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AN INTRODUCTION TO NUCLEAR SPACE

by

C. BESSAGA, B. HERNANDO BOTO and E. MARTIN-PEINADOR

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C. Bessaga, B. Hernando Boto and E. Martín-Peinador*

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Abstract.

These are slightly extended and revised notes of seminar lectures delivered by the first-named author at the Universidad Complutense in Madrid (Spain). they cover only special introductory topics on nuclear spaces, selected intentionally for the purpose of illustrating the role of Kolmogorov diameters and Kolmogorov numbers.

The discussion is restricted to real locally convex spaces and their subclasses. Nevertheless all the theorems presented are valid also in the complex case. The complex versions can either be proved in exactly the same way, or can be derived from the real theorems using some extra argument, (see Remark 1.7 and Exercise 2.4).

Many important subjects related to nuclear spaces are not even touched. For instance: Relations to the theory of absolutely summing operators; Discussion of concrete nuclear spaces and concrete bases; T. & Y. Komuras' theorem on universal spaces; Vogt's structure theory and in particular applications of his "splitting theorem"; Relations to the distribution theory; Tame structures and commections with Nash-Moser implicit function theorem; .....

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CONTENTS

§ o. Preliminaries

We shall concentrate on deviations from the standard terminology and notation of functional analysis and on the points which may cause a confusion.

$\mathbb{R}^+$ denotes the set of all non-negative reals.

Unless stated otherwise, X,Y,,... denote real topological vector spaces, and X^*, Y^*... their duals (i.e. the spaces of continuous linear functionals). By an <u>operator</u> $T:X \longrightarrow Y$ we mean a continuous linear mapping.A <u>functional</u> is an operator to the field of scalars. Here the arrow indicates the domain and codomain of T, while the expression "$x_n \longrightarrow x$" stands for "$\lim_n x_n = x$" ; and $x_n \longmapsto y_n$, denotes the mapping which sends each x_n to y_n.

The composition $T \circ S$ of operators will also be denoted by TS.

By a <u>subspace</u> of a topological vector space X we mean a linear (not necessarily closed) subspace of X. A subspace Y of X is said to be <u>complemented</u>, if there is a continuous linear projection of X with the range Y. For a subset $A \subset X$ the symbol [A] stands for the subspace of X generated by A.

<u>Seminorms</u> A seminorm defined on a vector space X is a function $p:X \longrightarrow \mathbb{R}^+$ such that $p(x+y) \le p(x) + p(y)$, $p(tx) = |t|p(x)$ for all $x,y \in X$, and for all scalars t. The symbol $U_p = \{x \in X;\ p(x) \le 1\}$, denotes the <u>seminorm-ball</u>. A <u>norm</u> on X is a seminorm $p:X \longrightarrow \mathbb{R}^+$ such that $p(x) = 0$ implies $x = 0$. A normed space is a pair $(X, \|\ \|)$ where X is a vector space and $\|\ \|:X \longrightarrow \mathbb{R}^+$ is a norm; often the normed space is denoted by the single letter X.

The symbol cpl X stands for the Banach space which is the completion of X in the norm.

$$B_X = \{x\in X; \|x\| \le 1\} \quad \text{and} \quad S_X = \{x\in X; \|x\| = 1\}$$

are the closed unit ball and the unit sphere of the normed space X.

To each seminorm p defined on a vector space X corresponds the quotient space $X_p = X/p^{-1}(0)$ whose elements are the cosets:

$$[x]_p = \{y\in X; p(y-x)=0\},$$

The space X_p will be regarded as a normed space with the norm $\|[x]_p\| = p(x)$, in the sequel denoted briefly by the same symbol p.

We shall also consider the Banach space $\tilde{X}_p = \text{cpl } X_p$, and we denote its norm again by the same symbol p.

If q is a seminorm on X dominating p, i.e., $p(x) \le cq(x)$ for some $c > 0$, then the linking operators

$$I_p : X \longrightarrow X_p, \quad I_p(x)=[x]_p; \quad I_{qp} : X_q \longrightarrow X_p, \quad I_{qp}([x]_q)=[x]_p$$

and $\tilde{I}_{qp} : \tilde{X}_q \longrightarrow \tilde{X}_p$, the continuous extension of I_{qp}, are well defined.

Let X be a topological vector space. Denote by $\mathcal{S}(X)$ and by $\mathcal{U}(X)$ the set of all continuous seminorms and the class of all zero-neigborhoods in the space X. We shall say that a subclass $\mathcal{W} \subset \mathcal{U}(X)$ is fundamental if there are constants $c(W) > 0$, $W\in\mathcal{W}$, such that the family $\{c(W)\cdot W; W\in\mathcal{W}\}$ is a base of zero-neighbourhoods of X; a subset $\mathcal{P} \subset \mathcal{S}(X)$ is said to be fundamental if the family $\{U_p; p\in\mathcal{P}\}$ is fundamental.

Clearly, each U_p is a convex, symmetric with respect to zero and closed zero-neighborhood, and to every convex centrally symmetric $U \in \mathcal{U}(X)$ corresponds a continuous seminorm p such that U_p is the closure of U, the gauge functional of U. However it may happen that $\mathcal{S}(X)$ consists of the zero seminorm only (e.g.for the space L_p with $0 \le p <1$).

Locally convex spaces and Fréchet spaces. The space X is locally convex iff $\mathcal{S}(X)$ is fundamental. Complete-metrizable locally convex spaces are called Fréchet spaces. A grading of a Fréchet space X is a non-decreasing sequence $\mathcal{G} = \{p_n\}$, $p_n \in \mathcal{P}(X)$ whose elements constitute a fundamental set.

Every Fréchet space, being metrizable, admits a countable fundamental set of seminorms $\{q_n; n\in\mathbb{N}\}$, therefore it also admits gradings, $\mathcal{G} = \{p_n\}$, where $p_n = \sup\{q_m;\ m\le n\}$ for $n\in\mathbb{N}$.

A Fréchet space X is said to be <u>locally radially bounded</u> if it admits a base of zero-neighbourhoods each of which does not contain any half-line, equivalently, if there exists a grading for X consisting of norms.

The following fact is referred to as the Banach - Steinhaus theorem for seminorms:

0.1 Let X be a Fréchet space. If $q_n \in \mathcal{P}(X)$, $n\in\mathbb{N}$ are such that $q(x) = \sup_n q_n(x) < \infty$, then the seminorm q is continuous.

<u>Proof</u>. By the assumption, the seminorm-balls $A_n = \{x\in X: q(x)\le n\}$ cover the (complete metric) space X. Therefore there is an n_0 such that $W = \operatorname{int} A_n$ is non empty. Since the set A_n is convex and centrally symmetric, it contains the open zero-neighbourhood $U = \frac{1}{2}(-W - W)$. Therefore, if $x \in \varepsilon n_0^{-1}U$ then $q(x) \le \varepsilon$, i.e., the seminorm q is continuous at zero and, being sublinear, is continuous everywhere. ■

0.2 <u>Corollary</u>. Let X be a Fréchet space and Y an arbitrary locally convex space. If $T_n: X \longrightarrow Y$, $n\in\mathbb{N}$, are continous linear operators such that $T(x) = \lim_n T_n(x)$ exists for every $x\in X$, then T is a continuous linear operator.

<u>Proof</u>. Let $p \in \mathcal{P}(Y)$ an arbitrary seminorm. Let $q_n(x) = p(T_n(x))$, and $q = \sup_n q_n$. By 0.1 q is continuous, which implies the continuity of T. ■

§1. Kolmogorov diameters

Definition. Let A,B be two nonempty subsets of a vector space X. For every $n\in\mathbb{N}$, the n-th Kolmogorov diameter of A with respect to B is defined by

$$\delta_n(A,B) := \inf_L \inf\{t>0;\ A\subset tB+L\},$$

the first infimum taken over all linear subspaces L of X with dim L <n. (Recall that the infimum of the empty set is ∞ !)

It should be clear that the Kolmogorov diameters depend also on the linear space X. In the situation $A,B \subset X \subset Y$ the diameters relative X may differ from those with respect to Y.

1.1 Remark. For a normed space X the Kolmogorov diameters with respect to the unit ball B_X can be expressed by

$$\delta_n(A) = \inf_{\dim L<n} \sup_{x\in A} \operatorname{dist}(x,L).$$

1.2. Elementary properies of Kolmogorov diameters:

(i) $\delta_1(A,B) \ge \delta_2(A,B) \ge \delta_3(A,B) \ge \dots$

(ii) $\delta_n(A_1,B_1) \le \delta_n(A,B)$ whenever $A_1\subset A$ and $B_1\supset B$.

(iii) $\delta_n(\alpha A,\beta B) = \alpha\beta^{-1}\delta_n(A,B)$ if $\alpha \ge 0$, $\beta > 0$.

(iv) $\delta_{n+m-1}(A,C) \le \delta_n(A,B)\cdot\delta_m(B,C)$.

(v) If X is a normed space, $A \subset X$, then $\delta_1(A,B_X) = \sup\{\|a\|;a\in A\}$.

The proof of (iv) is similar to that of 1.8 (i). The other properties are evident. ∎

Vanishing of Kolmogorov diameters of a set A is related to the dimension of its linear span.

1.3 Let A be a subset of a normed space X and let $n\in\mathbb{N}$. Then the following implications hold

(a) $\dim[A] < n$ implies $\delta_n(A) = 0$,

(b) $\dim[A] \geq n$ implies $\delta_n(A) > 0$.

In particular $\dim[A] = \infty$ iff $\delta_n(A) \neq 0$ for every $n \in \mathbb{N}$.

Proof. The statement (a) is obvious.

Assume that (b) is false, i.e., that there exist n linearly independent vectors $a_1, \ldots, a_n$ in A together with $\delta_n(A, B_X) = 0$. By Hahn-Banach theorem there are linear functionals $f_1, \ldots, f_n$ such that $f_i(a_j) = \delta_{ij}$, the Kronecker delta. Since $\det\{\delta_{ij}\} = 1$, there exists a $\lambda > 0$ with $\det\{\delta_{ij} - \alpha_{ij}\} \neq 0$ for $|\alpha_{ij}| \leq \lambda$.

