^{-1}\Pi - L_r \cdot \boldsymbol{I}^{-1}\Pi + \mathcal{V}(\Pi, r, s, \widetilde{r}, \widetilde{s})$$

where

$$\mathcal{V}(z) = -Gm_1m_2\frac{1}{|r|} - Gm_1\int_{S_0}\frac{dm(Q)}{\left|BQ + s + \frac{m_2}{M}r\right|} - Gm_2\int_{S_0}\frac{dm(Q)}{\left|BQ + s - \frac{m_1}{M}r\right|}, \tag{5}$$

where $M = m_0\overline{M}/M_t$ and $\widetilde{M} = m_1m_2/\overline{M}$.

3.2 Reduction by the rotation group

Here we take into account the free and proper action of the group $SO(3)$ on M_1

$$\Phi_2 : SO(3) \times M_1 \longrightarrow M_1(A, (B, B\widehat{\Pi}, r, \widetilde{r}, s, \widetilde{s})) \equiv (AB, AB\widehat{\Pi}, Ar, A\widetilde{r}, As, A\widetilde{s}).$$

Then, Φ_2 induces a Poisson structure in the quotient manifold $M_1/SO(3)$ with Poisson bracket

$$\{f,g\}_{M_1/SO(3)} \circ \pi_2 = \{f \circ \pi_2, g \circ \pi_2\}_{M_1}, \tag{6}$$

where $\pi_2 : M_1 \to M_1/SO(3)$ is the canonical projection and, f and g are in the space $C^\infty(M_1/SO(3))$. Because $\mathcal{H}_I$ is Φ_2-invariant the dynamics generated by $\mathcal{H}_I$ induces the dynamics generated by the reduced Hamiltonian $\mathcal{H}_{M_1/SO(3)}(\pi_2(z)) = \mathcal{H}_I(z)$.

It is not hard to show that the map

$$\begin{aligned} \Psi_2 \quad : \quad & \left(M_1/SO(3), \{\cdot,\cdot\}_{M_1/SO(3)}\right) \longrightarrow \left(\mathbb{R}^{15}, \{\cdot,\cdot\}_{II}\right) \\ & [(B, B\widehat{\Pi}, r, \widetilde{r}, s, \widetilde{s})] \mapsto (\Pi, \lambda, P_\lambda, \mu, P_\mu) \\ \lambda \;=\; & B^t r, \quad P_\lambda = B^t \widetilde{r}, \quad \mu = B^t s, \quad P_\mu = B^t \widetilde{s} \end{aligned}$$

is a Poisson diffeomorphism where the Poisson bracket $\{\cdot,\cdot\}_{II}$ is defined as follows. Let us take for each $f, g \in C^\infty(\mathbb{R}^{15})$ the associated functions $\overline{f}, \overline{g} \in C^\infty(M_1)$ and $\widetilde{f}, \widetilde{g} \in C^\infty(M_1/SO(3))$ defined by $\overline{f}(\alpha) = f(\Psi_2([\alpha]))$ and $\widetilde{f}(\beta) = f(\Psi_2(\beta))$. Then, by (6)

$$\begin{aligned} \{f,g\}_{II}(\Pi, \lambda, P_\lambda, \mu, P_\mu) &= \{\widetilde{f}, \widetilde{g}\}_{M_1/SO(3)}[(\Pi, \lambda, P_\lambda, \mu, P_\mu)] \\ &= \{\overline{f}, \overline{g}\}_I(B, B\widehat{\Pi}, r, \widetilde{r}, s, \widetilde{s}). \end{aligned} \tag{7}$$

The Poisson bracket given by (7) can be written using the two-contravariant tensor field Λ defined by the matrix

$$\Lambda(z) = \begin{pmatrix} \widehat{\Pi} & \widehat{\lambda} & \widehat{P_\lambda} & \widehat{\mu} & \widehat{P_\mu} \\ \widehat{\lambda} & 0 & Id & 0 & 0 \\ \widehat{P_\lambda} & -Id & 0 & 0 & 0 \\ \widehat{\mu} & 0 & 0 & 0 & Id \\ \widehat{P_\mu} & 0 & 0 & -Id & 0 \end{pmatrix}.$$

Then, the Poisson bracket (3) reads $\{f,g\}_{II}(z) = \nabla_z f^t \; \Lambda(z) \; \nabla_z g$.

The twice reduced Hamiltonian $\mathcal{H}_{II}(z) = \mathcal{H}_{M_1/SO(3)}\left(\Psi_2^{-1}(z)\right)$ is the function

$$\mathcal{H}_{II}(z) = \frac{|P_\mu|^2}{2M} + \frac{|P_\lambda|^2}{2\widetilde{M}} + \frac{1}{2}\Pi \cdot \boldsymbol{I}^{-1}\Pi - L_r \cdot \boldsymbol{I}^{-1}\Pi + \mathcal{V}(z) \tag{8}$$

where

$$\mathcal{V}(z) = -Gm_1m_2\frac{1}{|\lambda|} - Gm_1 \int_{S_0} \frac{dm(Q)}{\left|Q + \mu + \frac{m_2}{M}\lambda\right|} - Gm_2 \int_{S_0} \frac{dm(Q)}{\left|BQ + \mu - \frac{m_1}{M}\lambda\right|}. \tag{9}$$

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On the generalized Lissajous transformation

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Abstract

We present a transformation that generalizes the classic Lissajous transformation and that is valid for linear combinations of anharmonic oscillators.

The Lissajous transformation was invented by Deprit (1991) for handling perturbed elliptic oscillators in the 1:1 resonance, that is to say, for dynamical systems represented by Hamiltonians of the type Hamiltonians of the type $\mathcal{H} = \mathcal{H}_0 + \mathcal{P}$, with the principal part consisting of two harmonic oscillators

$$\mathcal{H}_0 = \frac{1}{2}(X^2 + \omega_1^2 x^2) + \frac{1}{2}(Y^2 + \omega_2^2 y^2), \tag{1}$$

and $\mathcal{P}(x, y, X, Y)$ being a weak perturbation, and equal frequencies, $\omega_1 = \omega_2 = \omega$. These Hamiltonians are quite common in non-linear dynamics. Indeed, they appear very often in galactic dynamics, atomic physics, in optics, etc. (see e.g. [11] and references therein).

The main purpose of the Lissajous transformation is to avoid the small divisors when a General Perturbation is applied to these kind of Hamiltonians. Indeed, the classical Lissajous transformation $\lambda : (\ell, g, L, G; \omega) \longmapsto (x, y, X, Y) : D \longmapsto \mathbb{R}^4$ is defined in the domain $D = [0, 2\pi) \times [0, 2\pi) \times \{L > 0\} \times \{|G| \leq L\}$, by

$$\begin{aligned}
x &= \sqrt{\frac{L+G}{2\omega}}\cos(\ell + g) - \sqrt{\frac{L-G}{2\omega}}\cos(\ell - g),\\
y &= \sqrt{\frac{L+G}{2\omega}}\sin(\ell + g) + \sqrt{\frac{L-G}{2\omega}}\sin(\ell - g),\\
X &= -\sqrt{\frac{\omega(L+G)}{2}}\sin(\ell + g) + \sqrt{\frac{\omega(L-G)}{2}}\sin(\ell - g),\\
Y &= \sqrt{\frac{\omega(L+G)}{2}}\cos(\ell + g) + \sqrt{\frac{\omega(L-G)}{2}}\cos(\ell - g).
\end{aligned} \tag{2}$$

This transformation is everywhere regular in the domain of definition, and as it can be easily checked, the pullback of the Hamiltonian (1) by the transformation is

$$\lambda^{\#}\mathcal{H}_0 = \omega L, \tag{3}$$

hence, the Lissajous transformation λ changes the Lie derivative L_0 associated with $\mathcal{H}_0$ into the single partial derivative $L_0 = \omega\partial/\partial\ell$. Therefore, the kernel of L_0 is characterized as the real algebra of functions independent of the angular variable ℓ.

Normalization of a Hamiltonian, we recall, is a operation (usually a Lie transformation) that makes the transformed Hamiltonian belongs to the kernel of the Lie derivative associated to $\mathcal{H}_0$. In the present case, normalization renders the Hamiltonian independent of the angle ℓ, hence, its conjugate moment, L, is an integral in the normalized Hamiltonian.

But resonance 1:1, although important, it but a particular case of the problems appearing in non linear dynamics. Remember, for instance the famous resonance 2:1 of Fermi [5] in the CO_2 molecule. And even more, one meets resonances in three degrees of freedom in Galactic dynamics (e.g. [2]) or in n-degrees of freedom in atomic and molecular physics (e.g. [6]). Besides, Nature is not restricted to just addition of harmonic oscillators, but one finds general linear combinations in problems like the geostationary satellite [10] or cosmology [1], to name but a few. Thus, since the discovered of the Lissajous transformation, an extension of it has been sought.

One of the first ones in partially succeeding in this task was Ferrer [7], who obtained the *Nodal-Lissajous* transformation for handling the 1:1:1 resonance. Ferrer and coworkers made an application use of this transformation to Hamiltonians of the type Hénon-Heiles. However, this transformation is valid for this specific resonance, and inherent to its definition, axially symmetry is required. Something very similar happens with the transformation given by Jalali [9]. An extension for the p:q resonance has been obtained by Gárate [8].

From our part [4], we took the problem from scratch. For only one oscillator, the classical Poincaré variables (ϕ, Φ) given by $x = \sqrt{(2\Phi)/\omega}\sin\phi, \quad X = \sqrt{2\omega\,\Phi}\cos\phi$, fit our problem since this transformation converts the harmonic oscillator into $\mathcal{H}_0 = \omega\Phi$. However, this is not the case for two oscillators in resonance p:q; indeed, Poincaré's transformation $x_i = \sqrt{(2\Phi_i)/\omega_i}\sin\phi_i, \quad X_i = \sqrt{2\omega_i\,\Phi_i}\cos\phi_i$, converts the Hamiltonian into the simple expression

$$\mathcal{H}_0 = \omega_1\Phi_1 + \omega_2\Phi_2, \tag{4}$$

but still is not in the form given by Lissajous variables, and zero divisors appear when a perturbation method is applied. Since we want to have the Hamiltonian in the simple form (3), we just built a new canonical transformation. We define a set of Lissajous canonical variables $(\psi_1, \psi_2, \Psi_1, \Psi_2)$, such that (4) be transformed into $\mathcal{H}_0 = \omega\Psi_1$. This requirement gives us one of the equations of the transformation, namely, $\Psi_1 = p\,\Phi_1 + q\,\Phi_2$.

For the second moment, we may choose among several possibilities. The one we select is $\Psi_2 = p\,\Phi_1 - q\,\Phi_2$. (N.B. With this election, $\Psi_1 \geq 0$, and $|\Psi_2| \leq \Psi_1$).

In order to have a canonical transformation, the transformation among actions must be completed with the corresponding transformation among the coordinates. We want the latter to be a Mathieu transformation, i.e., to satisfy the differential identity

$$\Phi_1 d\phi_1 + \Phi_2 d\phi_2 = \Psi_1 d\psi_1 + \Psi_2 d\psi_2,$$

hence,

$$\phi_1 = p(\psi_1 + \psi_2), \qquad \phi_2 = q(\psi_1 - \psi_2).$$

The composition of Poincaré transformation and the new Lissajous transformation, yields the Lissajous transformation from Cartesian to the Lissajous variables

$$\begin{aligned} x &= \sqrt{\frac{\Psi_1 + \Psi_2}{\omega_1 p}} \sin p\,(\psi_1 + \psi_2), \qquad X = \sqrt{\frac{\omega_1\,(\Psi_1 + \Psi_2)}{p}} \cos p\,(\psi_1 + \psi_2), \\ y &= \sqrt{\frac{\Psi_1 - \Psi_2}{\omega_2 q}} \sin q\,(\psi_1 - \psi_2), \qquad Y = \sqrt{\frac{\omega_2\,(\Psi_1 - \Psi_2)}{q}} \cos q\,(\psi_1 - \psi_2). \end{aligned} \tag{5}$$

For the resonance 1:1, ($\omega_1 = \omega_2 = \omega$, $p = q = 1$) this transformation coincides, precisely, with the second Lissajous transformation defined by Deprit [3, p. 218].