Let $t := \lambda / \max\{\|f_i\|;\ i = 1, \ldots, n\}$. Since $\delta_n(A, B_X) = 0$, there is a subspace $L \subset X$ of dimension at most $n-1$ such that $A \subset tB_X + L$. Thus $a_j = tx_j + \ell_j$, $j = 1, \ldots, n$, for some x_j in B_X and ℓ_j in L. As $\det\{f_i(\ell_j)\} = 0$ and $|tf_i(x_j)| \leq \lambda$ we have

$$0 \neq \det\{f_i(a_j) - tf_i(x_j)\} = \det\{f_i(\ell_j)\} = 0,$$

a contradiction. ■

Now it seems natural to study bounded sets A of a normed space X such that $\dim[A] = \infty$ but $\lim_n \delta_n(A, B_X) = 0$. The following proposition characterizes these sets:

1.4 A bounded subset A of a normed space X is precompact if and only if $\lim_n \delta_n(A, B_X) = 0$.

Proof. We recall that A is precompact if for every $\varepsilon > 0$ there exists a finite set $F \subset X$ such that $A \subset \varepsilon B_X + F$. Therefore, if A is precompact, then $\lim_n \delta_n(A, B_X) = 0$. The converse implication follows from the fact that every bounded subset of a finite-dimensional normed space is precompact. ■

The following theorem of Krasnoselski, Krein and Milman is the main technical tool for computing and estimating Kolmogorov diameters, cf [T].

1.5 Theorem. For every n-dimensional ($n\geq 1$) subspace H of a normed space X it is satisfied that $\delta_n(B_H, B_X) = 1$.

Proof. (following [DG]) As we have observed, the n-th Kolmogorov diameter of an arbitrary set A with respect to the unit ball B_X is equal to

$$\inf_{\dim L<n} \sup_{x\in A} \operatorname{dist}(x,L),$$

and since the vector 0 belongs to every subspace L, it follows that $\delta_n(B_H, B_X) \leq \sup_{x\in B} \|x-0\| = 1$. So it remains to prove the following fact:

(kkm) Let H and L be finite-dimensional subspaces of X. If $\dim L < \dim H$, then there is an $x_0 \in S_H$ such that $\operatorname{dist}(x_0, L) = \|x_0\| = 1$.

Proof. There is no loss of generality in assuming that $\dim H = \dim L + 1$. Suppose first that the norm $\| \ \|$ is strictly convex, i.e., $\|x+y\| < \|x\| + \|y\|$ whenever x, y are linearly independent. It is then easily seen that each $x \in X$, in particular, $x \in S_H$ has a unique nearest point $y = \varphi(x)$ and that the metric projection $\varphi: S_H \longrightarrow L$ is continuous. (Use standard Bolzano-Weierstrass argument: If $x_n \longrightarrow x$ in S_H then every convergent subsequence of the sequence $(\varphi(x_n))$ has the same limit $\varphi(x)$, and every subsequence of $(\varphi(x_n))$ contains a convergent sub-subsequence. Hence $\varphi(x_n) \longrightarrow \varphi(x)$). Furthermore the mapping φ has the property $\varphi(-x) = -\varphi(x)$ for all $x \in S_H$. Consequently by the Borsuk theorem ([DG],3.5.2), there is an $x_0 \in S_H$ such that $\varphi(x_0) = 0$. Clearly x_0 is the required point.

In the general case choose a basis $f_1, f_2, \ldots, f_n$, $n = \dim H$, in H^* and define

$$\|x\|_m = \sqrt{\|x\|^2 + m^{-1}(f_1(x)^2+\ldots+ f_n(x)^2)}$$

$\|x\|_m$ is a new and strictly convex norm in H For each $m\in\mathbb{N}$ there is $x_m \in S_H$ with $\text{dist}_m(x_m,L) = \|x_m - 0\|_m = 1$. Since $\|x_m\| < \|x_m\|_m = 1$, the sequence $\{x_m\}$ contains a convergent subsequence relative $\|\ \|$; the limit x_0 of this subsequence satisfies the requirements of the theorem. ∎

1.6 <u>Exercise.</u> Under the additional assumption, that the subspace is a range of a contractive projection of X, prove the last theorem without referring to the Borsuk theorem.

The additional assumption above is met in the applications presented in these Notes.

1.7 <u>Remark.</u> Assume now that X is a complex normed space. Let $\delta_n(A,B)$ be the n-th Kolmogorov diameter of A with respect to B, and let $\delta_n^{\mathbb{R}}(A,B)$ denote the n-th real Kolmogorov diameter, i.e., with the infimum taken over all real subspaces L of real dimension less than n. Obviously $\delta_{2n}^{\mathbb{R}}(A,B) \le \delta_n(A,B)$. Hence, by Theorem 1.5, if H is a subspace of X of dimension n, (and real dimension 2n) then $\delta_n(S_H,B_X) \ge \delta_{2n}^{\mathbb{R}}(S_H,B_X) \ge 1$. That means that Theorem 1.5 implies its complex version.

<u>Definition.</u> Let $T:X \longrightarrow Y$ be an operator acting between normed spaces. The n-th <u>Kolmogorov number</u> of T is defined by

$$d_n(T) := \delta_n(T(B_X),B_Y).$$

Obviously, $\|T\| = d_1(T) \ge d_2(T) \ge \ldots$

1.8 <u>Basic properties of Kolmogorov numbers</u>

Assume that $T: X \longrightarrow Y$ and $S: Y \longrightarrow Z$. Then

(i) $d_{n+m-1}(ST) \le d_n(T)d_m(S)$, in particular, $d_n(ST) \le d_n(T)\cdot\|S\|$ and $d_m(ST) \le \|T\|\cdot d_m(S)$ for every $n,m\in\mathbb{N}$.

(ii) If $\tilde{T}:\text{cpl } X \longrightarrow \text{cpl } Y$ is the continuous extension of the operator T, then $d_n(\tilde{T}) = d_n(T)$ for all $n\in\mathbb{N}$.

(iii) If Y is a linear subspace of Z and $J:Y \longrightarrow Z$ is the canonical inclusion then $d_n(JT) \le d_n(T)$ for all $n\in\mathbb{N}$; if Y is a range of a contractive linear projection of Z, in particular , if Z is a Hilbert space, then $d_n(JT) = d_n(T)$.

(iv) An operator T is compact, if an only if $\lim_n d_n(T) = 0$.

Proof. (i) Ignoring the trivial case when one of the right hand side factors is infinite, assume that $\alpha > d_n(T)$ and $\beta > d_m(T)$. By definition of the Kolmogorov numbers there exist subspaces L and Λ of Y and Z, respectively, with $\dim L < n$ and $\dim \Lambda < m$, such that

$$T(B_X)\subset L + \alpha B_Y \text{ and } S(B_Y)\subset \Lambda + \beta B_Z.$$

Therefore

$$ST(B_X)\subset S(L)+\alpha(\Lambda + \beta B_Z) \subset S(L) + \Lambda + \alpha\beta B_Z \text{ and } \dim(S(L)+\Lambda)<n+m-1$$

The statements (ii) and (iii) are obvious and (iv) is a direct consequence of 1.4. ■

We end this section with a useful fact concerning seminorm-balls.

1.9 Let X be a vector space, $p,q \in \mathcal{S}(X)$. If q dominates p then $d_n(\tilde{I}_{qp}) = d_n(I_{qp}) = \delta_n(U_q,U_p)$ for every $n\in\mathbb{N}$.

Proof. The first equality directly follows from 1.8 (ii). The second equality is a consequence of the following three easily verifiable facts:

$$x\in U_p \text{ iff } [x]_p\in B_{X_p} \; ; \; x \in U_q \text{ iff } [x]_q\in B_{X_q}.$$

For every subspace $L \subset X$ with $\dim L < n$ and for every $t > 0$ the condition $U_q \subset tU_p + L$ implies $B_{X_q} \subset tB_{X_p} + I_p(L)$.

For every subspace $H \subset X_p$ with $\dim H = k < n$ and for every $t > 0$ the condition $I_{qp}(B_{X_q}) \subset tB_{X_p} + H$ implies $U_q \subset tU_p + L$, L being any subspace of X generated by the vectors $x_1, \ldots, x_k$ such that $[x_1]_p, \ldots, [x_k]_p$ are linearly independent in the space H. ■

§2. Nuclear operators

We shall be concerned with the relations between the property $\sum_n n^{\alpha} d_n(T) < \infty$, $\alpha > 0$ of an operator and its nuclearity

<u>Definition.</u> Let X and Y be normed spaces. An operator $T: X \longrightarrow Y$ is said to be nuclear if there exist two sequences $\{f_n\} \subset X^*$ and $\{y_n\} \subset Y$ such that

$$\sum_n \|f_n\| \, \|y_n\| < \infty \quad \text{and} \quad T(x) = \sum_n f_n(x) y_n \quad \text{for all } x \in X.$$

The mentioned sequences are not unique and the expression $T = \sum_n f_n \otimes y_n$ is called a nuclear representation of T. The set $N(X,Y)$ of all nuclear operators from X into Y is a linear space and the function $\gamma: N(X,Y) \longrightarrow \mathbb{R}$ defined by

$$\gamma(T) := \inf \left\{ \sum_n \|f_n\| \, \|y_n\| \; ; \; T = \sum_n f_n \otimes y_n \right\}$$

is a norm on $N(X,Y)$.

It is convenient to assume that γ is defined for all operators and $\gamma(T) = \infty$ if T is not nuclear.

2.1 <u>Elementary properties of nuclear operators</u>. Assume that $T: X \longrightarrow Y$ is an operator. Then

(i) $\|T\| \le \gamma(T)$, whence nuclear operators are the $\| \; \|$-limits of finite rank operators, therefore they are compact.

(ii) If $S: Y \longrightarrow Z$, then $\gamma(ST) \le \min\{\gamma(S)\|T\|, \gamma(T)\|S\|\}$.

(iii) If Y is a Banach space and $S_n: X \longrightarrow Y$ are such that the series

$\sum_n \gamma(S_n) < \infty$, then $T := \sum_n S_n$ is a nuclear operator.

(iv) If Y is complete then $(N(X,Y),\gamma)$ is a Banach space.

(v) If $\tilde{T}$:cpl X $\longrightarrow$ cpl Y is the continuous extension of the operator T, then $\gamma(\tilde{T}) = \gamma(T)$.

(vi) Let Y_1 be a subspace of Y and let $T:X \longrightarrow Y$ be a nuclear operator with values in Y_1, and let $T_1:X \longrightarrow Y_1$ be the astriction of T, i.e., $T_1(x) = T(x)$ for $x \in X$. If either Y_1 is dense in Y or Y_1 is the range of a contractive linear projection, then T_1 is nuclear.

Proof of (iii). For each $n \in \mathbb{N}$ take a nuclear represenation

$$S_n = \sum_m f_m^n \otimes y_m^n \text{ with } \sum_m \|f_m^n\| \, \|y_m^n\| \le \gamma(S_n) + 2^{-n}.$$

Then $T(x) = \sum_n \sum_m f_m^n(x) y_m^n$ for all $x \in X$ and therefore $T \in N(X,Y)$.

(iv) easily follows from (iii).

(v) The inequality $\gamma(\tilde{T}_1) \le \gamma(T)$ is obvious. The other inequality follows from the argument below.

(vi). Assume that $T = \sum_n f_n \otimes y_n$ with $y_n \in Y$ is a nuclear representation of T. If Y_1 is dense in Y, then, for every $\varepsilon > 0$, each y_n is a sum of a series $\sum_m y_{nm}$ such that $\sum_m \|y_{nm}\| \le \|y_n\| + \varepsilon \cdot 2^{-n}$, whence $T_1 = \sum_{m,n} f_n \otimes y_{nm}$ is the nuclear representation; if $P:Y \longrightarrow Y_1$ is a contractive projection then $T_1 = \sum_n f_n \otimes P(y_n)$, the required nuclear representation.

Routine proofs of (i) and (ii) are left to the reader. ■

Let us note that the astrictions of nuclear operators need not be nuclear; they are called quasinuclear, see [P], sect. 3.2.

2.2. Lemma. Let Y be a normed space and Z one of its subspaces of dimension n. Then there exists a linear operator $P:Y \longrightarrow Z$ such that $P(z) = z$ for all $z \in Z$ and $\gamma(P) \le n$.

Proof. Let us take an Auerbach basis for Z, i.e. vectors $z_1,\ldots,z_n \in Z$ and functionals $f_1,\ldots,f_n \in Z^*$ such that $f_i(z_j) = \delta_{ij}$ and $\|z_i\| = \|f_j\|$ for $i,j \in \{1,\ldots,n\}$, cf. [W], 2E.11. Let g_i, $\|g_i\| = 1$, be the Hahn-Banach extension of f_i for $i=1,\ldots,n$. Then $P = \sum g_i \otimes z_i$ has the required property. ∎

2.3 If $T:X \longrightarrow Y$ is a rank n operator, i.e., $\dim T(X) = n$, then $\gamma(T) \le n\|T\|$.

Proof. $T = JT$, where J is the operator of the previous Lemma with $Z = T(X)$. Hence, by 2.1 (i), $\gamma(T) \le \gamma(J)\|T\|$. ∎

2.4 Exercise. Let X and Y be complex normed spaces, $T:X \longrightarrow Y$ an operator which admits a nuclear representation by means of real linear functionals defined on X. Show that T is nuclear.

Hint. If $T:X \longrightarrow Y$ is a complex linear operator, i.e., such that $T(x) = -iT(ix)$, and $T = \sum f_n \otimes y_n$ with real linear functionals f_n, $n \in \mathbb{N}$, then $T(x) = \sum \varphi_n(x) y_n$, where $\varphi_n(x) = \frac{1}{2}(f_n(x) - if_n(ix))$ are complex linear functionals with $\|\varphi_n\| \le \|f_n\|$.

An excursion to Hilbert spaces. We shall be concerned with real Hilbert spaces X, Y, Z, ... and compact operators acting between them. Inexplained terminology and the proofs which are omitted can be found in Schatten's book [Sch]. The reader interested in generalizations of the stated results to the cases of noncompact operator and to complex Hilbert spaces is also referred there.

Recall that for an operator $T: X \longrightarrow Y$, its Hilbert conjugate $T^*: Y \longrightarrow X$ is the only operator such that $(T(x)|y) = (x|T^*(y))$ for all $x \in X$, $y \in Y$. An operator $T: X \longrightarrow X$ is said to be hermitian if $T = T^*$. An hermitian operator T is positive if $(T(x)|x) \ge 0$ for every $x \in X$.

Recall the spectral theorem for compact hermitian operators.

2.5 Every compact hermitian operator $K: X \longrightarrow X$ can be expressed in the form

$$K(x) = \sum_n \lambda_n (x|f_n) f_n$$

where λ_n, $n\in\mathbb{N}$, are non-zero eigenvalues of K, repeated according to their multiplicity and (f_n) is the sequence of eigenvectors corresponding to these eigenvalues, i.e., $K(f_n) = \lambda_n f_n$. The values λ_n are real and $\lambda_n \longrightarrow 0$.

Clearly, the operator K is positive iff $\lambda \geq 0$ for all $n\in\mathbb{N}$.

<u>Definition</u>. The n-th <u>singular</u> <u>number</u> of a compact operator $T:X \longrightarrow Y$ is defined by $s_n(T) = \sqrt{\lambda_n}$, where $\{\lambda_n\}$ is the sequence of eigenvalues of the positive compact selfadjoint operator $K = T^*T$ put in the non-increasing order.

A consequence of the spectral theorem is the following representation theorem for compact operators:

2.6 Every compact operator $T: X \longrightarrow Y$ can be represented in the form

$$\sum_n s_n (x|e_n) f_n, \quad \text{where} \quad s_n = s_n(T),$$

where (e_n) and (f_n) are orthonormal systems in X and Y, respectively, for each $n\in\mathbb{N}$, e_n is the eigenvector of the operator T^*T corresponding to the eigenvalue s_n^2 and $Te_n = s_n f_n$.

2.7 For a compact operator $T: X \longrightarrow Y$ acting between Hilbert spaces the singular numbers are exactly the Kolmogorov numbers.

<u>Proof</u> Let $T = \sum_n s_n (x|e_n) f_n$, $H(m) = [f_1, \ldots, f_{m-1}]$. Then

$$T(B_X) = \{ \sum_n s_n t_n f_n ; \sum_n t_n^2 \leq 1\},$$

and, for every $x \in T(B_X)$, we have

$$\operatorname{dist}(x,H(m)) \le \sup\{\|\sum_{n\ge m} s_n t_n f_n\|;\ \sum_n t_n^2 \le 1\}$$

$$\le s_m \cdot m \cdot \sup\{\|\sum_{n\ge m} t_n e_n\|; \sum_n t_n^2 \le 1\} = s_m .$$

Hence $d_m(T) \le s_m$. On the other hand, $T(B_X)\cap H(m+1) \supset s_m B_{h(m+1)}$, and therefore, by theorem 1.5, $d_m(T) \ge s_m$. ∎

2.8 A compact operator $T: X \longrightarrow Y$ acting between Hilbert spaces is nuclear iff $\sum_n d_n(T) < \infty$. Furthermore $\gamma(T) = \sum_n d_n(T)$.

<u>Proof</u>. By 2.6, $T = \sum_n s_n f_n \otimes e_n$, whence $\gamma(T) \le \sum_n \|s_n f_n\|\cdot\|e_n\| = \sum_n s_n = \sum_n d_n(T)$. Therefore the condition $\sum_n d_n(T) < \infty$ implies the nuclearity of the operator T.

Now suppose that T admits the nuclear representation $T = \sum_n g_n \otimes y_n$. Let s_n, e_n, f_n, be those of 2.6. Fix an $n\in\mathbb{N}$. By Bessel's inequality we have

$$\|g_n\|^2 \ge \sum_j |(g_n|e_j)|^2 \quad ; \quad \|y_n\|^2 \ge \sum_j |(y_n|f_j)|^2$$

and, by the Cauchy-Schwartz inequality,

$$\sum_j |(g_n|e_j)|\,|(y_n|f_j)| \le \Big(\sum_j |(g_n|e_j)|^2\Big)^{1/2}\Big(\sum_j |(y_n|f_j)|^2\Big)^{1/2} \le \|g_n\|\ \|y_n\| .$$

Thus

$$\sum_j d_j(T) = \sum_j s_j = \sum_j (Te_j|f_j) = \sum_j \Big(\sum_n (g_n|e_j)y_n|f_j\Big) =$$

$$= \sum_j \sum_n (g_n|e_j)\ (y_n|f_j) \le \sum_n \sum_j |(g_n|e_j)|\ |(y_n|f_j)| \le \sum \|g_n\|\ \|y_n\| < \infty. \ ∎$$

<u>Back to normed spaces</u>. The following proposition is a link connecting nuclear operators acting in Hilbert spaces with those in Banach spaces.

2.9 <u>Lemma</u>. If X and Y are Banach spaces, then every nuclear operator $T:X \longrightarrow Y$ can be factored through the Hilbert space ℓ_2.

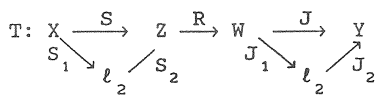
<u>Proof</u>. Choose a nuclear representation $T = \sum_n f_n \otimes e_n$ such that

$\|f_n\| = \|e_n\|$. Obviously $\sum_n \|f_n\|^2 = \sum_n \|e_n\|^2 < \infty$. The operators $T_1 : X \longrightarrow \ell_2$ and $T_2 : \ell_2 \longrightarrow Y$ defined by $T_1(x) = (f_n(x))$ and $T_2((a_n)) = \sum_n a_n e_n$ are both continuous and satisfy $T = T_2 T_1$. ■

Now we are ready to prove the main results of this section,

2.10 Let X and Y be normed spaces and $T : X \longrightarrow Y$ a composition of three nuclear operators. Then $\sum_n d_n(T) < \infty$.