Generalization to a Hamiltonian of the type

$$\mathcal{H}_0 = \frac{1}{2} \sum_{1 \le i \le n} (X_i^2 + \omega^2 p_i^2 x_i^2), \qquad \text{with} \quad p_i \in \mathbb{N}, \tag{6}$$

may be obtained by induction. The canonical transformation

$$\lambda : (\boldsymbol{\psi}, \boldsymbol{\Psi}) \longmapsto (\boldsymbol{x}, \boldsymbol{X}) : T^n \times (\mathbf{D} \subset \mathbb{R}^n) \longmapsto \mathbb{R}^{2n}$$

given by

$$\left.\begin{aligned} x_1 &= \frac{1}{\omega^{1/2} p_1} ((2-n)\Psi_1 + \Sigma)^{1/2} \sin p_1\,\sigma, \\ X_1 &= \omega^{1/2} ((2-n)\Psi_1 + \Sigma)^{1/2} \cos p_1\,\sigma, \end{aligned}\right\}$$

and for $1 < j \le n$, (7)

$$\left.\begin{aligned} x_j &= \frac{1}{\omega^{1/2} p_j} (\Psi_1 - \Psi_j)^{1/2} \sin p_j(-2\psi_j + \sigma), \\ X_j &= \omega^{1/2} (\Psi_1 - \Psi_j)^{1/2} \cos p_j(-2\psi_j + \sigma), \end{aligned}\right\}$$

reduces the Hamiltonian (6) to the function

$$\lambda^{\#}\mathcal{H}_0 = \omega\,\Psi_1.$$

In the preceding formulas, we introduced the shorthands $\Sigma = \sum_{1 \le i \le n} \Psi_i$ and $\sigma = \sum_{1 \le i \le n} \psi_i$, and the domain $\mathbf{D}$ is

$$\mathbf{D} = \Big\{ (\Psi_1, \ldots, \Psi_n) \in \mathbf{D}^n \;\Big|\; 0 \le |\Psi_j| \le \Psi_1,\ 1 < j \le n \Big\}.$$

Note that the generalized Lissajous transformation here presented, is not only valid for Hamiltonians of the type (6), but also for any linear combination of oscillators. Indeed, let us consider, for instance, a two degrees of freedom made of the subtraction of two harmonic oscillators

$$\mathcal{H}_0 = \frac{1}{2}(X^2 + \omega_1^2 x^2) - \frac{1}{2}(Y^2 + \omega_2^2 y^2).$$

Our extended Lissajous transformation converts it into

$$\mathcal{H}_0 = \omega\Psi_2.$$

In the general case, the obtaining of a similar transformation for an arbitrary linear combination of oscillators, raises no difficulty. The guidelines above shown provides the transformation.

In conclusion, we found a generalization of the Lissajous transformation valid for any resonance, for n degrees of freedom and for any linear combination of oscillators. Applications to particular cases are in progress.

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Perturbed Ion Traps: A Generalization of the Hénon-Heiles Problem

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1. Introduction: Trapping Ions

Ion traps are experimental devices developed to accumulate beams of charged particles or ions. The most popular schemes to trap and hold charges are based on combinations of static and radio-frequency electric fields (Paul trap) [5] or static electric and magnetic fields (Penning trap) [6]. In particular, the Penning trap provides three-dimensional trapping by means of a quadrupole electric field plus a static magnetic field [7]. The quadrupole electric potential is achieved by means of a set of three electrodes. One of the electrodes, called the ring, is similar to the form of the inner surface of a toroid. The other two electrodes are like hemispheres placed above and below the ring. In this arrangement, the quadrupole potential acts as a trap only in one dimension, along the axis between the hemispheres (we call this axis z); while the motion in the radial plane ($x-y$ plane) is unstable. The presence of the magnetic field along the z axis can provide the complete trapping. For a single ion of mass m and charge q, the quadrupole electric potential is given by

$$\Phi(x,y,z) = \frac{mw_z^2}{4q}(2z^2 - x^2 - y^2), \tag{1}$$

where $w_z^2 = \frac{4V_o q}{m(r_o^2+2z_o^2)}$, V_o is the potential of hemispheres with respect to the ring, and r_o and z_o are the physical dimensions of the trap. We assume the product $V_o q$ to be always positive. The magnetic field is $\vec{B} = B\widehat{z}$, which introduces the cyclotron frequency $w_c = \frac{qB}{m}$. The Hamiltonian for the charge q in these fields is

$$\mathcal{H} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}w_c(xp_y - yp_x) + \frac{m}{8}(w_c^2 - 2w_z^2)(x^2+y^2) + \frac{m}{2}w_z^2 z^2. \tag{2}$$

¿From Hamiltonian (2) we get the *trapping condition*; that is to say, the factor $w_c^2 - 2w_z^2$ must be positive in order to obtain stable motion in the radial plane. Hereafter, we assume this condition

and we define $w_1^2 = w_c^2 - 2w_z^2 > 0$. Under this consideration, the Hamiltonian (2) becomes

$$\mathcal{H} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}w_c(xp_y - yp_x) + \frac{m}{8}w_1^2(x^2 + y^2) + \frac{m}{2}w_z^2 z^2. \tag{3}$$

The dynamics arising from this system has been studied for several authors [8], and its main feature is the harmonicity of the motion.

2. Field Imperfections: The Generalized Hénon-Heiles Model

We can separate field imperfections in two groups: harmonic and anharmonic imperfections. From the mathematical point of view, the second group is more interesting because they lead to nonlinear motion. In particular, the electrostatic perturbations arise from imperfections in the physical design of the electrodes as well as for misalignments in the mounting. We model the electrostatic imperfections by means of the multipole expansion of the electric potential [4]. This expansion in cylindrical (ρ, θ, ϕ) coordinates takes the form

$$V(\rho, \theta, \phi) = \sum_{l,m} a_{l,m}\rho^l \mathcal{P}_l^m(\cos\theta)\cos(m\phi), \tag{4}$$

where $\mathcal{P}_l^m$ are the Legendre polynomials with $0 \leq m \leq l$. The first term $V_0 = a_{0,0}$ is the origin of the electrostatic potentials. The linear term $V_1 = a_{1,0}z + a_{1,1}x$ gives rise to a constant force. The first important term is the *quadrupole* term V_2 (in cartesian coordinates)

$$V_2 = \frac{1}{2}a_{2,0}(2z^2 - x^2 - y^2) - 3a_{2,1}xz + 3a_{2,2}(x^2 - y^2). \tag{5}$$

However, since all terms in (5) are quadratic, the motion remains harmonic. All higher orders in (4) will introduce nonlinearities in the motion.

After dropping the linear terms in (4), the Hamiltonian of the perturbed system is (in cylindrical coordinates)

$$\mathcal{H} = \frac{1}{2m}(p_\rho^2 + p_z^2) + \frac{p_\phi^2}{2m\rho^2} + \frac{1}{2}w_c p_\phi + \frac{m}{8}w_c^2\rho^2 + \sum_{l\geq 2,m} a_{l,m}\rho^l \mathcal{P}_l^m(\cos\theta)\cos(m\phi). \tag{6}$$

In general, Hamiltonian (6) represents a three-dimensional dynamical system. However, some restrictions allow to reduce the dimensionality of the problem. The first one is to suppose axial z symmetry. Under this reduction, the Hamiltonian (6) becomes the function

$$\mathcal{H} = \frac{1}{2m}(p_\rho^2 + p_z^2) + \frac{p_\phi^2}{2m\rho^2} + \frac{1}{2}w_c p_\phi + \frac{m}{8}w_c^2\rho^2 + \sum_{l\geq 2} a_l\rho^l \mathcal{P}_l^m(\cos\theta). \tag{7}$$

The second one is to assume axial z symmetry and symmetry with respect to the plane $z = 0$, and the Hamiltonian (7) takes the form

$$\mathcal{H} = \frac{1}{2m}(p_\rho^2 + p_z^2) + \frac{p_\phi^2}{2m\rho^2} + \frac{1}{2}w_c p_\phi + \frac{m}{8}w_c^2\rho^2 + \sum_{l\geq 2,even} a_l\rho^l \mathcal{P}_l^m(\cos\theta). \tag{8}$$

In both cases, the Hamiltonians (7) and (8) represent two-dimensional dynamical systems, because the z component of the angular momentum p_ϕ is a new constant of the motion.

At this point, we take the first two terms in the Hamiltonian (8). These terms are, respectively, the quadrupole and the sextupole terms, and the corresponding Hamiltonian is (assuming trapping condition)

$$\mathcal{H} = \frac{1}{2m}(p_\rho^2 + p_z^2) + \frac{p_\phi^2}{2m\rho^2} + \frac{1}{2}w_c p_\phi + \frac{m}{8}w_1^2\rho^2 + \frac{m}{2}w_z^2 z^2 + a_6(2z^3 - 3\rho^2 z). \tag{9}$$

In the above Hamiltonian it is possible to eliminate the linear term in w_c (*paramagnetic*) going to a reference frame rotating with the frequency w_c. After this transformation, the Hamiltonian (9) becomes

$$\mathcal{H}_{rot} = \frac{1}{2m}(p_\rho^2 + p_z^2) + \frac{p_\phi^2}{2m\rho^2} + \frac{m}{8}w_1^2\rho^2 + \frac{m}{2}w_z^2 z^2 + a_6(2z^3 - 3\rho^2 z). \tag{10}$$

Hence, the dynamics depends on the four parameters p_ϕ, w_1, w_z and a_6 as well as on the energy $\mathcal{H}_{rot}$. However, we can reduce the number of the parameters by introducing the unit of length $\lambda = \frac{mw_1^2}{a_6}$ and the dimensionless time $\tau = w_1 t$. After this transformation, equation (10) becomes the following dimensionless Hamiltonian

$$\frac{\mathcal{H}_{rot}}{m\lambda^2 w_1^2} = \mathcal{H}' = \frac{1}{2}(p_\rho^2 + p_z^2) + \frac{p_\phi^2}{2\rho^2} + \frac{1}{8}\rho^2 + \frac{w^2}{2}z^2 + (2z^3 - 3\rho^2 z), \tag{11}$$

where $w = w_z/w_1$, and the dynamics depends only on the two parameters p_ϕ, w and the energy $\mathcal{H}'$.

We call the system defined by the Hamiltonian (11) the *Generalized Hénon-Heiles Problem* (*GHHP*), because for the special case $p_\phi = 0$ and $w = 1$ (1:1 resonance) we get the well-known Hénon-Heiles problem [2].

The *GHHP* opens the possibility of study the classical mechanics of a three-dimensional system by means of tools that usually are applied to two-dimensional systems. The presence of two parameters in the Hamiltonian, besides the energy, will allow us to uncover how the dynamics evolves as the parameters change. In this sense, for the special case $p_\phi = 0$ (the *Polar Case*), the evolution of the Poincaré surfaces of section, as a function of w, reveals the appearance of bifurcations that dramatically change the phase portrait. This feature indicates a rich dynamics depending on the parameters of the problem, specially on the frequency, which can be outlayed through a normalized system. However, the presence of resonances yields an extra difficulty in the normalization process and, hence, in the study of the problem. In this way, appropriate coordinates, like the extended Lissajouss variables [1], are needed in order to carry out the normalization.