<u>Proof</u>. According to 1.8 (ii),(iii) and 2.1 (v),(vi), we may pass to the completions of the spaces and assume that $T = JRS$ is the product of three nuclear operators: $S : X \longrightarrow Z$, $R : Z \longrightarrow W$ and $J : W \longrightarrow Y$, where X, Z, W, Y are Banach spaces. The previous lemma allows us to write T as the composition of the following operators:



Hence $T = J_2 H S_1$, where $H = J_1 R S_2$ is a nuclear operator. Since, by 2.8, $\sum_n d_n(H) < \infty$, we conclude that

$$\sum_n d_n(T) \le \sum_n \|J_2\| d_n(H) \|S_1\| < \infty. \quad ■$$

2.11 Let X,Y be normed spaces. If an operator $T : X \longrightarrow Y$ satisfies $\sum n^2 d_n(T) < \infty$, then T is nuclear.

<u>Proof</u>. Take an (n-1)-dimensional subspace $L_{n-1} \subset Y$ such that $\sup\{\mathrm{dist}(y, L_{n-1});\ y \in T(B_X)\} < 2d_n(T)$. By Lemma 2.2 there exists a projection $P_{n-1} : Y \longrightarrow L_{n-1}$ such that $\|P_{n-1}\| \le \gamma(P_{n-1}) \le n-1$.

We claim now that: $T(X) = \lim_n P_{n-1} T(x)$.

This can be proved as follows: take $x \in B_X$ and let $z \in L_{n-1}$ be the nearest point to $T(x)$. Then

$\|T(x) - P_{n-1}T(x)\| \leq \|T(x) - z\| + \|P_{n-1}T(x) - P_{n-1}(z)\| \leq 2d_n(T)$
$+ 2(n-1)d_n(T) = 2nd_n(T)$.

By the assumption on T, $\lim_n 2nd_n(T) = 0$.

Let us write $T(x) = P_1T(x) + \sum_2^\infty (P_n - P_{n-1})T(x)$. We have

$\|(P_n - P_{n-1})T\| = \|T-P_{n-1}T - (T-P_nT)\| \leq \|T-P_{n-1}T\| + \|T-P_nT\| \leq$
$\leq 2nd_n(T) + 2(n+1)d_{n+1}(T) \leq (4n+2)d_n(T)$.

Now, since $\text{rank}(P_nT - P_{n-1}T) \leq 2n-1$, by Proposition 2.3 we obtain:
$\gamma(P_nT - P_{n-1}T) \leq (2n-1)\|P_nT - P_{n-1}T\| \leq (2n-1)(4n+2)d_n(T) \leq 8n^2d_n(T)$.
Since T is a sum of a series of nuclear operators such that the sum of their nuclear norms is convergent, we conclude that T is nuclear. ■

§ 3. Locally convex nuclear spaces. Mitiagin's characterization

We recall that a topological vector space X is <u>locally convex</u> iff $\mathcal{S}(X)$ is fundamental.

It is clear that if X is locally convex and if $\mathcal{P} \subset \mathcal{S}(X)$ is fundamental, then every $p \in \mathcal{S}(X)$ is dominated by a seminorm $q \in \mathcal{P}$.

<u>Definition</u>. A locally convex space X is said to be <u>nuclear</u> if for every $p \in \mathcal{S}(X)$ there exists a $q \in \mathcal{S}(X)$ such that the linking operator I_{qp} is nuclear.

In the above definition the expression "$p \in \mathcal{S}(X)$" can be replaced by "$p \in \mathcal{P}$, a fundamental set of seminorms", and the operator I_{qp} can be replaced $\tilde{I}_{qp}$. This is a direct consequence of 2.1 and 1.8.

Now we are ready to state the Mitiagin characterization theorem.

3.1 <u>Theorem</u>. For a locally convex space X the following statements are equivalent:

(i) X is nuclear,

(ii) there is an $a > 0$ such that for every $p \in \mathcal{P}(X)$ there is a $q \in \mathcal{P}(X)$ with $q > p$ and $\sup_n n^a d_n(I_{qp}) < \infty$,

(*ii) there is an $a > 0$ such that for every $U \in \mathcal{U}(X)$ there is a $V \in \mathcal{U}(X)$ with $\sup_n n^a \delta_n(V,U) < \infty$,

(iii) for every $a > 0$ and for every $p \in \mathcal{P}(X)$ there is a $q \in \mathcal{P}(X)$ with $q > p$ and $\sup_n n^a d_n(I_{qp}) < \infty$,

(*iii) for every $a > 0$ and for every $U \in \mathcal{U}(X)$ there exists a $V \in \mathcal{U}(X)$ with $\sup_n n^a \delta_n(V,U) < \infty$,

3.2 Remark. Assume that $\mathcal{W} \subset \mathcal{U}(X)$ and $\mathcal{P} \subset \mathcal{P}(X)$ are fundamental. Then, by 1.2 and 1.8, in the statements (*ii), (*iii) above the $\mathcal{U}(X)$ may be replaced by $\mathcal{W}$ and in (ii), (iii) the $\mathcal{P}(X)$ may be replaced by $\mathcal{P}$. Also the link operators I_{qp} may be replaced by $\tilde{I}_{qp}$.

Proof. (i) $\Longrightarrow$(ii). For a fixed $p \in \mathcal{P}(X)$ let r, s and q be three continuous seminorms on X with $p \le r \le s \le q$ such that the corresponding operators I_{rp}, I_{sr} and I_{qs} are nuclear. Then, by 2.10, the operator $I_{qp} = I_{rp} I_{sr} I_{qs}$ satisfies $\sum_n d_n(I_{qp}) < \infty$. Hence the sequence $\{d_n^a(I_{qp})\}$ is bounded for $a = 1$.

(ii) $\Longrightarrow$(iii). Given $p \in \mathcal{P}(X)$ there exist a $q_1 \in \mathcal{P}(X)$, $q_1 > p$ and a constant $C_1 > 0$ such that $d_n(I_{q_1 p}) \le C_1 n^{-a}$ for all $n \in \mathbb{N}$. Next, there exist $q_2 \in \mathcal{P}(X)$, $q_2 > q_1$ and $C_2 > 0$ such that $d_n(I_{q_2 q_1}) \le C_2 n^{-a}$ for all n, whence $q_2 > p$ and, by 1.2,

$$d_{2n}(I_{q_2 p}) \le d_n(I_{q_2 q_1}) \cdot d_n(I_{q_1 p}) \le C_1 C_2 n^{-2a} \text{ for all } n.$$

Proceeding in the same way, for every $k \in \mathbb{N}$, we can find a $q_k \in \mathcal{P}(X)$, $q_k > p$, and $D_k > 0$ such that

$$d_{kn}(I_{q_k p}) \le D_k n^{-k \cdot a} \quad \text{for all } n \in \mathbb{N}.$$

Now, given $b > 0$, if $k \in \mathbb{N}$ is so big that $k \cdot a \ge 3b$, then

$$\sup m^b d_m(I_{q_k p}) < \infty$$

because for every $m \geq k^2$ there is an $n \geq k$ such that $kn \leq m \leq k(n+1)$ Thus $m^b d_m(I_{q_k p}) \leq m^b d_{kn}(I_{q_k p})$ and $m^b \leq k^b(n+1)^b \leq n^b(n^2)^b = n^{3b} \leq n^{k \cdot a}$.

(iii) $\Longrightarrow$(i). Let $p \in \mathcal{S}(X)$. By the hypothesis, we can pick a $q \in \mathcal{S}(X)$ so that $\sup n^4 d_n(I_{qp}) < \infty$. Then, by 2.11, I_{qp} is nuclear.

(*ii) $\Longleftrightarrow$(ii) & (*iii) $\Longleftrightarrow$(iii). By 1.9, $d_n(I_{qp}) = \delta_n(U_q, U_p)$. We complete the proof by applying Remark 3.2 with $\mathcal{P} = \{U_p;\ p \in \mathcal{S}(X)\}$.

§ 4 Nuclear Fréchet spaces with bases. Absoluteness

The aim of this paragraph is the Dynin Mitiagin criterion characterizing nuclear spaces among Fréchet spaces with bases, which implies that all bases of nuclear Fréchet spaces are absolute.

Assume that X is an infinite-dimensional Fréchet space. Recall that $\mathcal{S}(X)$ denotes the set of all continuous seminorms defined on X.

A sequence $(x_n) \subset X$ is said to be a basis [an absolute basis] of X if every $x \in X$ has a unique representation $x = \sum_n t_n x_n$ [and moreover $\sum_n p(t_n x_n) < \infty$ for every seminorm $p \in \mathcal{S}(X)$].

It is known that the <u>coeficient</u> <u>functionals</u> (f_n) of the basis defined by

$$f_k(\sum t_n x_n) = t_k, \text{ for } k \in \mathbb{N},$$

are continuous, cf. [R], sect. 2.6. Therefore by the Banach-Steinhau theorem the projectors P_n and $P_{n,m}$ defined by

$$P_n(x) = \sum_{i=1}^{n} t_i x_i, \quad P_{n,m} = P_{n+m} - P_n, \quad n,m \in \mathbb{N},$$

are equicontinuous, that means:

(*) $\forall p \in \mathcal{S}(X)\ \exists r \in \mathcal{S}(X)$ such that $p(P_{n,m}(x)) \leq r(x)\quad \forall x \in X,\ n,m \in \mathbb{N}$

In the sequel when writing "a basis (x_n, f_n)" we shall have in mind that (x_n) is a basis and (f_n) the sequence of its coeficient functionals.

Definition. A seminorm $p \in \mathcal{P}(X)$ is said to be *adjusted* [resp. ℓ_1 *adjusted*] to the basis (x_n, f_n) if $\| f_n \otimes x_n : X_p \longrightarrow X_p \| \le 1$ for every $n \in \mathbb{N}$, that is, if $p(f_n(x)x_n) \le p(x)$ [resp. $\sum_n p(f_n(x)x_n) = p(x)$] for all $x \in X$, $n \in \mathbb{N}$. A grading $\mathcal{G}$ is said to be *adjusted* [ℓ_1 *adjusted*] if each seminorm of the grading is adjusted [ℓ_1 adjusted].

4.2 Let X be a Fréchet space with a basis (x_n). For every seminorm $p \in \mathcal{P}(X)$ there is a seminorm $q \in \mathcal{P}(X)$ which is adjusted to the basis and such that $q \ge p$.

Proof Given p, let

$$q(x) = \sup \{p(P_{n,m}(x));\ \ n,m \in \mathbb{N}\}.$$

Then, clearly, $q \ge p$, and since $f_n \otimes x_n = P_{n-1,n}$, we get $q(f_n(x)x_n) = = p(f_n(x)x_n) \le q(x)$. The continuity of q follows from $q(x) \le r(x)$, with r selected according to the condition (*) above. ∎

The proof of the main result is based on a lemma concerning operators acting between Banach spaces. Recall that a rank one operator $F: X \longrightarrow X$ is an operator of the form $T = f \otimes x$, where $f \in X^*$, $x \in X$, $f \ne 0$, $x \ne 0$.

4.3 *Lemma*. Let Z and Y be normed spaces, $T: Z \longrightarrow Y$ an operator such that

(1) $$\sum_m m d_m(T) < \infty.$$

If $F_n: Z \longrightarrow Z$, $G_n: Y \longrightarrow Y$ are rank 1 operators such that:

(i) $F_n F_m = \delta_{nm} F_n$; $G_n G_m = \delta_{nm} G_n$,

(ii) $\|G_n\| = 1$ and the operators $P_n = \sum_{i=1}^{n} G_i$ are equicontinuous,

(iii) the set $[G_1(Y) \cup G_2(Y) \cup \ldots]$ is dense in Y,

(iv) $TF_n = G_n T;\quad T(F_n(Z)) = G_n(Y);\quad \|F_n\| = 1$

for all $n \in \mathbb{N}$. Then $\sum_n \|G_n T\| < \infty$.

<u>Proof</u> Denote $A_n = \|G_n T\|$, $n\in\mathbb{N}$, and observe that, by (iv), the image $G_n T(Z) = T(F_n(Z)) = G_n(Y)$ is one-dimensional and therefore $A_n \neq 0$.

Further argument will proceed in five steps.

1° $y = \lim_n P_n(y)$ for every $y \in Y$.

If $y \in [G_1(Y) \cup G_2(Y) \cup \ldots]$, say $y = G_1(y_1) + \ldots + G_m(y_m)$, then, by (i), $G_n(y) = G_n(y_n)$ for $n \le m$ and $G_n(y) = 0$ for $n > m$. Therefore

$$y = \lim_n P_n(y) \text{ for every } y \in [G_1(Y) \cup G_2(Y) \cup \ldots]$$

By (ii) and (iii) the same is true for every $y \in Y$.

2° $A_n \longrightarrow 0$.

In fact, if not, there would exist a bounded sequence $\{z_n\}$ in Z, a subsequence $\{k(n)\}$ of indices, and an $\varepsilon>0$, such that

$$\|G_{k(n)} T(z_n)\| \ge \varepsilon. \tag{2}$$

Since, by (1), the operator T is compact, we may assume without loss of generality that

$$G_{k(n)} T(z_n) \longrightarrow y \in Y.$$

But, by (i), $G_m(y) = \lim_n G_m G_{k(n)}(T(z_n)) = 0$ for every fixed $m\in\mathbb{N}$, whence, by 1°, $y = 0$, a contradiction with (2).

3° $\|T(u)\| = A_n \cdot \|u\|$ for every $u \in F_n(Z)$, $n\in\mathbb{N}$.

In fact, by (iv),

$$A_n = \sup\{\|TF_n(z)\|;\ z\in B_Z\} = \sup\{\|TF_n(F_n(z))\|;\ z\in B_Z\}$$

and $F_n(B_Z) \subset B_Z$. Therefore $A_n = \sup\{\|TF_n(u)\|; u \in B_Z \cap F_n(Z)\} = \|TF_n(u_0)\|$ for some $u_0 \in B_Z \cap F_n(Z)$. But since the range $F_n(Z)$ is one-dimensional, we get the equality 3°.

According to 2°, after passing to a permutation of indices, we may assume that

(3) $$A_1 \geq A_2 \geq A_3 \geq \dots$$

Consider the m-dimensional subspace $H(m) = G_1(Y) + \dots + G_m(Y)$, for a fixed $m \in \mathbb{N}$. We claim that

4° $B_{H(m)} \subset m \cdot A_m^{-1} T(B_Z)$.

Proof of 4°. Pick an arbitrary $y = G_1(y_1) + \dots + G_m(y_m) \in B_{H(m)}$. By (i), $G_n(y) = G_n(y_n)$ for $n \leq m$, i.e

$$y = G_1(y) + \dots + G_m(y).$$

By (iv) there are $u_n \in F_n(Z)$ such that $G_n(y) = TF_n(u_n)$ for $n \leq m$, whence $y = T(z)$ with $z = u_1 + \dots + u_m$. By 3°, $\|u_n\| = A_n^{-1}\|T(u_n)\| = A_n^{-1}\|G_n(y)\| \leq$ $\leq A_n^{-1}\|y\|$. Hence, by (3), $\|z\| \leq (A_1^{-1} + \dots + A_m^{-1})\|y\| \leq mA_m^{-1}\|y\|$, it means that $z \in mA_m^{-1} B_Z$.

5° Now we complete the proof of the lemma. Combining (3) with theorem 1.2 we get

$$md_m(T) = \delta_m(mT(B_Z), B_Y) = A_m \delta_m(mA_m^{-1} T(B_Z), B_Y) \geq A_m \delta_m(B_{H(m)}, B_Y) = A_m.$$

Hence the assumption (1) implies $\sum_m A_m < \infty$. ■

Now we can state the Dynin-Mitiagin criterion

4.4 Theorem. Let X be a Fréchet space with a basis (x_n, f_n). Then the following statements are equivalent:

(n1) X is nuclear,

(n2) $\forall\, p \in \mathcal{P}(X)\ \exists\, q \in \mathcal{P}(X)$, $q \geq p$, such that $\sum_n \|f_n \otimes x_n : X_q \longrightarrow X_p\| < \infty$.

(n3) $\forall\, p \in \mathcal{P}(X)\ \exists\, q \in \mathcal{P}(X)$, $q \geq p$, such that $\sum_n p(x_n)/q(x_n) < \infty$.

Proof. (n1) $\Longrightarrow$ (n2). Let $p \in \mathcal{P}(X)$. Choose $r, s, q \in \mathcal{P}(X)$, so that $p \leq r \leq s \leq q$, r and q are adjusted to the basis and $\sum_m md_m(I_{sr}) < \infty$. (This is possible because of 3.1(ii) and 4.2.), whence

(4) $$\sum_m md_m(I_{qp}) < \infty.$$

Let $M = \{n\in\mathbb{N};\ r(x_n) \neq 0\}$ and let $\{k(n)\}$ be the increasing sequence of all the the indices belonging to M. Let $Y = \tilde{X}_r$, $y_n = [x_{k(n)}]_r$, $z_n = [x_{k(n)}]_q$ for $n \in \mathbb{N}$, Z = the closure of the set $[\{z_n;n\in\mathbb{N}\}]$ in the Banach space $\tilde{X}_q$. Let $T:Z \longrightarrow Y$, $F_n:Z \longrightarrow Z$, $G_n:Y \longrightarrow Y$, $n\in\mathbb{N}$, be the continuous extensions of the operators defined by the formulas:

$$T([x]_q) = I_{qr}([x]_q) = \sum_n f_{k(n)}(x)[x_{k(n)}]_r = \sum_n f_{k(n)}(x)y_n,$$

$$F_n([x]_q) = f_{k(n)}(x)z_n,\quad G_n([x]_r) = f_{k(n)}(x)y_n.$$

Since $\{x_n,f_n\}$ is a basis in X and the seminorms r,q are adjusted, it easily follows that the conditions (i) - (iv) of the lemma **4.3** are met. The condition (4) is nothing else but the hypothesis (1).

Hence, applying the lemma with the specified above data, we get the statement (n2).

(n2) $\Longrightarrow$(n1). Obvious.

(n2) $\Longleftrightarrow$(n3). By **4.2**, the two statements remain unchanged when restricting to $p,q\in\mathcal{P}(X)$, adjusted to the basis. Hence the following observation completes the proof:

4.5 Let X be a Fréchet space with a basis $\{x_n,f_n\}$. If $p,q\in\mathcal{P}(X)$, $p\leq q$ and q is adjusted to the basis then, for every $n\in\mathbb{N}$,

$$\| f_n \otimes x_n : X_q \longrightarrow X_p \| = p(x_n)/q(x_n) \quad (0/0 = 0).$$

<u>Proof</u> Fix $n\in\mathbb{N}$, denote $F = f_n \otimes x_n$. Since q is adjusted, we have $q(F(x)) \leq q(x)$ for every $x\in X$. Hence

$$p(f_n(x)x_n) = q(f_n(x)x_n)\cdot p(x_n)/q(x_n) \leq q(x)\cdot p(x_n)/q(x_n).$$

On the other hand, $p(f_n(x_n)x_n) = p(x_n) = q(x_n)\cdot p(x_n)/q(x_n)$. That means:

$$p(F(x)) \leq q(x)\cdot p(x_n)/q(x_n) \text{ and } p(F(x_n)) = q(x_n)\cdot p(x_n)/q(x_n),$$

i.e., $\|F:X_q \longrightarrow X_p\| = p(x_n)/q(x_n)$. ∎

4.6 Corollary. Every basis in a nuclear Fréchet space is absolute.

Proof. Assume that (x_n, f_n) is a basis in a nuclear Fréchet space X. Let $p \in \mathscr{P}(X)$. Choose q according to the condition (n2) of 4.4. Since, for each $x \in X$, the series $\sum_n f_n(x)x_n$ is convergent, we conclude that $\sup_n q(f_n(x)x_n) < \infty$, whence by (n2) $\sum_n p(f_n(x)x_n) < \infty$. ■

There is only one type of absolute bases of infinite-dimensional, separable Banach spaces: the unit vector basis of the space ℓ_1. More precisely:

If $(x_n, f_n)_{n\in\mathbb{N}}$ is an absolute basis in a Banach space X with $\|x_n\| = 1$ for all $n \in \mathbb{N}$, then the map

$$X \ni x \longmapsto (f_n(x)) \in \ell_1$$

is an isomorphism which takes the basis (x_n) of X onto the unit vector basis (e_n) of the space ℓ_1.