The study of the mechanism of escape in the *GHHP* also appears as an interesting question because this process could be closely related with the phase space structure. Hence, a changing phase space will affect the escape process; that is to say, the escape probability and the number and morphology of the accessible channels of escape. This study is currently being in progress in our group for the Polar Case.

Finally, we can consider the *GHHP* as an scattering system. This subject has recently attracted much attention because dynamical instabilities and chaos have been discovered even in the simplest scattering systems [9].

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Algebra Computacional y Mecánica Celeste

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1. Introducción

A nadie se le oculta la importancia adquirida, en la mayor parte de las disciplinas científicas, por las técnicas de manipulación simbólica y algebraica por ordenador. Por un lado, dichas técnicas ayudan en el quehacer diario del científico proporcionándole un medio flexible y cómodo de trabajo que mejora notablemente la productividad de su investigación. Por otro, problemas aparentemente estancados por la complejidad de sus desarrollos analíticos han podido ser reconsiderados.

La Mecánica Celeste no solo no ha sido ajena a este movimiento, sino que, como en muchas otros momentos de su historia, ha contribuido al desarrollo de estas nuevas técnicas proporcionando complicados problemas a los que aplicarlas y forzando al desarrollo de nuevos métodos de resolución.

La dificultad de la implementación en un ordenador de los métodos manuales tradicionales ha multiplicado la investigación de nuevas técnicas simbólicas y algebraicas adaptadas al tratamiento computacional, dando lugar a una nueva disciplina científica llamada Algebra Computacional que ha venido de la mano de numerosos sistemas de cálculo comerciales o no. No es nuestro propósito hablar aquí de los fundamentos teóricos del Algebra Computacional, sino revisar la interacción que ha existido durante los últimos años entre el Algebra Computacional y la Mecánica Celeste.

Ya en el año 1959, Herget y Musen [12] publican en el *Astronomical Journal* uno de los primeros ejemplos de uso del ordenador en un cálculo no numérico. A partir de ese momento y coincidiendo con la aparición de los lenguajes de ordenador FORTRAN y LISP comienzan a aparecer los primeros sistemas de cálculo algebraico y simbólico SAC (Symbolic and Algebraic Computations) que poco a poco van dividiéndose en dos tipos de programas, por una parte los sistemas de carácter general que pretenden ser útiles para una amplia gama de usuarios y por otra los sistemas especializados en la resolución de algún problema concreto que aprovechan las propiedades particulares del problema para lograr más eficiencia en su resolución.

Aunque desde el comienzo la Mecánica Celeste usó alguno de los sistemas generales como FORMAC y CAMAL, algunos autores son conscientes de la necesidad de usar herramientas más especializadas. De hecho, Danby, Deprit y Rom [6] identifican en 1965 el objeto matemático

clave en el desarrollo analítico de la mayor parte de los problemas, no solo de la Mecánica Celeste y Astrodinámica, sino de la Mecánica No–Lineal en general, llamándole Serie de Poisson, que puede describirse como una serie de Fourier multivariada cuyos coeficientes son series de Laurent multivariadas. Las series de Poisson pueden representarse en la forma siguiente

$$\sum_{i_0,\ldots,i_{n-1},j_0,\ldots,j_{m-1}} C_{i_0,\ldots,i_{n-1}}^{j_0,\ldots,j_{m-1}} x_0^{i_0}\ldots x_{n-1}^{i_{n-1}} \begin{pmatrix} \text{sen} \\ \cos \end{pmatrix} (j_0 y_0 + \ldots + j_{m-1} y_{m-1}).$$

A los programas especializados en manipular estos objetos se les ha llamado Procesadores de Series de Poisson (PSP) y constituyen la herramienta más usada para el tratamiento analítico computacional en los grandes problemas de la Mecánica Celeste.

A partir de la generalización del uso de los ordenadores personales y de la aparición de dos de los SAC generales más usados en la actualidad, *Mathematica* y MAPLE, se crea un estado de opinión que hace pensar en la inutilidad de sistemas especializados como son los PSP. Sin embargo, aunque los SAC generales siguen siendo muy usados y producen importantes resultados en las primeras etapas de desarrollo de un nuevo problema teórico, todavía no han podido sustituir en potencia de cálculo a los sistemas especializados en las fases intermedias y finales de resolución del problema, cuando el volumen de cálculos se hace crítico.

En este punto final de desarrollo no solo es necesario un buen diseño de la herramienta de cálculo, el PSP, sino que el problema debe ser enfocado de manera adecuada para su tratamiento simbólico. Una buena estrategia en este sentido puede marcar diferencias, no solo en tiempos de cálculo, sino también en ocasiones entre poder y no poder resolver el problema. De hecho, métodos como el de integración de Lie–Deprit han sido creados para poder integrar automáticamente problemas que no podrían integrarse de otro modo usando un ordenador.

En este trabajo analizaremos la historia y desarrollo de las herramientas de cálculo simbólico tanto generales como especializadas. También desarrollaremos las principales ideas a considerar en el diseño de herramientas especializadas en el tratamiento computacional de objetos matemáticos. Aplicaremos dichas ideas a las series de Poisson aunque éstas pueden servir a cualquier otro tipo de objeto. Finalmente analizaremos como el enfoque y planteamiento del problema pueden ayudarnos en su tratamiento simbólico.

2. Breve historia de los sistemas de cálculo simbólico y algebraico (SAC)

Con objeto de situar en el tiempo los primeros ensayos de sistemas de cálculo algebraico y simbólico es preciso recordar que la aparición del lenguaje de programación FORTRAN tiene lugar en el año 1958, mientras que ALGOL y LISP aparecen en el año 1960.

Los primeros sistemas de este tipo responden a la búsqueda de soluciones para el tratamiento de problemas muy determinados. Así en 1961, J. Slage, en el M.I.T., desarrolla en lenguaje LISP el sistema SAINT (Symbolic Automatic INTegration) para la obtención simbólica de integrales. Entre ese año y el año 1966, G. Collins desarrolla en IBM y la universidad de Wisconsin un programa llamado PM para la manipulación de polinomios.

Merece la pena destacar entre estos primeros programas el sistema FORMAC, escrito entre los años 1962 y 1964 en FORTRAN y PL/1 por J Sammet y R. Tobbey, para el manejo de funciones elementales, incluyendo polinomios y funciones racionales y mencionado y usado en las primeras aplicaciones del cálculo simbólico a la Mecánica Celeste, así como ALPAK, una colección de rutinas, también para manejar funciones polinómicas y racionales, escritas en 1964 en lenguaje ensamblador y que podían ser llamadas desde un programa FORTRAN.

Fruto de estos primeros esfuerzos y ante el convencimiento de la utilidad y necesidad de estos programas el año 1966 se celebra en Washington D.C. el 1^{st} Symposium on Symbolic and Algebraic Manipulation que pone las bases para el futuro desarrollo del Algebra Computacional como disciplina científica y para el desarrollo de los sistemas de cálculo algebraico y simbólico SAC.

Entre el final de los años sesenta y toda la década de los setenta puede decirse que los SAC alcanzan su mayoría de edad con la aparición de los primeros sistemas de carácter general. Entre los primeros de estos sistemas podemos citar ALTRAN como sucesor de ALPAK, así como SAC, sucesor de PM y SAC/ALDES, sucesor del anterior SAC. Este último está escrito en un lenguaje especial llamado ALDES (ALgebraic DEScription) con un traductor para convertir los resultados a FORTRAN. Otro sistema de gran importancia durante este periodo es CAMAL (Cambridge Algebra System) desarrollado por Barton, Bourne y Fitchen la universidad de Cambridge y muy usado en Mecánica Celeste y teoría de la relatividad. CAMAL fue escrito en lenguaje BCPL que es un antecesor del actual C.

Más importantes que los anteriores por su incidencia en el desarrollo posterior de este tipo de programas son los cuatro sistemas que se mencionan a continuación.

Por un lado REDUCE que es un sistema escrito en LISP cuya primera versión fue desarrollada por T. Hearn en 1968 en la Universidad de Stanford. Este sistema y sus posteriores revisiones es el más usado en la década de los setenta especialmente por su portabilidad.

El sistema que más impulsa el desarrollo del algebra computacional por su potencia y posibilidades en el desarrollo de nuevos algoritmos es MACSYMA escrito en 1971 por J. Moses, también en lenguaje LISP.

Otro sistema desarrollado en aquella época es SCRATCHPAD, que a pesar de sus grandes posibilidades no alcanza un desarrollo como los anteriores al funcionar únicamente en grandes ordenadores IBM.

El cuarto sistema general a destacar durante esta época es MuMATH escrito por D. Stoutemeyer y A. Rich en lenguaje MuSIMP que es un subconjunto de LISP para ordenadores personales. A pesar de las limitaciones de este sistema frente a sus grandes hermanos como MACSYMA, tiene la ventaja de poder utilizarse en pequeños ordenadores compatibles lo que lo hace accesible a un mayor número de usuarios potenciales, no necesariamente especialistas.

Por otro lado junto con estas sistemas generales se comprueba la utilidad de sistemas especializados en la resolución de problemas concretos que siguen desarrollándose en determinadas áreas como SHEEP para manipulación de tensores o SHOONSHIP usado en física de altas energías.

Con la aparición en los años ochenta de lenguajes como C, potentes y flexibles y muy adaptados para el cálculo simbólico, junto con el importante desarrollo de la informática personal que generaliza en todos los ámbitos el uso de ordenadores, el cálculo simbólico alcanza su madurez bajo dos ideas que caracterizan los sistemas de ésta época: portabilidad y eficiencia.

Además del desarrollo continuado de REDUCE y MACSYMA, que no ha cesado, aparecen otros cuatro importantes sistemas generales: MAPLE, *Mathematica*, DERIVE y AXIOM. MAPLE fue desarrollado por G. Gonnes y K. Geddes en la Universidad de Waterloo. *Mathematica* constituye la versión comercial de SMP (Symbolic Manipulator Program) desarrollado por S. Wolfram en la Universidad de Caltech. DERIVE y AXIOM son los sucesores respectivos de MuMATH y SCRATCHPAD. Estos sistemas, junto con los dos anteriores y el reciente MUPAD, están cada vez más extendidos y son usados en casi todos los ámbitos que requieren el uso de herramientas matemáticas desde la educación a la investigación pasando por disciplinas como las Matemáticas, la Física, la Ingeniería, etc.

La elección de uno de estos sistemas depende de una serie de condiciones que van desde el tipo de problema a resolver hasta la disponibilidad de medios materiales del usuario. Sin embargo, a pesar de una apariencia en ocasiones similar y de resolver problemas parecidos el enfoque interno para la resolución de los problemas es totalmente distinto y pueden encontrarse cuatro tendencias que probablemente marcarán el futuro de los SAC.

Por un lado REDUCE y MACSYMA están basados en el uso de LISP y por tanto en las ideas del tratamiento de listas para la resolución de los problemas. MAPLE se encuentra más próximo al lenguaje procedural de los lenguajes de programación clásicos dotándonos de gran número de funciones para la resolución de un amplio número de problemas simbólicos, numéricos, gráficos, etc. *Mathematica* se acerca más al lenguaje simbólico del álgebra al estar basado esencialmente en el reconocimiento de patrones simbólicos y la aplicación de reglas asociadas a cada patrón. AXIOM y MUPAD, por su parte, toman las ideas de la moderna programación orientada a objetos y basan su estrategia en el tratamiento de las estructuras algebraicas de los objetos matemáticos.