Definition. Let $A = (a_{kn})_{k,n\in\mathbb{N}}$ be a Köthe matrix, i.e. a matrix of real numbers such that, for every $n \in \mathbb{N}$,

$$0 \le a_{1n} \le a_{2n} \le \ldots \text{ and } \lim_k a_{kn} > 0.$$

The Köthe space $\ell_1(A)$ is the linear space of all numerical sequences $x = (x(n))$ such that, for each $k \in \mathbb{N}$, $p_k(x) = \sum_n |a_{kn}x(n)| < \infty$, regarded as a Fréchet space with the grading $\mathscr{G} = (p_k)_{k\in\mathbb{N}}$.

Clearly, the sequence (e_m) of unit vectors, i.e., $e_m(n) = \delta_{nm}$, is an absolute basis of the space $\ell_1(A)$ and the grading $\mathscr{G}$ is ℓ_1 adjusted to this basis.

The theorem 4.7 below provides the complete description of absolute bases in Fréchet spaces in terms of Köthe spaces $\ell_1(A)$.

4.7 Theorem. Let $(x_n, f_n)_{n\in\mathbb{N}}$ be an absolute basis in a Fréchet space X and let (p_k) a grading for X, and $A = (a_{kn}) = (p_k(x_n))$. Then the map

$$X \ni x \longmapsto (f_n(x)) \in \ell_1(A)$$

is an isomorphism which takes the basis (x_n) of X onto the unit vector basis (e_n) of the space $\ell_1(A)$.

Proof. Obviously the linear mapping $T:X \longrightarrow \ell_1(A)$ defined by the formula $T(x) = (f_n(x))$ is bijective. Since, for, for every $k\in\mathbb{N}$, $p_k(x) \le \le \sum_n p_k(f_n(x)x_n)$, the operator T^{-1} is continuous. And since

$$T(x) = \sum_n f_n(x)e_n \text{ for } x\in X,$$

the operator T is the limit of continuous linear mappings (the partial sums of the series) and, by 0.2, is continuous. ■

An immediate corollary of the last theorem is the following:

4.8 A basis (x_n) of a Fréchet space X is absolute if and only if X admits a grading ℓ_1 adjusted to the basis.

§ 5. The uniqueness problem. Regular bases

In this section we consider only infinite-dimensional Fréchet spaces.

Corollary **4.6** together with the observation concerning absolute bases of Banach spaces, motivates the study of uniqeness of bases in nuclear Fréchet spaces. We precise the sense of the word "unique":

Definition. Let (x_n, f_n) be a basis of a Fréchet space X. The coordinate range of the basis is the set of coeficients:

$$\text{Cr } (x_n) = \{(t_n);\ \sum_n t_n x_n \text{ converges}\} = \{(f_n(x));\ x\in X\}.$$

Bases (x_n) and (y_n) of Fréchet spaces X and Y, respectively, are said to be equivalent if $\text{Cr } (x_n) = \text{Cr } (y_n)$; are said to be diagonally

$\underline{\text{equivalent}}$ if there exists a sequence of non-zero scalars (t_n) such that the (x_n) is equivalent to the basis $(t_n y_n)$; finally (x_n) and (y_n) are called quasi-equivalent if there is a permutation $\pi:\mathbb{N} \longrightarrow \mathbb{N}$ such that $(y_{\pi(n)})$ is a basis diagonally equivalent to (x_n).

We have

5.1 If (x_n, f_n) and (y_n, g_n) are bases of Fréchet spaces X and Y, respectively, and if Cr (x_n) = Cr (y_n), then the mapping $x_n \longmapsto y_n$, $n \in \mathbb{N}$, uniquely extends to an isomorphism $T:X \longrightarrow Y$.

Proof. Let $T(x) = \sum_n f_n(x) y_n$, whence $T^{-1}(y) = \sum_n g_n(x) x_n$. The continuity of T and T^{-1} follows from 0.2. ∎

Let us note that the theorem 4.7 together with corollary 4.6 say that every basis of an infinite-dimensional nuclear Fréchet space is equivalent to the unit vector basis of a space $\ell_1(A)$ with a suitable matrix A.

Except the case where X is isomorphic to the space $\mathbb{R}^{\mathbb{N}}$ of all numerical sequences, if (x_n) is a basis in X one can always find a sequence (t_n) of positive scalars such that (x_n) and $(t_n x_n)$ are not equivalent; also it can be proved that a permutation can change the diagonal-equivalence type of the basis. Hence, the most convenient is the concept of the quasi-equivalence. The general question of whether in every nuclear Fréchet space with a basis all the bases are quasi-equivalent is still open. One of the partial solutions is the theorem, established independently by Kondakov and by Crone and Robinson, related to the concept of a regular basis.

Definition. Assume that X is a locally radially bounded Fréchet space and $\mathcal{N}(X)$ the fundamental set of seminorms consisting of all continuous norms. A basis (x_n) of the space X is said to be <u>regular</u> if for every $p \in \mathcal{N}(X)$ there is a $q \in \mathcal{N}(X)$ such that

$$p(x_n)/q(x_n) \geq p(x_{n+1})/q(x_{n+1}) \quad \text{for all } n \in \mathbb{N}.$$

5.2 <u>Theorem</u>. Any two regular absolute bases (x_n) and (y_n) of an arbitrary Fréchet space are diagonally equivalent.

<u>Proof</u>. The argument presented here, based on the Kolmogorov diameters, is due to Djakov [Dj].

We can take gradings $\mathcal{P} = (p_k) \subset \mathcal{N}(X)$ and $\mathcal{H} = (q_k) \subset \mathcal{N}(X)$ which are ℓ_1 adjusted to the bases (x_n) and (y_n), respectively, and such that

(1) the sequences $(p_k(x_n)/p_{k+1}(x_n))_{n\in\mathbb{N}}$ and $(q_k(y_n)/q_{k+1}(y_n))_{n\in\mathbb{N}}$ are non-increasing for all $k\in\mathbb{N}$.

Without loss of generality we may assume that

$$p_1 \leq q_1 \leq p_2 \leq q_2 \leq \dots,$$

for otherwise, we pass to suitable subsequences of the norms and replace them by their positive multiples.

Let $W_k = U_{p_k}$ and $V_k = U_{q_k}$, the unit balls of the normed spaces (X,p_k) and (X,q_k) respectively. The respective Kolmogorov diameters are expressed by

$$d_n(W_{k+h},W_k) = p_k(x_n)/p_{k+h}(x_n), \quad d_n(V_{k+h},V_k) = q_k(y_n)/q_{k+h}(y_n),$$

for $k,h,n\in\mathbb{N}$. This follows from the results of § 1 and the inclusions:

$$W_{k+h} \subset \rho W_k + Z(n), \quad \rho B_{Z(n)} \subset W_{k+h},$$

where $\rho = p_k(x_n)/p_{k+h}(x_n)$, $Z(n) = [x_1,\dots,x_{n-1}]$, $B_{Z(n)} = W_k \cap Z(n)$, and from the corresponding inclusions for V_{k+h} and V_k.

Therefore

(2) if $k \leq j$ then $W_k \supset V_k \supset V_j \supset W_{j+1}$ and $d_n(W_{j+1},W_k) \leq d_n(V_j,V_k)$, whence $p_k(x_n)/p_{j+1}(x_n) \leq q_k(y_n)/q_j(y_n)$,

(3) if $k > j$ then $V_j \supset W_{j+1} \supset W_k \supset V_k$ and $d_n(V_k,V_j) \le d_n(W_k,W_{j+1})$,

whence $q_j(y_n)/q_k(y_n) \le p_{j+1}(x_n)/p_k(x_n)$.

By (2) and (3)

$$p_k(x_n)/q_k(y_n) \le p_{j+1}(x_n)/q_j(y_n) \quad \text{for all } k,j,n \in \mathbb{N}.$$

Let $r_n = \sup \{p_k(x_n)/q_k(y_n); k\in\mathbb{N}\}$. Then $p_k(x_n) \le r_n q_k(y_n) \le p_{k+1}(x_n)$ for all $k,n \in \mathbb{N}$.

Hence $\mathrm{Cr}\,\{x_n\} = \ell_1(\{p_k(x_n)\}) = \ell_1(\{r_n q_k(y_n)\}) = \mathrm{Cr}\,\{r_n y_n\}$. ■

5.3 <u>Theorem</u>. Let X be a nuclear Fréchet space with a regular basis $\{x_n,f_n\}$. Then, for every basis $\{y_n,g_n\}$ of the space X there is a permutation $\pi:\mathbb{N} \longrightarrow \mathbb{N}$ such that $\{y_{\pi(n)}\}$ is a regular basis. Consequently all the bases in X are quasi-equivalent.

For the proof we need two lemmas concerning rank one projections.

5.4 <u>Lemma</u>. Let X be a Fréchet space with an absolute basis $\{x_n,f_n\}$ and let $\mathcal{G} = \{p_k\}$ be a grading ℓ_1 adjusted to the basis and such that

(i) $p_{k+1} \ge 2^k p_k$ for every $k\in\mathbb{N}$.

Assume that $F = g \otimes y: X \longrightarrow X$ is a rank one projection, i.e.,

(ii) $g(y) = 1$,

such that

(iii) $p_k \circ F \le p_{k+1}$ for all $k\in\mathbb{N}$.