3. Evolución de los procesadores de series de Poisson (PSP)

La aplicación del Algebra Computacional a la Mecánica Celeste ha estado asociada al uso de sistemas especializados en el tratamiento de las llamadas series de Poisson desde el año 1965 cuando Deprit, Rom y Danby [6] definen dicho objeto de manera formal y lo identifican como el objeto matemático más general, con una estructura algebraica bien definida, que aparece en los problemas de la Mecánica Celeste, Astrodinámica y Dinámica No–Lineal. En el mismo artículo describen el primer procesador de series de Poisson , llamado MAO (Mechanized Algebraic Operations) que posteriormente es mejorado y extendido por Rom [19, 20]. MAO fue escrito parte en FORTRAN y parte en Assembler (lenguaje de la máquina) y puede tratar hasta diez variables polinómicas y seis angulares simultáneamente. La estructura básica para el almacenamiento de las series es la *Pila*, aunque ya destacan la necesidad de uso de estructuras como listas o árboles para una mejor implementación, de hecho emulan dicha estructura mediante el uso de una pila

de datos.

Dentro de lo que podemos llamar la primera generación de procesadores de series de Poisson debemos también mencionar el de Broucke y Garthwaite [4] que está también escrito en FORTRAN y Assembler y utiliza la estructura de Pila. Existen dos versiones de este procesador que pueden manejar tres o seis variables polinómicas y otras tantas angulares.

Una verdadera estructura de *lista* para el tratamiento de las series es usada en TRIGMAN, escrito por Jeffreys [13], y que junto con esta importante novedad añade un programa escrito enteramente en FORTRAN con lo que renuncia a parte de la velocidad proporcionada por el uso del lenguaje ensamblador en aras de una mayor compatibilidad del programa. Esta razón condujo a un amplio uso de este sistema en Mecánica Celeste. Posteriores mejoras conducen a la construcción de un preprocesador basado en un lenguaje de cadenas de texto, llamado SNOBOL, que permite el uso de un lenguaje propio en el que escribir de forma más sencilla los programas de tratamiento de las series de Poisson, para posteriormente ser traducidos a FORTRAN por el preprocesador [14]. La última mejora de este sistema es llevada a cabo por Rickfles, Jeffreys y Broucke [18] que construyen TRIGPROG que añade, en apariencia, nuevos tipos de datos como el tipo `SERIE`.

En todos estos programas hay que destacar la dificultad de la implementación en lenguajes no preparados para ello, como FORTRAN, de ideas, como el uso de estructuras y tipos de datos especiales, que posteriormente constituirán el núcleo de los modernos PSP.

Dentro de esta primera generación de PSP debemos destacar el procesador escrito por Dasenbrock [7, 8] en 1973, aunque documentado en 1982. Dicho procesador también escrito en FORTRAN es el más simple, potente y mejor documentado de la época y tiene unos límites, 5000 términos de series, 100 series, 24 variables polinómicas y 8 variables angulares, que permiten su uso en problemas con grandes requerimientos.

Alrededor del año 1988 y coincidiendo con el 109th IAU Colloquium celebrado en Gaithersburg (USA), [22], se van haciendo públicos lo que podemos llamar la segunda generación de procesadores de series de Poisson. Aunque algunos de ellos siguen escritos en FORTRAN, como el programa MSTN de J.C. Agnese o MS de Henrard y Moons, aparecen otros basados bien en más modernos lenguajes y compiladores o bien construidos aprovechando las particularidades de algún tipo de ordenador.

Como secuelas de MAO aparecen MAO II (Deprit y Miller, [17]) escrito en LISP y preparado para su uso en una Lisp–Machine y MAO!! (Deprit y Deprit, [10]) versión de MAO para la Connection–Machine, ordenador paralelo masivo. Por otro lado, Richardson D.L. prepara PARSEC [22], una aplicación para el tratamiento de series de Poisson en ordenadores personales IBM. El lenguaje C, mucho mejor adaptado que FORTRAN para este tipo de problemas, es utilizado en TRIP, de J. Laskar [16] y finalmente por nuestro procesador PSPC [1], desarrollado en las Universidades de Zaragoza y La Rioja.

A pesar de la importancia que para nuestra disciplina científica posee el uso de este tipo de programas, su desarrollo y uso se encuentra restringido al entorno de cada grupo investigador. No existe una herramienta común, salvo los SAC de carácter general, que permita un intercambio

cómodo de resultados y evite el esfuerzo que para un grupo aislado supone la creación de sus propias herramientas. A cambio, la ventaja de esta forma de trabajar es la gran adaptación de la herramienta a los problemas tratados por cada grupo particular.

4. Manipulación de objetos matemáticos en un ordenador

Cuando se aborda la construcción de un programa de ordenador para el tratamiento de un objeto matemático deben tenerse en cuenta tres aspectos del mismo que condicionan la implementación de dicho objeto: su estructura algebraica, su representación simbólica y su representación computacional. Para una mejor comprensión de estas tres facetas del objeto a manipular estudiaremos tres de estos objetos: los números, los polinomios y por último las series de Poisson.

De manera tradicional se piensa en un ordenador como una máquina capaz de manejar números de manera rápida y eficiente, sin embargo la implementación de los números como objetos matemáticos es un claro ejemplo de mala adaptación entre las matemáticas y la tecnología.

En efecto, pensemos en el tratamiento de los números realizado en un lenguaje moderno como C. Este tratamiento, similar al realizado por otros lenguajes está basado en la implementación de dos tipos de datos: `int` para el tratamiento de enteros y `double` para el tratamiento de números reales. El resto de tipos como `short`, `long`, `float` o `long double` no son sino distintas versiones del mismo tipo de datos. La primera consecuencia, derivada de limitaciones tecnológicas, es la necesidad de limitar el tamaño de los números, lo que obliga a trabajar con un subconjunto de Z y R que ni siquiera es cerrado respecto a las operaciones habituales, por lo que no forma una subestructura de la estructura algebraica original de los números. De hecho, se ha demostrado en [15] que la suma datos del tipo `double` no es conmutativa.

Para tratar esta limitación en la implementación de los números ha sido necesario crear toda un teoría del tratamiento de errores y aproximaciones con la que nos hemos visto obligados a convivir para el tratamiento numérico de los problemas. Actualmente también se trabaja en otra dirección con la implementación de tipos de datos numéricos extendidos como enteros, racionales y reales de precisión múltiple, no limitados por el tamaño de una *palabra* del ordenador, sino por la capacidad de su memoria. Este tratamiento, que no suele ser considerado en problemas de tipo numérico por su gran coste computacional, es fundamental en los modernos SAC o PSP pues evita, en lo posible, los errores derivados de una mala implementación de la estructura algebraica en la definición de objetos matemáticos más complejos.

Los polinomios nos dan una idea general más clara de la aproximación computacional a los objetos matemáticos. En primer lugar es precisa una correcta identificación del objeto y de sus propiedades matemáticas. Pensemos, por ejemplo, en el tratamiento de los elementos del conjunto

$$\mathrm{R}[x] = \{\, p(x) = \sum_{j=0}^{k} a_j x^j; \quad x \in \mathrm{R},\, a_j \in \mathrm{R} \}$$

que como sabemos tiene, respecto a la suma, el producto y el producto por un escalar, una estructura de algebra conmutativa con elemento unidad. Un programa que maneje dichos ele-

mentos debe poder realizar las tres operaciones y verificar todas las propiedades asociadas a su estructura de álgebra.

El siguiente aspecto a considerar es la representación simbólica de los polinomios que permitirá el intercambio de información entre el usuario de dicho programa y el ordenador. Pensemos que un polinomio como $9x^2-4$ puede ser representado bien con las potencias en orden decreciente, bien en orden creciente como $-4+9x^2$, o bien en forma de factores como $9(x-2/3)(x+2/3)$. Cualquiera de las tres formas podría servir como representación simbólica diferente del mismo polinomio.

Finalmente la representación computacional nos da la forma en que el ordenador almacena la información básica que identifica el polinomio. En este caso la información básica vendrá dada por un símbolo, x que representa la variable y por tres números reales que pueden ser bien los coeficientes si se elige una de las dos primeras representaciones simbólicas o bien el coeficiente del término de mayor exponente y las dos raíces si se elige la representación factorizada. En cualquier caso estos tres números deben ser almacenados en forma ordenada por medio de una estructura de lista $\{9,0,-4\}, \{-4,0,9\}, \{9,2/3,-2/3\}$. Las modernas técnicas de programación para el tratamiento de estructuras de datos junto con las facilidades de los lenguajes moderno para definir nuevos tipos de datos nos ayudan en la elección adecuada de la estructura y el tratamiento de los datos básicos.

Un concepto íntimamente ligado con la representación simbólica de los objetos matemáticos es el de *simplificación*. Dicho concepto es de gran importancia cuando en el tratamiento de objetos matemáticos sin una clara representación simbólica, sin embargo, debido a su ambigüedad debe ser desterrado del cálculo simbólico, [11, 8], y sustituido por el concepto de *función canónica* que construye el único representante de la clase de equivalencia formada por objetos matemáticos idénticos con distinta representación simbólica. Un ejemplo de una función de este tipo es la función **Expand** de *Mathematica* que aplicada a un polinomio lo convierte en el polinomio expresado por la segunda de las representaciones simbólicas anteriores.

5. Tratamiento computacional de las series de Poisson

Antes de considerar el tratamiento de las series de Poisson es necesaria una rigurosa definición de las mismas así como un estudio de sus propiedades algebraicas.

Llamaremos **serie de Poisson** de n variables polinómicas $\boldsymbol{x} = (x_1, \ldots, x_n)$ y m angulares $\boldsymbol{y} = (y_1, \ldots, y_m)$, a una aplicación

$$(\boldsymbol{x}, \boldsymbol{y}) \to \mathcal{S}(\boldsymbol{x}, \boldsymbol{y}) : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R},$$

definida por

$$\mathcal{S}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{i}\in\mathcal{I}, \boldsymbol{j}\in\mathcal{J}} C_{\boldsymbol{i}}^{\boldsymbol{j}} P_{\boldsymbol{i}} T_{\boldsymbol{j}}, \quad C_{\boldsymbol{i}}^{\boldsymbol{j}} \in \mathbf{R},$$

donde $\boldsymbol{i} = (i_0, \ldots, i_{n-1})$ y $\boldsymbol{j} = (j_0, \ldots, j_{m-1})$ son elementos de $\mathbf{Z}^n$ y $\mathbf{Z}^m$ respectivamente, y además

$$\begin{aligned} P_i &= x_0^{i_0} \dots x_{n-1}^{i_{n-1}} \\ T_j &= \begin{pmatrix} \text{sen} \\ \cos \end{pmatrix} (j_0 y_0 + \dots + j_{m-1} y_{m-1}). \end{aligned}$$

El cardinal de $\mathcal{I}$ y $\mathcal{J}$ puede ser finito o infinito, aunque en el tratamiento computacional únicamente podrán ser tratadas series con un número finito de términos.

Para estudiar la estructura algebraica de este objeto matemático es preciso considerar antes el tipo de operaciones que queremos realizar con estas series. Par ello pensemos en la ecuación básica del método de Lie–Deprit, el llamado *triángulo de Lie*

$$\mathcal{H}_{p,q} = \mathcal{H}_{p+1,q-1} + \sum_{j=0}^{p} \binom{p}{j} (\mathcal{H}_{p-j,q-1}\,;\, \mathcal{W}_{j+1})$$

que permite la construcción del hamiltoniano transformado por una transformación de Lie.

Esta ecuación nos indica las operaciones básicas a realizar por el procesador de series de Poisson: suma, producto, producto por un escalar y paréntesis de Poisson. Con respecto a las tres primeras se comprueba sin dificultad la estructura de álgebra conmutativa con elemento unidad de las series de Poisson. La necesidad de calcular paréntesis de Poisson amplía los requerimientos de nuestro programa. Esto llevó a la definición de un concepto más amplio que el de álgebra, llamada *álgebra de Poisson*, como un álgebra tal que dados dos elementos cualesquiera, su paréntesis de Poisson pertenece también al álgebra. Teniendo en cuenta las propiedades de las series de Poisson y la regla de la cadena que permite poner

$$\frac{dS(\boldsymbol{x},\boldsymbol{y})}{dt} = \frac{\partial S}{\partial \boldsymbol{x}} \cdot \frac{d\boldsymbol{x}}{dt} + \frac{\partial S}{\partial \boldsymbol{y}} \cdot \frac{d\boldsymbol{y}}{dt}$$

llegamos a la conclusión de que las series de Poisson forman una álgebra de Poisson cuando las derivadas $d\boldsymbol{x}/dt, d\boldsymbol{y}/dt$ son también series de Poisson.

La definición de las series de Poisson está basada en cada término como unidad básica, siendo una serie un conjunto de términos que se suman. Esto permite reconocer el término como unidad básica de la serie e identificar la información básica del mismo a partir de los siguientes elementos:

- un coeficiente C_i^j que puede ser racional o real.
- n enteros $(i_0, \dots, i_{n-1})$ que representan los exponentes de las variables polinómicas.
- m enteros $(j_0, \dots, j_{m-1})$ que representan los coeficientes de las variables así como la información que indica si el término es sin o cos.

La representación simbólica clásica

$$\sum_{i \in \mathcal{I}, j \in \mathcal{J}} C_i^j P_i T_j$$

está basada en una ordenación de los términos en forma lexicográfica, esto es, similar a la ordenación alfabética de las palabras donde cada término se identifica con una palabra formada

por $n+m$ letras que son respectivamente los n exponentes $(i_0,\ldots,i_{n-1})$ y los m coeficientes $(j_0,\ldots,j_{m-1})$. Esta es la ordenación obtenida en *Mathematica* al aplicar la función canónica `Expand[TrigReduce[_]]`.

Por su utilidad desde el punto de vista práctico debemos mencionar otras dos representaciones simbólicas de una serie de Poisson. Por un lado la resultante sacar factor común los términos trigonométricos resultando

$$S(\boldsymbol{x},\boldsymbol{y}) = \sum_{j=1}^{\mathrm{card}(\mathcal{J})} \left(\sum_{i=1}^{\mathrm{card}(\mathcal{I})} C_i^j P_i \right) T_j.$$

así como la representación matricial

$$S(\boldsymbol{x},\boldsymbol{y}) = P \cdot C \cdot T.$$

donde P y T son los vectores formados por los términos polinómicos y trigonométricos, respectivamente, ordenados lexicográficamente y C es la matriz de coeficientes.

Para estudiar la representación de las series debe considerarse en primer lugar la información básica que define cada término de la serie y decidir el tipo de dato que usaremos para almacenar esta información básica. En cuanto a los coeficientes una representación racional es la más adecuada, aunque esto requiere la creación de este nuevo tipo de dato no considerado en los actuales compiladores. Para almacenar los exponentes y coeficientes de las variables polinómicas y angulares debe tenerse en cuenta que habitualmente éstos tienen un valor pequeño, de hecho unos límites entre -128 y 127 se consideran suficientes. Esto permite un gran ahorro de memoria al bastar una variable tipo `char` de 8 bits en lugar de un entero `int` de 32 bits para almacenar cada uno de estos elementos.

Una vez establecido el método de almacenamiento de cada término hay que definir la estructura de datos que relacione todos los términos entre si para almacenar la serie. Asociada con la representación simbólica clásica se encuentra la estructura de lista mostrada en la figura 1 siguiente

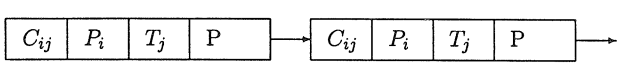

Figura 1.—Estructura de lista unidimensional clásica

Hasta donde nosotros conocemos todos los PSP actuales, excepto PSPC, han sido implementados utilizando la estructura de lista unidimensional anterior, sin embargo, Dasenbrock ya señala en [8] las ventajas de una representación computacional con una estructura de lista bidimensional como la mostrada en la figura 2. Dicha representación, aunque es más complicada desde el punto de vista de la programación resulta mucho más eficiente en el tratamiento de las series. Hay que hacer notar además que la matriz C es almacenada en PSPC forma dispersa, esto es sin almacenar los ceros.

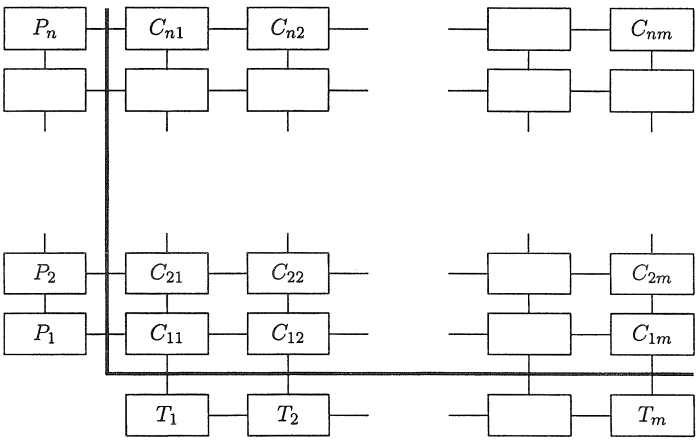

Figura 2.—Estructura de lista bidimensional

6. Propiedades simbólicas de los problemas

Hasta aquí se ha destacado la importancia de la herramienta en el tratamiento simbólico de los problemas de Mecánica Celeste, sin embargo para que este tratamiento sea realmente eficiente debe ser realizado un esfuerzo en el acondicionamiento del problema a su tratamiento simbólico. La eliminación de la paralaje en el problema del satélite artificial constituye un claro ejemplo de adaptación entre técnicas matemáticas y tratamiento informático del problema.

Consideraremos el movimiento del satélite artificial sometido a la atracción de los términos zonales del potencial gravitacional terrestre. La derivada de Lie en variables polares–nodales para este problema puede escribirse como

$$\mathcal{L}_0 = R\frac{\partial}{\partial r} - \left(\frac{\mu}{r^2} - \frac{\Theta}{r^3}\right)\frac{\partial}{\partial R} + \frac{\Theta}{r^2}\frac{\partial}{\partial \theta}$$

y es mucho más complicada que cuando se expresa en variables de Delaunay, sobre todo si se tiene en cuenta que debe ser usada para calcular, de acuerde con el método de Lie–Deprit, una integral primera de la ecuación en derivadas parciales

$$\mathcal{L}_0(\mathcal{W}_n) = \tilde{\mathcal{H}}_{n,0} - \mathcal{H}_{0,n}. \tag{1}$$

Pensando en la estructura algebraica que soporta la resolución del triángulo de Lie, esto es el álgebra de Poisson, se observa que el conjunto de funciones de la forma

$$\mathcal{F} = \{F = \sum_{j\geq 0}(C_j \cos j\theta + S_j \sin j\theta)\,, \quad C_j, S_j \in ker(\mathcal{L}_0)\} \tag{2}$$

constituye un álgebra de Poisson que contiene el hamiltoniano del problema zonal del satélite artificial. Para ello, además de la verificación de las propiedades correspondientes, basta comprobar que definiendo las funciones de estado como $C = e\cos g$, $S = e\sin g$, donde $C, S \in ker(\mathcal{L}_0)$,

podremos poner $1/r$ y R como funciones de $\mathcal{F}$ en la forma

$$\frac{1}{r} = \frac{1}{p} + \frac{C}{p}\cos\theta + \frac{S}{p}\sin\theta, \quad R = \frac{C\Theta}{p}\sin\theta - \frac{S\Theta}{p}\cos\theta. \tag{3}$$

Las relaciones anteriores, sirven para obtener otra representación simbólica de $\mathcal{F}$

$$\mathcal{F} = \{\sum_{i\geq 0}\sum_{j\geq 0} X_{ij}\frac{R^j}{r^i}, \quad X_{ij} \in ker(\mathcal{L}_0)\}$$

El método de eliminación de la paralaje está basado en la siguiente relación

$$\mathcal{L}_0\left[\sum_{j\geq 0}\frac{1}{j}\left(C_j \sin j\theta - S_j \cos j\theta\right)\right] + \frac{\Theta}{r^2}C_0 = \frac{\Theta}{r^2}F,$$

válida para cualquier función $F \in \mathcal{F}$ expresada simbólicamente en la forma dada por (2). La ecuación anterior puede identificarse con (1) llamando

$$F = \frac{r^2}{\Theta}\tilde{\mathcal{H}}_{n,0}, \quad \mathcal{H}_{0,n} = \frac{\Theta}{r^2}C_0, \quad \mathcal{W}_n = \sum_{j\geq 0}\frac{1}{j}\left(C_j \sin j\theta - S_j \cos j\theta\right)$$

Así pues, basta partir de $\frac{r^2}{\Theta}\tilde{\mathcal{H}}_{n,0}$, sustituyendo las potencias de $1/r$ y R por las expresiones (3) y una vez expresada en la forma (2) separar en esta expresión la parte que no depende de θ, C_0 del resto. A partir de esta segunda expresión una simple reordenación de términos permite construir la función $\mathcal{W}_n$.

Hay que destacar en este proceso la identidad entre el algoritmo y la representación simbólica definida para las funciones, así como la construcción de la función generatriz sin necesidad de realizar la integración.

7. Conclusiones

Para terminar destacaremos de nuevo la importancia de las técnicas de cálculo simbólico en el tratamiento de problemas de Mecánica Celeste y Astrodinámica. Sin embargo, es preciso tener en cuenta que para que estas técnicas den los frutos deseados es necesario un estudio previo del problema en el que se analicen en forma detallada las características del mismo que puedan ayudar a su correcto tratamiento computacional. Dicho estudio, para el que los SAC de tipo general serán de gran ayuda, permite una formulación del problema compatible con herramientas especializadas como los PSP. Así mismo, las características particulares encontradas en el estudio previo conducirán, en ocasiones, a mejoras de la herramienta básica, el PSP.

Agradecimientos

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PSPCLINK: Un nuevo kernel para *Mathematica*

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Abstract

Poisson Series are mathematical objects frequently used in celestial mechanics. We had developed PSPC, a special Poisson Series processor, that take advantage of the algebraical structure of Poisson series.

In this paper we describe a new tool: PSPCLink. It permits to connect PSPC with *Mathematica*, in order to combine the efficiency of PSPC in handling Poisson series and the flexibility of a general symbolic processor.

1. Introducción

El objeto matemático de nuestro estudio, las series de Poisson [6], se define como una serie de Fourier multivariada cuyos coeficientes son series de Laurent multivariadas en la forma

$$\sum_{i_0,\ldots,i_{n-1},j_0,\ldots,j_{m-1}} C_{i_0,\ldots,i_{n-1}}^{j_0,\ldots,j_{m-1}} x_0^{i_0}\ldots x_{n-1}^{i_{n-1}} \begin{pmatrix}\text{sen}\\ \cos\end{pmatrix}(j_0y_0+\ldots+j_{m-1}y_{m-1}).$$

Los Procesadores de Series de Poisson, PSP, son manipuladores algebraicos que tratan eficientemente estos objetos matemáticos. En [1], [2], [3] y [8], se describe la estructura algebraica, las posibles representaciones simbólicas, la creación de un PSP que implementa dicha estructura, llamado PSPC, y la utilización de éste en la resolución de problemas de Mecánica Celeste.