Then there exist an index $m \in \mathrm{Supp}\, y := \{n\in\mathbb{N};\ f_n(y) \ne 0\}$ and a vector ty in the range of F such that

$$p_k(x_m) \le p_{k+1}(ty) \le p_{k+2}(x_m) \quad \text{for all } k\in\mathbb{N}.$$

<u>Proof</u>. Let $\nu\in\mathbb{N}$ be the first integer such that $p_{\nu+1}(y) \ne 0$. Then, by (i) and the fact that the grading $\mathcal{G}$ is ℓ_1 adjusted, we get

$$p_{k+1}(y)^{-1}\sum_n |f_n(y)|\,p_k(x_n) \le p_k(y)/p_{k+1}(y) \le 2^{-k} \text{ for every } k \ge \nu$$

whence

(A) $\sum_n |f_n(y)| \sup_{k\geq\nu} p_k(x_n)/p_{k+1}(y) \leq \sum_n |f_n(y)| \sum_{k\geq\nu} p_k(x_n)/p_{k+1}(y) \leq$

$\leq \sum_{k\geq\nu} p_{k+1}(y)^{-1} \sum_n p_k(x_n)|f_n(y)| \leq 1.$

By (iii), $|g(x_n)| \leq p_{k+1}(x_n)/p_k(y)$ for every $n\in\mathbb{N}$ and $k>\nu$. Hence,

$$|g(x_n)| \leq \inf_{k>\nu} p_{k+1}(x_n)/p_k(y)$$

Therefore, by (ii),

(B) $1 = \sum_n f_n(y)g(x_n) \leq \sum_n |f_n(y)||g(x_n)| \leq \sum_n |f_n(y)| \inf_{k>\nu} p_{k+1}(x_n)/p_k(y)$

Comparing (A) and (B), we conclude that there is an $m \in \operatorname{Supp} y$ such that

$$\sup_{k\geq\nu} p_k(x_m)/p_{k+1}(y) \leq \inf_{k>\nu} p_{k+1}(x_m)/p_k(y) > 0.$$

Taking $t = \inf_{k>\nu} p_{k+1}(x_m)/p_k(y)$ we get

$$p_k(x_m) \leq tp_{k+1}(y) \leq p_{k+2}(x_m), \text{ for every } k \geq \nu.$$

If $k < \nu$ then $p_k(y) = p_{k+1}(y) = 0$ and, since $m \in \operatorname{Supp} y$ and $\{p_k\}$ is ℓ_1 adjusted, also $p_k(x_m) = 0$. Therefore the assertion is proved. ■

We shall need the following concept

<u>Definition</u> Let X be a Fréchet space and let $\{x_n\}$, $\{y_n\}$ be two arbitrary sequences of elements of X. We say that $\{y_n\}$ is <u>pseudodominated</u> by $\{x_n\}$ if there exist a sequence of indices $\{m(n)\}$ with $m(n) \longrightarrow \infty$, a sequence $\{t_n\}$ of positive numbers and a grading $\mathcal{G} = \{p_n\}$ for the space X such that

(pd) $\quad p_k(x_{m(n)}) \leq p_{k+1}(t_n y_n) \leq p_{k+2}(x_{m(n)})$ for all $k,n \in \mathbb{N}$.

5.5 <u>Lemma</u>. Let X be a Fréchet space and Y a closed subspace with dim Y = ∞ such that there exists a continuous linear projection P of X onto Y. Let $\{x_n\}$ and $\{y_n\}$ be bases of X and Y, respectively. If X does not contain any subspace isomorphic to the Banach space ℓ_1, in particular if X is nuclear, then $\{y_n\}$ is pseudodominated by $\{x_n\}$.

Proof. Since all the rank one projections $F_n = g_n \otimes y_n$ are equicontinuous (see (*) in § 4), there is a grading $\mathcal{P} = (p_n)$ satisfying the condition (i) of lemma 5.4 and such that each F_n, $n \in \mathbb{N}$, fulfills (ii) and (iii). Hence, for each $n \in \mathbb{N}$, we can select an $m(n)$ and t_n to satisfy the condition (pd). It remains to show that $m(n) \longrightarrow \infty$. Otherwise there would exist an m_0 with $m(n) = m_0$ for infinitely many indices n, and the corresponding subsequence of the basis $(t_n y_n)$ would be equivalent to the unit vector basis of ℓ_1. Finally let us remark that, if X is nuclear, then so is Y and Y cannot be isomorphic to the the infinite-dimensional Banach space ℓ_1. ■

Now to complete the proof of the theorem 5.3 it is enough to make the following trivial observation

5.6. If (x_n) and (y_n) are absolute bases in the spaces X and Y respectively, (x_n) is regular and (y_n) is pseudodominated by (x_n) and if $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ is a permutation which makes the sequence $(m(n))$ appearing in the condition (pd) to tend non-decreasingly to infinity, then $(y_{\pi(n)})$ is a regular basis. ■

Another consequence of lemma 5.5 is the following fact (stated already in [B] in terms of infinite systems of equations):

5.7 Theorem. Every basis (x_n) of the space $\mathbb{R}^{\mathbb{N}}$ is equivalent to the unit vector basis.

Proof. By 5.5, $Cr(x_n) = \{ (c_n) ; \sum_n c_n t_n e_{m(n)}$ is convergent$\} = \mathbb{R}^{\mathbb{N}}$, the set of all numerical sequences. ■

§ 6 Nuclear Fréchet spaces without bases

Recall that, by the theorem 4.4, to each basis of a nuclear

Fréchet space corresponds a sequence of rank one operators $\{f_n \otimes x_n\}$ such that

(npi) $\forall\, p \in \mathcal{P}(X)\ \exists\, q \in \mathcal{P}(X),\ q \geq p,$ such that $\sum_n \| f_n \otimes x_n : X_q \longrightarrow X_p \| < \infty.$

Such a sequence of operators will be called an <u>npi</u>, the abbreviation for nuclear partition of the identity.

We shall present a nuclear Fréchet space E without any npi, which is a sleight modification of Djakov - Mitiagin example [DjM], cf. also [Be] and [BeDu]. The construction is based on a geometrical property of the 2-dimensional space $\mathbb{R}^2$.

Let $\{e_1, e_2\}$ be the canonical basis of $\mathbb{R}^2$ and let $e_1^*, e_2^* \in (\mathbb{R}^2)^*$ its coeficient functionals. Denote $w_1 = e_1 + e_2$, $w_1^* = e_1^* + e_2^*$ and $w_2^* = e_2^* - e_1^*$

6.1 <u>Lemma</u>. For every $u \in \mathbb{R}^2$, $v^* \in (\mathbb{R}^2)^*$ the rank ($\leq$) one operator $T = v^* \otimes u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ satisfies the inequality

$$|e_1^* T(e_1)| \leq |e_2^* T(e_1)| + |e_1^* T(e_2)| + |w_2^* T(w_1)|.$$

<u>Proof</u>. Assume that the left-hand side of the inequality is not zero. The substitution $s = |e_2^*(u)/e_1^*(u)|$, $t = |v^*(e_2)/v^*(e_1)|$ reduces the inequality to the elementary fact: $1 \leq s+t+(1-s)(1-t)$ for $s,t \geq 0$. ■

For a <u>fixed</u> $n \in \mathbb{N}$ define on the space $\mathbb{R}^2$ the three norms:

$$|x|_1 = |e_1^*(x)| + 2^n |e_2^*(x)|,$$
$$|x|_2 = 4^n |e_1^*(x)| + 2^n |e_2^*(x)|,$$
$$|x|_3 = 4^n |w_1^*(x)| + 8^n |w_2^*(x)|.$$

For an operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $i \in \{1,2,3\}$ we denote

$$|T|_{ii} = \sup \{ |T(x)|_i ;\ |x|_i \leq 1 \}.$$

It is straightforward to check that

$$|x|_1 \leq |x|_2 \leq |x|_3 \text{ for all } x \in \mathbb{R}^2$$

and

$$|e_1^* \otimes e_1|_{11} = |e_1^* \otimes e_2|_{22} = \tfrac{1}{2}\ |w_2^* \otimes w_1|_{33} = 2^{-n}.$$

6.2. <u>Lemma</u>. Let $T_m:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $m\in\mathbb{N}$, be rank one operators and let $\sum_m T_m(x) = x$ for every $x\in\mathbb{R}$. If there exists a constant $C < \infty$ such that

$$\sum_m |v^* T_m(u)| \le C|v^* \otimes u|_{jj} \text{ for every } u \in \mathbb{R}^2 X, v^* \in (\mathbb{R}^2)^*, \ j\in\{1,2,3\}$$

then $C \ge 2^n/4$.

<u>Proof</u>. Taking $v^*\otimes u = e_2^*\otimes e_1$, next $e_1^*\otimes e_2$, and next $w_2^*\otimes w_1$ we get

$$\sum_m |e_2^* T_m(e_1)| \le 2^{-n}C \ , \ \sum_m |e_1^* T_m(e_2)| \le 2^{-n}C \ , \ \sum_m |w_2^* T_m(w_1)| \le 2^{1-n}C \ .$$

Hence by lemma 6.1, $1 = e_1^*(e_1) = \sum_m e_1^* T_m(e_1) \le \sum_m |e_1^* T_m(e_1)| \le 4C2^{-n}$.

Therefore $C \ge 2^n/4$. ■

Let $\mathbb{K} = \{(i,j) \in \mathbb{N}\times\mathbb{N};\ i+1 < j\}$ and let $\sigma:\mathbb{N} \longrightarrow \mathbb{K}$ be a surjective mapping such that, for every $(i,j) \in \mathbb{K}$, the set $\sigma^{-1}(i,j)$ is infinite.

With the same fixed $n\in\mathbb{N}$ we denote by X_n the space $\mathbb{R}^2$ equipped with the sequence of norms $\{\|\cdot\|_k\}$ defined by

$$\|x\|_k = \begin{cases} |x|_1 & \text{for } k \le i \\ |x|_2 & \text{for } i + 1 \le k \le j \\ |x|_3 & \text{for } j > k \end{cases}$$

where $(i,j) = \sigma(n)$.

Till this moment we have considered a fixed $n \in \mathbb{N}$ and a fixed space X_n, from now on we shall be dealing with the sequence $\{X_n\}_{n\in\mathbb{N}}$. Of course, if $n,m\in\mathbb{N}$ with $\sigma(n) \ne \sigma(m)$, then the norms $\{\|\ \|_k\}$ on the space X_n are not the same as the norms on the space X_m denoted by the same symbol. This should not create confusions !

Here is the promissed example:

6.3 <u>Example</u>. The space

$$E = \{\mathbf{x} = \{x_n\};\ x_n\in X_n \text{ and } p_k(\mathbf{x}) := \sum_n n^k \|x_n\|_k < \infty \text{ for every } k\in\mathbb{N}\}$$

equipped with the grading $\mathcal{G} = \{p_k\}$ is a nuclear Fréchet space which does not admit any npi; therefore E has no basis in.