La principal ventaja de los PSP sobre los sistemas de carácter general se encuentran en la utilización de las propiedades algebraicas del objeto que implementan. Esto permite ganar eficiencia en tiempo y memoria, frente a herramientas generales como *Mathematica* [9] que deben manipular los objetos de forma mucho más general y que por ello no pueden sacar partido de sus propiedades particulares.

Por otro lado, el uso de procesadores, como PSPC, nos hace renunciar a las ventajas que poseen los de carácter general como son un entorno de trabajo agradable, proporcionado por el

Front–End, y la posibilidad de tratamiento simultáneo de distintos objetos matemáticos, como funciones especiales [7] o no matemáticos como gráficas, etc.

Para no renunciar a las ventajas que nos proporcionan ambas herramientas, hemos utilizado el protocolo de comunicaciones *MathLink* [10], integrado dentro de *Mathematica*. Éste permite conectar el kernel de *Mathematica* con aplicaciones implementadas en lenguaje C, a través del Front–End de *Mathematica*. Así se ha creado PSPCLink, que permite el uso simultáneo de PSPC y *Mathematica*. Como se puede observar en la figura 1 desde el Front–End de *Mathematica* se envían las expresiones al kernel o a PSPCLink según el tipo de operación y el tipo de objetos involucrados.

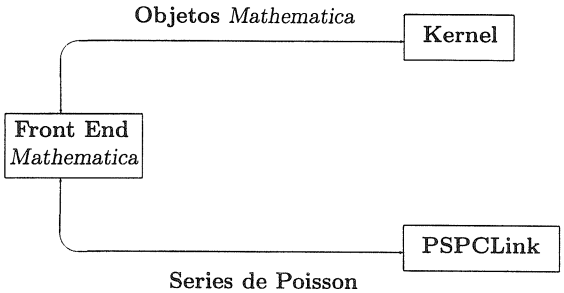

Figura 1.—Flujo de información desde el Front End de *Mathematica*.

PSPCLink ha sido desarrollado en dos etapas. En la primera se conectó el Front–End de *Mathematica* y una aplicación creada a partir de PSPC, con la que se establece una transmisión de datos para solicitar la realización de una operación entre series de Poisson. Mientras que en la segunda se realizó la integración, dentro del estilo propio de *Mathematica*, de los mecanismos de comunicación.

2. Procesadores simbólicos y manipuladores algebraicos

El tratamiento dado a las series de Poisson por *Mathematica* es completamente diferente al realizado por un procesador de series de Poisson, de hecho, dicho tratamiento nos muestra las diferencias que existen entre los llamados procesadores simbólicos, como *Mathematica*, y los manipuladores algebraicos como PSPC.

A continuación analizaremos el comportamiento de los dos tipos de sistemas. Tomemos como ejemplo una serie sencilla

In[1]:=

```
s1 = 1 + (x+y) Sin[a-b];
```

Si calculamos su cuadrado

In[2]:=

```
s2 = s1^2
```

Out[2]=

```
                              2
(1 + (x + y) Sin[a - b])
```

Mathematica nos devuelve la expresión anterior, en la que podemos observar que el cuadrado de `s1` no es para *Mathematica* una serie de Poisson, sino un objeto con la estructura general que maneja *Mathematica*, esto es, *un árbol*, en el que no se ha realizado la operación algebraica solicitada. Sólo si forzamos a *Mathematica* a expandir la expresión y pasarla a ángulos múltiples con la expresión `TrigReduce` obtenemos otra serie de Poisson.

Sin embargo, un procesador de series de Poisson trabaja con la estructura algebraica definida por este objeto matemático, esto es, una estructura de álgebra conmutativa. Los objetos manejados, independientemente de la forma en que son almacenados en el computador, deben ser todos elementos de dicha estructura, esto es series de Poisson. De esta forma, la única representación posible para el cuadrado de una serie es el resultado de realizar la operación, lo que transforma este resultado en una serie de Poisson.

3. Construcción de un nuevo kernel

PSPCLink es una aplicación que permite la utilización desde *Mathematica*, vía *MathLink*, de PSPC, con objeto de una manipulación eficiente del álgebra de series de Poisson combinado con las ventajas del tratamiento general de objetos matemáticos proporcionado por *Mathematica*.

La conexión entre los dos sistemas planteó una serie de problemas derivados principalmente de la diferencia apuntada en el apartado anterior, sistemas simbólicos frente a los procesadores algebraicos, y del lenguaje empleado en la implementación de las operaciones de PSPC, lenguaje procedural frente al estilo utilizado por *Mathematica*. Estas cuestiones están ampliadas en [4] y [5]

A continuación se muestra la eficiencia de PSPCLink, frente al uso exclusivo de las funciones proporcionadas por *Mathematica*, hemos realizado un simple ejemplo donde se mide el

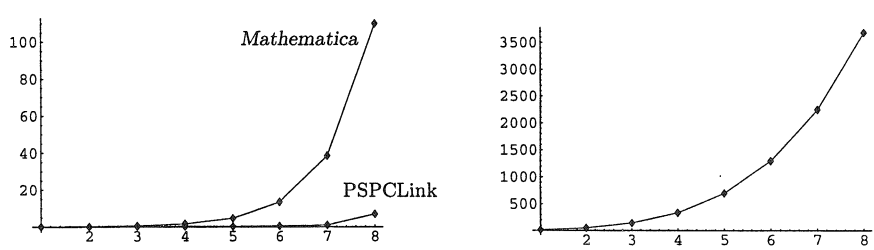

Figura 2.—A la izquierda se muestra la comparación del tiempo de cálculo, en segundos, entre *Mathematica* y PSPCLink. A la derecha, el número de términos del cómputo.

tiempo empleado en elevar a la quinta potencia una serie de Poisson, en la que progresivamente aumentamos el número de variables polinómicas

$$\left[1+\left(\sum_{i=1}^{n} x_i\right)\cos\left(a+b\right)\right]^5 .$$

En la gráfica se observa cómo al aumentar el número de términos manipulados, la eficiencia de PSPCLink es considerablemente superior.

Todos los cálculos presentados en este trabajo han sido realizados usando la versión 3.0 de *Mathematica* en un computador con procesador PowerPC 750 G3 a 266 Mhz.

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On the evaluation of quadratures containing trascendental universal functions

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Abstract

For a universal and uniform analytical formulation and treatment of the two–body problem, the main dynamical quantities and variables of interest depend, in a simple way, on some low–order members of certain families of transcendental universal functions (e. g. the *Battin universal U–functions* and the *Stumpff c–functions*) which generalize the standard and hyperbolic trigonometric functions.

We show how certain integrals of expressions depending on those functions are transformed into expressions with simpler *polynomial arguments*. Appropriate changes of the integration variable and application of analytical properties of universal functions lead to this reduction. This approach might be applied when generalized Sundman–type transformations of the independent variable are performed to obtain either analytical linearization (and /or regularization) of equations of motion or automatic stepsize regulation in numerical integrations for orbit computation.

1. Introduction

Floría & Caballero (1995) proved how the intermediate anomaly of Keplerian motion, introduced by Nacozy (1977) for the elliptic Kepler problem, can be extended to any type of conic–section orbit. To this end, we used certain classes of special functions: the *Stumpff c–functions* (Stumpff 1959, §37, §41; Stiefel & Scheifele 1971, §11, pp. 43–45) and the *universal U–functions* (Battin 1987, §4.5, §4.6), which generalize the standard and hyperbolic trigonometric functions. Following Nacozy's steps, and generalizing his analytical treatment, the *Sundman–type transformation* (Sundman 1912, p. 127) of the time variable, given by a differential relation and integrated by Nacozy for the pure elliptic motion, was integrated for the three main cases of conic–section Keplerian orbits.

Floría (1997) proposed a systematic derivation, via universal functions, of the expression for the differential time transformation introducing the length of orbital arc as the independent vari-

able, irrespective of the type of Keplerian orbit. The resulting transformation was analytically integrated, in closed form, by means of elliptic functions.

That treatment is comprised in a more general handling of integrands containing universal functions. In order to generalize our previous particular developments when dealing with integrals involving some universal functions, in the present paper we show that appropriate *changes of the integration variable* reduce the integrands to respective *integrands of polynomials*. This procedure is similar to the elementary substitution technique, usually applied to reduce trigonometric integrands or integrands containing hyperbolic functions.

We will concentrate on the use of the *Battin U–functions*. According to our experience in dealing with *Keplerian–like systems*, the dynamical quantities and variables of interest depend on the functions U_0, U_1, U_2 and U_3. However, since time t usually enters under differentials, the basic universal functions involved in calculations are U_0, U_1 and U_2.

The treatment proposed here is particularly useful when the reduction yields square roots of third– and fourth–degree polynomials and their products with other rational functions. For instance, when generalized Sundman–type transformations of the independent variable are performed to deal with Keplerian–like systems.

2. On the Definition and Some Useful Properties of Universal Functions

The *Stumpff c_n–functions* can be defined by the relation

$$c_n(z) \;=\; \sum_{k=0}^{\infty}(-1)^k z^k/(2k+n)!\;, \qquad n \;=\; 0,\,1,\,2\ldots. \tag{1}$$

If ϱ is a real parameter, and $z = \varrho s^2$, the alternative *Battin universal functions* are

$$U_n(s,\varrho) \;\equiv\; s^n c_n(\varrho s^2) \;=\; \sum_{k=0}^{\infty}(-1)^k \varrho^k s^{2k+n}/(2k+n)!\,. \tag{2}$$

In applications to the two–body problem, the parameter ϱ is related to the orbital energy of the system, and one usually takes $\varrho = 2L$. As for the *notations*, the symbol μ represents the gravitational bodycentric parameter of the two–body system, while (e, q) will refer to Keplerian orbital elements: eccentricity and distance of the pericentre, regardless of the type of trajectory. According to Stiefel & Scheifele (1971, p. 50, Formula [64]), the *negative of the total energy* of a Keplerian system will be the quantity

$$L \;=\; [\mu(1-e)]/(2q)\,. \tag{3}$$

Next we give some analytical and dynamical properties and relations between universal functions (Stiefel & Scheifele 1971, pp. 50–51; Battin 1987, §4.5, §4.6):

$$r \;=\; q + \mu e s^2 c_2\left(2Ls^2\right) \;=\; q + \mu e U_2(s, 2L)\,, \text{ radial distance;} \tag{4}$$

$$dt \;=\; r\,ds, \text{ Sundman's transformation;} \tag{5}$$

$$t \;=\; qs + \mu e U_3(s, 2L), \text{ Kepler's equation;} \tag{6}$$

$$dU_0/ds = -\varrho U_1\,; \quad dU_n/ds = U_{n-1}\,, \quad n = 1, 2, 3, \ldots\,; \tag{7}$$

$$1 = U_0^2 + \varrho\, U_1^2\,; \tag{8}$$

$$1 = U_0 + \varrho\, U_2\,; \tag{9}$$

$$U_1^2 = U_2\,(1 + U_0)\,; \tag{10}$$

$$U_1^2 = 2U_2 - \varrho\, U_2^2\,. \tag{11}$$

The fictitious time s, a *universal eccentric-like anomaly* proportional to the classical eccentric anomaly in the cases of elliptic and hyperbolic motion, is introduced through Stumpff's generalization (1959, §41) of Sundman's regularizing transformation (5).

3. The Integral and Its Transformation

Let $\Phi\,(U_0, U_1, U_2)$ be a function of the transcendental universal functions $U_j\,(s\,,\,\varrho)$, with $j = 0, 1, 2$, as its arguments. Consider the problem of evaluating the quadrature

$$\mathcal{I} = \int_0^s \Phi\,(U_0(s\,,\,\varrho)\,,\, U_1(s\,,\,\varrho)\,,\, U_2(s\,,\,\varrho))\; ds\,. \tag{12}$$

We handle this integral by *direct substitution*: one simplifies the integrand by replacing an expression appearing in it with a single variable. Thus, to arrive at integrands conceivable as *functions of polynomial arguments*, we express each U_j and the differential of s in terms of one of the other universal functions. The possible *choices for the basic universal function* from which we develop the remaining elements involved in the quadrature are:

3.1 Reduction in terms of U_2

We start the detail of our derivation by studying the reduction of (12) with the help of U_2. For this purpose, we wish to express U_0, U_1 and ds in terms of U_2. To this end, we perform the *change of integration variable* $s \longrightarrow v$ given by

$$U_2(s\,,\,\varrho) = v \Longrightarrow U_1(s\,,\,\varrho)\, ds = dv\,, \quad s = 0 \Longrightarrow v(0) = U_2(0\,,\,\varrho) = 0\,, \tag{13}$$

where we have used the rule (7). By virtue of identities (11) and (8) or (9) we obtain

$$U_1 = \sqrt{2v - \varrho v^2}\,, \quad U_0 = 1 - \varrho v\,, \quad dv = \sqrt{2v - \varrho v^2}\; ds\,. \tag{14}$$

The quadrature (12) is converted into

$$\mathcal{I} = \int_0^v \frac{\Phi\left(1 - \varrho v\,,\, \sqrt{2v - \varrho v^2}\,,\, v\right)}{\sqrt{2v - \varrho v^2}}\; dv\,, \tag{15}$$

and the new integrand turns out to be a function of the variable v.

3.2 Reduction in terms of U_1

We transform the integration variable $s \longrightarrow v$ according to the rule

$$U_1(s, \varrho) = v \Longrightarrow U_0(s, \varrho)\, ds = dv, \quad s = 0 \Longrightarrow v(0) = U_1(0, \varrho) = 0. \tag{16}$$

Application of identities (8) and (10) yields

$$U_0 = \sqrt{1 - \varrho v^2}, \quad U_2 = \frac{v^2}{1 + \sqrt{1 - \varrho v^2}}, \quad dv = \sqrt{1 - \varrho v^2}\, ds. \tag{17}$$

Inserting all these intermediate expressions into the function Φ and replacing the differential of s in terms of v achieves the desired reduction of (12).

3.3 Reduction in terms of U_0

Change the integration variable $s \longrightarrow v$ by means of

$$U_0(s, \varrho) = v \Rightarrow -\varrho U_1(s, \varrho)\, ds = dv, \quad s = 0 \Rightarrow v(0) = U_0(0, \varrho) = 1, \tag{18}$$

taking into account the special formula (7) for the derivative of U_0. Properties (8) and (10) or (9) lead us to

$$U_1 = \sqrt{\frac{1 - v^2}{\varrho}}, \quad U_2 = \frac{1 - v}{\varrho}, \quad dv = -\sqrt{\varrho\,(1 - v^2)}\, ds. \tag{19}$$

These ingredients allow us to produce the transformed expression for the integrand in (12).

Here we have *systematized and generalized* our previous developments (Floría 1997). This approach is specially advisable when Φ is an algebraic function of its arguments, since the usual techniques to evaluate algebraic integrands in v can be readily applied.

Acknowledgments

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Compresión de efemérides lunares: Análisis espectral de errores

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La Sección de Efemérides del Real Instituto y Observatorio de la Armada utiliza las DE200/LE200 como efemérides básicas de los objetos mayores del sistema solar, para la generación de los almanaques que publica. A partir de ellas se calculan las coordenadas esféricas aparentes de dichos cuerpos, coordenadas sobre las que se realizan aproximaciones polinómicas uniformes en la norma de Chebyshev. Estas aproximaciones constituyen la base de cálculo del *Almanaque Náutico* y del ANDI y aseguran que el error cometido con la aproximación está acotado en el intervalo de validez. Los polinomios se calculan para 32 días, con lo que con 12 ajustes por coordenada se cubre un año completo[2, 6].

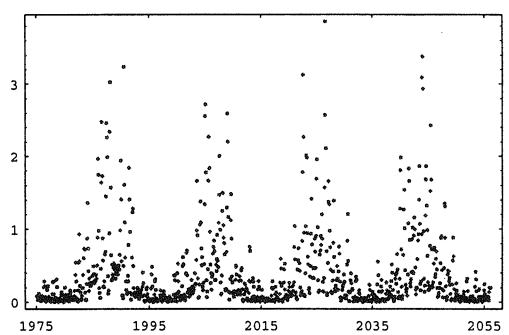

Figura 1.—Error máximo en la ascensión recta de la Luna. Cada punto corresponde a un mes. Las abscisas están en años y las ordenadas en segundos de arco.

El grado del polinomio necesario para alcanzar la precisión requerida por las publicaciones varía según los cuerpos; la Luna es la que mayor lo requiere, siendo necesarios polinomios de grado 27 para la ascensión recta y la declinación y de grado 11 para la distancia geocéntrica. Estos grados se han determinado analizando los errores máximos cometidos en un período de tiempo grande, puesto que dicho error varía notablemente aún dentro del mismo año. En la Figura 1 se presentan los errores máximos de cada mes para la ascensión recta de la Luna durante el período 1975–2056.

[1]Los autores se citan en orden alfabético

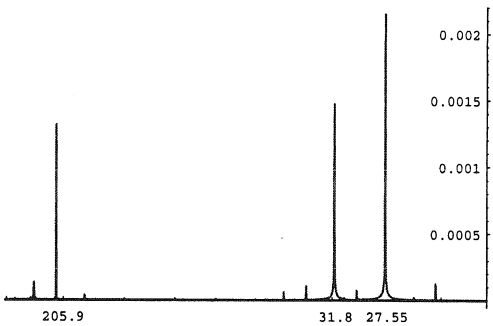

Figura 2.—Parte principal del espectro de los errores máximos de la aproximación uniforme a la distancia geocéntrica de la Luna con polinomios de grado 11 para un período de 32 días, muestreados cada 2 días. En el eje de abscisas se representan en días los períodos de las frecuencias en escala hiperbólica. Las líneas principales corresponden al movimiento del perigeo y la excentricidad, 205.9 días, al perigeo, 31.8 días, y al período anomalístico (perigeo a perigeo) de 27.55 días

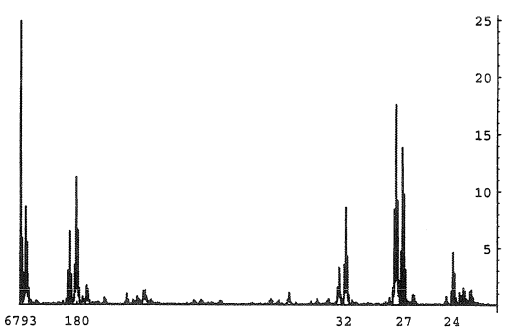

Figura 3.—Parte principal del espectro de los errores máximos de la aproximación uniforme a la ascensión recta de la Luna con polinomios de grado 27 para un período de 32 días, muestreados cada 2 días. En el eje de abscisas se representan en días los períodos de las frecuencias en escala hiperbólica.

La considerable variación de los errores máximos podría ser debida a una incorrecta implementación del algoritmo de aproximación; no obstante, la experiencia adquirida en el uso de los programas de aproximación sugiere que éstos son altamente fiables, por lo que hay que justificar la variación de los errores con otros argumentos. Un indicio de cómo justificar esta variación se obtiene, de forma casi inmediata, de la Figura 1, en la que puede apreciarse un ciclo de período aproximado de 19 años, que sugiere la posibilidad de un reflejo del período de 18.6 años de la longitud del nodo ascendente de la Luna. Este hecho nos lleva a la hipótesis de que la variación se debe a las irregularidades del movimiento lunar. Para corroborarla hemos efectuado un análisis espectral de los errores y hemos identificado las frecuencias obtenidas con las que caracterizan el movimiento de la Luna. Aunque con otros fines, el análisis espectral ya ha sido usado en mecánica celeste, por ejemplo, en los trabajos de Carpino et al. [1] o Chapront [3] para la elaboración de teorías sintéticas del movimiento planetario.

El análisis espectral lo efectuaremos aplicando la Transformada Discreta de Fourier (DFT) a la sucesión formada por los errores máximos de los polinomios de aproximación. Al aplicar la DFT a una muestra de una función temporal, puede aparecer el fenómeno de *aliasing*, dependiendo de cuales sean la frecuencia de muestreo y el ancho de banda del espectro de dicha función. Pero además, la necesidad de tratar con un número finito de datos, restringiéndonos a un intervalo de tiempo determinado, distorsiona el espectro resultante, a menos que la función sea periódica y analicemos un número exacto de períodos.

Con objeto de eliminar en lo posible el *aliasing* que aparece en el muestreo original (período de muestreo de un mes), hemos muestreado la función error cada dos días, calculando el error máximo cometido al ajustar polinomios válidos para un intervalo de 32 días centrados en la fecha en cuestión. De esta forma, el *aliasing* se producirá para frecuencias de período inferior a 4 días, de escaso significado en la Teoría de la Luna.

Para reducir los errores introducidos al analizar un intervalo de tiempo finito hemos trabajado con 13596 datos, que prácticamente coincide con 4 períodos de 18.6 años, intervalo que es coherente con la periodicidad que se aprecia en la Figura 1.

No obstante lo anterior, el espectro que obtenemos no refleja exactamente los distintos períodos que caracterizan el movimiento de la Luna, aunque sí somos capaces de identificar las frecuencias más significativas, de modo que podemos establecer la bondad de nuestra hipótesis. Un análisis más riguroso requeriría realizar filtrados previos de la función error que permitiesen examinar exhaustivamente todo el espectro de frecuencias.

Nótese que, mientras que habitualmente las frecuencias principales del movimiento de la Luna se obtienen al estudiar las perturbaciones de los elementos orbitales, en las coordenadas que estamos utilizando deberán aparecer mezcladas todas ellas, excepto para la distancia, que fundamentalmente se ve afectada por las perturbaciones del perigeo, la excentricidad y el semieje de la órbita lunar, tal y como se aprecia en el espectro de la Figura 2. El análisis de la declinación ofrece un espectro cualitativamente igual al de la ascensión recta, cuya parte más significativa se presenta en la Figura 3. Para mayor comodidad, hemos hecho la identificación comparando con aquellos términos de la Teoría de la Nutación UAI 1980 [4] que tienen dependencia exclusiva

Término Nutación UAI		Espectro	Término Nutación UAI		Espectro
N°	Período	Período	N°	Período	Período
	d	d		d	d
1	6798.4	6793.0	32	27.6	27.6
2	3399.2	3396.5	35	31.8	31.8
3	1305.5	1293.9	36	27.1	27.1
5	1615.7	1598.4	38	27.7	27.7
9	182.6	182.4	39	27.4	27.5
13	177.8	177.6	44	23.9	23.9
14	205.9	205.9	47	27.0	27.0

Tabla 1.—Correspondencia entre los términos más significativos del espectro de la Figura 3 y aquéllos de la Teoría de la Nutación UAI 1980 que tienen dependencia exclusiva de la Luna.

de la Luna, en vez de comparar con otra fuente tal como las Tablas de la Luna de Brown, que complicaría mucho el problema para el fin que perseguimos. En la Tabla 3. se presenta la correspondencia de las líneas más importantes del espectro.

A modo de conclusión, vemos que el movimiento del nodo de la órbita de la Luna es el principal responsable de la irregularidad del error máximo conseguido con la aproximación uniforme al ajustar la ascensión recta y la declinación de la Luna. Tal perturbación provoca que los errores de dicho ajuste puedan ser insuficientes en ciertas épocas para el grado del polinomio y el intervalo de validez que se están utilizando; por este motivo, para futuras ediciones del ANDI se está considerando la compresión de la base de datos en coordenadas cartesianas, que permiten una mejor aproximación [5].

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Electronic traps in a circularly polarized microwave field and a static magnetic field: Stability analysis

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1. Introduction

The interaction of a hydrogen or Rydberg atom with a circularly polarized (CP) microwave field leads, with finely tuned parameters, to the creation of stable equilibrium positions similar to that of gravitational equilibrium points well known in celestial mechanics [1, 2]. Besides, the addition of a static magnetic field (**B**), perpendicular to the plane of polarization, can be used to manipulate the stability properties of the equilibria [4, 5].

The aim of this paper is to deal with the linear stability properties of the equilibrium points and show that some of these points need further analysis to establish Lyapunov stability, because the quadratic approximation in a vicinity of the equilibrium is not a definite form. In fact the last point is the mean result of the paper which is intended to clarify some misleading in the literature, where no special attention is paid to the quadratic approximation, giving rise to some potential errors [3]

2. Hamiltonian and equilibria

In atomic units, the Hamiltonian for the CP $\times$ **B** problem, in the dipole approximation, is given by

$$\begin{aligned}\mathcal{H} = \tfrac{1}{2}(P_x^2 + P_y^2 + P_z^2) - \tfrac{1}{\sqrt{x^2+y^2+z^2}} \pm \tfrac{\omega_c}{2}(xP_y - yP_x) + \tfrac{\omega_c^2(x^2+y^2)}{8} \\ + f(x\cos\omega_f t + y\sin\omega_f t),\end{aligned} \tag{1}$$

where the magnetic field is taken to lie along the positive z direction. In Eq. (1) ω_c is the cyclotron frequency, ω_f is the CP field frequency and f is the electric field strength. Going to a frame rotating with the CP frequency ω_f, it is possible to eliminate the explicit time dependence in Eq. (1), producing the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(P_x^2 + P_y^2 + P_z^2) - \frac{1}{\sqrt{x^2+y^2+z^2}} - (\omega_f \pm \frac{\omega_c}{2})(xP_y - yP_x) + \frac{\omega_c^2(x^2+y^2)}{8} \pm fx. \tag{2}$$

where x, y and z are assumed to refer to the rotating frame. Now, due to the fact that trajectories with initial conditions $z = P_z = 0$ remain always in the plane $x - y$, we focus our attention on the planar ($2D$) model. The corresponding $2D$ Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{2}(P_x^2 + P_y^2) - \frac{1}{\sqrt{x^2+y^2}} - (\omega_f \pm \frac{\omega_c}{2})(xP_y - yP_x) + \frac{\omega_c^2(x^2+y^2)}{8} \pm fx. \tag{3}$$

Then, the equations of the motion are

$$\begin{aligned} \dot{x} &= p_x + \omega y \\ \dot{y} &= p_y - \omega x \\ \dot{p}_x &= -\tfrac{x}{r^3} + \omega p_y \mp f - \tfrac{\omega_c^2}{4}x \\ \dot{p}_y &= -\tfrac{y}{r^3} - \omega p_x - \tfrac{\omega_c^2}{4}y \end{aligned} \tag{4}$$

where $\omega = \omega_f \pm \frac{\omega_c}{2}$. The equilibrium points are the roots of the system made of the right-hand side of (4) equal to 0. In this way, the coordinates of the equilibrium points satisfy

$$\begin{aligned} &p_x = y = 0, \qquad p_y = \omega x, \\ &\omega_f(\omega_f \pm \omega_c)x^3 \mp fx^2 - 1 = 0, \quad (x > 0), \\ &\omega_f(\omega_f \pm \omega_c)x^3 \mp fx^2 + 1 = 0, \quad (x < 0). \end{aligned}$$

Attending to the sign (plus or minus) different situations are accounted depending on the number of real roots of the cubic equation that the x coordinate satisfies. Thus, for the minus sign and $\omega_f(\omega_f - \omega_c) > 0$ there are two equilibria, one of them with positive x coordinate and the other one with negative x coordinate; if $\omega_f(\omega_f - \omega_c) < 0$ there is not any equilibria with negative x coordinate but two, one or none equilibria with positive x coordinate if $f > F_c$, $f = F_c$ or $f < F_c$ respectively, where $F_c = \sqrt[3]{\frac{27}{4}\omega_f^2(\omega_f - \omega_c)^2}$. On the other hand, for the plus sign there are always two equilibria one with positive x coordinate and the other one with negative x coordinate. These points are precisely the critical points of the zero velocity surface defined as $\mathcal{H} - \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = -\frac{1}{r} \pm fx - \frac{1}{2}\omega_f(\omega_f \pm \omega_c)(x^2+y^2)$. It is worthy to note that, from the analysis of the critical points in the zero velocity surface, only two relevant configurations are needed to account. The first one corresponds to the presence of a maximum and a saddle (plus sign or minus sign and $\omega_f(\omega_f - \omega_c) > 0$) and the other one corresponds to a minimum and a saddle.

3. Linear stability analysis

Linear approximation in a neighborhood of the critical points yields to the quadratic Hamiltonian

$$\mathcal{H} = \mathcal{H}_0(x_0, 0, 0, \omega x_0) + \frac{px^2 + py^2}{2} - \omega(xpy - ypx) + \frac{\omega^2}{2}(\alpha x^2 + \beta y^2),$$

where

$$\alpha = \left(\frac{\omega_c^2}{4} - 3\frac{x_0^2}{r_0^5} + \frac{1}{r_0^3}\right)\frac{1}{\omega^2}, \qquad \beta = \left(\frac{\omega_c^2}{4} + \frac{1}{r_0^3}\right)\frac{1}{\omega^2},$$

x_0 stands for the x coordinate of the equilibrium and $r_0 = x_0$ if $x_0 > 0$ and $r_0 = -x_0$ if $x_0 < 0$, being $\mathcal{H}_0 = \frac{\omega_c^2}{8}x_0^2 - \frac{\omega^2}{2}x_0^2 \mp fx_0 - \frac{1}{r_0}$.

Linear stability is obtained if one of the two following conditions is satisfied (see e.g. [6])

a) $\alpha > 1$ and $\beta > 1$. In this case the quadratic part is positive defined and linear stability implies Lyapunov stability, by virtue of Morse's Lemma [7].

b) $\alpha < 1$, $\beta < 1$ and $(\alpha - \beta)^2 + 8(\alpha + \beta) > 0$. In this case the quadratic part is undefined and further analysis is needed to establish Lyapunov stability.

Let us consider the stability of the critical points in the two general situations.

The saddle-minimum configuration

This configuration is obtained for the minus sign and the two additional conditions $\omega_f(\omega_f - \omega_c) < 0$ and $f > F_c = \sqrt[3]{\frac{27\omega_f^2(\omega_f-\omega_c)^2}{4}}$. Besides, $0 < x_s < \sqrt{\frac{-2}{\omega_f(\omega_f-\omega_c)}} < x_m$, for x_s and x_m the x coordinate of the saddle and the minimum. Moreover, both x_s and x_m satisfies the cubic equation $\omega_f(\omega_f - \omega_c)x^3 + fx^2 - 1 = 0$.

From the cubic equation results

$$\frac{1}{x^3} = \omega_f(\omega_f - \omega_c) + \frac{f}{x} \Longrightarrow \frac{1}{x^3} > \omega_f(\omega_f - \omega_c)$$

and then, for the saddle and the minimum,

$$\beta = \frac{1}{\omega^2}\left(\frac{w_c^2}{4} + \frac{1}{x_0^3}\right) > \frac{1}{\omega^2}\left(\frac{w_c^2}{4} + \omega_f^2(\omega_f - \omega_c)^2\right) = 1.$$

On the other hand, taking into account $x_s < \sqrt{\frac{-2}{\omega_f(\omega_f-\omega_c}} < x_m$

$$\frac{-2}{x_m^3} > \omega_f(\omega_f - \omega_c), \qquad \text{and} \qquad \frac{-2}{x_s^3} < \omega_f(\omega_f - \omega_c),$$

and then

$$\alpha = \frac{1}{\omega^2}\left(\frac{w_c^2}{4} - \frac{2}{x_0^3}\right) \begin{cases} > 1, & x_0 = x_m \\ < 1, & x_0 = x_s \end{cases}$$

Consequently, the minimum satisfies the stability condition *a)* and it is Lyapunov stable while the saddle point is unstable.

The saddle-maximum configuration

Although this configuration may be obtained with the plus sign as well as the minus sign, we will focus on the plus sign (the analysis of the minus sign is analogous). In this case, the x coordinates of the saddle and the maximum verify the cubic equations

$$\omega_f(\omega_f - \omega_c)x_s^3 - fx_s^2 + 1 = 0, \qquad \omega_f(\omega_f - \omega_c)x_s^3 - fx_s^2 - 1 = 0,$$

where x_s and x_m stand for the x coordinate of the saddle and the maximum respectively. Besides $x_s < 0$ and $x_m > 0$. Taking this into account we have for the saddle the followings bounds

$$\frac{2}{x_s^3} < w_f(\omega_f + \omega_c), \qquad \frac{-1}{x_s^3} > w_f(\omega_f + \omega_c)$$

and then

$$\alpha = \frac{1}{\omega^2}\left(\frac{w_c^2}{4} + \frac{2}{x_s^3}\right) < 1, \quad \beta = \frac{1}{\omega^2}\left(\frac{w_c^2}{4} - \frac{1}{x_s^3}\right) > 1.$$

This implies that the saddle point is unstable.

On the other hand, for the maximum we have the bounds

$$\frac{-2}{x_s^3} < w_f(\omega_f + \omega_c), \qquad \frac{1}{x_s^3} < w_f(\omega_f + \omega_c)$$

and then

$$\alpha = \frac{1}{\omega^2}\left(\frac{w_c^2}{4} - \frac{2}{x_m^3}\right) < 1, \quad \beta = \frac{1}{\omega^2}\left(\frac{w_c^2}{4} + \frac{1}{x_m^3}\right) < 1.$$

The last result does not imply stability nor instability. In fact, if the stability condition *b)* is fulfilled we obtain linear stability when

$$(\alpha - \beta)^2 + 8(\alpha + \beta) = \frac{9}{\omega^4 x_m^6} + \frac{1}{\omega^2}\left(4w_c^2 - \frac{8}{x_m^3}\right) > 0.$$

Solving the last equation for x_s and going to the cubic equation we obtain that the stability region, in the parameter space, is delimited by the surfaces

$$f = \frac{\omega_f(\omega_f - \omega_c)(2\omega \pm \sqrt{4\omega^2 - 9\omega_c^2}) - 2\omega\omega_c^2}{(2\omega\omega_c^2)^{1/3}(2\omega \pm \sqrt{4\omega^2 - 9\omega_c^2})^{2/3}}.$$

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