We omit the routine verification that E is complete. The nuclearity of E follows from the fact that, for each $k\in\mathbb{N}$, the linking operator $T = I_{p_{k+3},p_k}$ is a sum of a series $\sum_n T_n$ of rank two operators such that $\|T_n\| \le n^{-3}$ and therefore, by 2.3, the nuclear norms $\gamma(T_n) \le n^{-2}$, whence $\sum_n \gamma(T_n) < \infty$. Thus T is a nuclear operator.

Suppose that E has an npi $\{F_m\}$, $F_m = y_m^* \otimes y_m$ for $m\in\mathbb{N}$. Then

(1) $$\sum_m F_m(y) = y \text{ for all } y \in E,$$

and acording to the condition **(npi)** we have:

(2) for $p = p_1$ there is $q = p_i$ $(i > 1)$ such that

$$\sum_m p_1(F_m(y)) \le p_1(y)\cdot \sum_m \|y_m^* \otimes y_m : E_{p_i} \longrightarrow E_{p_1}\|$$

(3) for $p = p_{i+1}$ there is $q = p_j$ $(j > i+1)$ such that

$$\sum_m p_{i+1}(F_m(y)) \le p_j(y)\cdot \sum_m \|y_m^* \otimes y_m : E_{p_{i+1}} \longrightarrow E_{p_j}\|$$

(4) for $p = p_{j+1}$ there is $q = p_k$ $(k > j+1)$ such that

$$\sum_m p_{j+1}(F_m(y)) \le p(y)\cdot \sum_m \|y_m^* \otimes y_m : E_{j+1} \longrightarrow E_{p_k}\|.$$

Let i, j be those appearing in the last estimates. Take an arbitrary fixed $n \in \sigma^{-1}(i,j)$ and let $\iota_n : X_n \longrightarrow E$ be the canonical embedding and $P_n : E \longrightarrow X_n$ the canonical projection: $P_n(y) = y_n$. Finally let $T_m = P_n F_m \iota_n$ regarded as an operator acting on $\mathbb{R}^2$. Then, from the definition of the norms $\{p_n\}$ together with the statements (1), (2), (3) we conclude that every $x \in \mathbb{R}^2$ is the sum

$$x = \sum_m T_m(x) \text{ and}$$

and

$$\sum_m n|T_m(x)|_1 \le Cn^i|x|_1,\ \sum_m n^{i+1}|T_m(x)|_2 \le Cn^j|x|_2,\ \sum_m n^{j+1}|T_m(x)|_3 \le Cn^k|x|_3.$$

Hence, with $s = \max\{i-1, j-i-1, k-j-1\}$, we have

$$\sum_m |T_m(x)|_\alpha \le Cn^s|x|_\alpha \text{ for } \alpha \in \{1,2,3\},$$

and by lemma 6.2, $C \ge 2^{n-2}n^{-s}$, and the last estimate must hold for all n in the infinite set $\sigma^{-1}(i,j)$, a contradiction. ∎

§ 7. Notes and comments

Ad § 0

Locally convex spaces have been distinguished by Tychonoff [T]. Fréchet spaces (called B_0) spaces have been defined by S. Mazur and W. Orlicz in the context of summability theory and later re-discovered by French mathematicians from the Bourbaki circle. On the best of my knowledge the first published paper in which the term "B_0 space" appears is Eidelheit's [E], 1936; the treatise [MO] of Mazur and Orlicz devoted to a systematic study of these spaces appeared only after the Second World War. The French School, in contrast to Mazur and Orlicz, put the emphasis on infinite-dimensional locally convex spaces with such properties which are shared by Banach spaces of a finite dimension only, rather than looking for analogies with the general (infinite-dimensional) Banach space theory. In this context the classes of Montel, Schwartz and nuclear spaces have been defined.

Theorem 0.1 expresses the well-known fact that Fréchet spaces are barrelled.

Ad § 1

More about Kolmogorov diameters and their applications in the approximation theory can be found in V. Tikhomirov's paper [Ti]. For relations of Kolmogorov diameters and numbers with similar parameters, e.g. Gelfand diameters and numbers, see A. Pietsch [Pi].

Ad §§ 2 & 3

Nuclear spaces and nuclear operators were introduced in early fifties by A. Grothendieck in, see [G]. The attempts to understand the so called Red Book [G] stimulated the study of nuclear spaces and related topics of the operator theory in Eastern Europe, in G.D.R., Poland and the Soviet Union. The appearance of the beautiful monograph [P] of Albrecht Pietsch, helped to clarify the general notion of an operator ideal (Pietsch [Pi]) and was a beginning of the intensive research on absolutely summing operators (Kwapień, Lindenstrauss, Pełczyński, Pietsch and others).

Ad § 4

According to corollary 4.5 every basis in a nuclear Fréchet space is absolute. This property characterizes nuclear Fréchet spaces among Fréchet spaces [Wo].

Every absolute basis is <u>unconditional</u>, i.e., the expansions with respect to the basis are unconditionally convergent. For a discussion on unconditional bases in Banach spaces and the existence and uniqueness problems for them see [D].

Pełczyński and Singer [PS] found that every infinite-dimensional Banach space with basis admits two bases which are not diagonally equivalent and, at least, one of them is not unconditional.

* * *

For a Köthe matrix $A = \{a_{kn}\}$ one can also define the spaces $\ell_\alpha(A) = \{x = (\xi_n); \{a_{kn}\xi_n\} \in \ell_\alpha$ for every $k\in\mathbb{N}\}$; $p_k(x) = \|\{a_{kn}\xi_n\}\|_{\ell_\alpha}$. for $1 \le \alpha \le \infty$, and similarly $c_0(A)$. These spaces are nuclear, if and only if the matrix A satisfies the additional condition

(Kn) $\forall j\in\mathbb{N}\ \exists k\in\mathbb{N}$ such that $\sum_n a_{jn}/a_{kn} < \infty$.

Under this condition the unit vectors $\{e_n\}$ constitute an absolute basis in each of the spaces and therefore $c_0(A) = \ell_\alpha(A) = \ell_1(A)$ for every $1 \le p \le \infty$; the spaces are isomorphic under the identity mapping.

Ad § 5

We do not know any "natural" example of a nuclear Fréchet space without a basis. In this context it is important to be able to represent a given functional nuclear Fréchet as a Köthe spaces $\ell_1(A)$ with a relatively simple structure.

Specially interesting are the spaces $\ell_1(A)$ which are generated by a single sequence, among them power series spaces of types ∞ and 0.

Let $\{a_n\}$ and $\{b_n\}$ be sequences of reals such that

(1) $1 \le a_1 \le a_2 \le \dots\ ;\ 1 \ge b_1 \ge b_2 \ge \dots \ge 0.$

Consider the Köthe matrices $A = \{a_n^k\}$ and $B = \{b_n^{1/k}\}$. The spaces $\ell_1(A)$ and $\ell_1(B)$ are called <u>power</u> <u>series</u> <u>spaces</u> of <u>type</u> ∞ and 0 respectively.

The nuclearity of power series spaces is characterized by the conditions

$\exists k\in\mathbb{N}$ with $\sum_n a_n^{-k} < \infty$, and $\forall s>0\ \sum_n b_n^s < \infty$, respectively.

Observe that the unit vectors $\{e_n\}$ of a power series space constitute an absolute and regular basis. Using 5.2 it is not difficult to show that a power series space of type ∞ is never isomorphic to the one of type 0. We also have

7.1 Power series spaces X and Y of the same type are isomorphic if and only if they are equal as sets of numerical sequences.

<u>Proof</u> (Sketch). Recall that $W_k = \{x; p_k(x)<1\}$ for $k \in \mathbb{N}$. By (1), if $i < j$, then

$$d_n(W_j, W_i) = a_n^{i-j} \quad \text{for the space } \ell_1(\{a_n^k\}),$$

and

$$d_n(W_j, W_i) = b_n^{1/i-1/j} \quad \text{for the space } \ell_1(\{b_n^{1/k}\}).$$

Hence

$$\ell_1(\{a_n^k\}) = \{\{t_n\};\ \exists i\ \forall j \text{ with } \sum_n |t_n|/d_n(W_j, W_i) < \infty\} =$$

$$= \{\{t_n\};\ \exists U\ \forall V \text{ with } \sum_n |t_n|/d_n(V, U) < \infty\},$$

and similarly

$$\ell_1(\{b_n^{1/k}\}) = \{\{t_n\};\ \forall i\ \exists j \text{ with } \sum_n |t_n|\cdot d_n(W_j, W_i) < \infty\} =$$

$$= \{\{t_n\};\ \forall U\ \exists V \text{ with } \sum_n |t_n|\cdot d_n(V, U) < \infty\},$$

where V and U run over all zero-neighbourhoods. That means that the two classes of numerical sequences can be described invariantly. ■

The products X × Y of power series spaces of different types have bases but no regular bases.

Standard examples of power series spaces are:

$$\ell_1(\{n^k\}),\quad \ell_1(\{(2^n)^k\}),$$

They isomorphically represent the functional spaces: $C^\infty(T)$ of periodic infinitely differentiable functions on the line, $H(\mathbb{C})$ of entire functions, $H(\mathbb{D})$ of holomorphic functions on the open disk. To get the representation we use the trigonometric system as the basis for the first space and the sequence $\{z^{n-1}\}$ for the second and third.

We notice that the spaces $H(\mathbb{C}^r)$ of entire functions of r variables and $H(\mathbb{D}^r)$ of holomorphic functions on the r-dimensional open polydisk are represented by the power series spaces $\ell_1(A)$ and $\ell_1(B)$,

$$A = \left\{2^{kn^{1/r}}\right\},\quad B = \left\{2^{k^{-1}n^{1/r}}\right\}.$$

The dimension r of the domain is an isomorphic invariant of the spaces of holomorphic functions, while every space $C^{\infty}(M)$ of all infinitely differentiable functions on a smooth compact manifold, regardless of the dimension of M, is isomorphic to $C^{\infty}(T)$.

More information on the isomorphical classification of Banach and Fréchet spaces of functions can be found in the survey article [Pe].

* * *

The concept of the quasi-equivalence of bases as well as the first result relating this concept to nuclearity (all bases in each $H(\mathbb{C})$ and $H(\mathbb{D})$ are quasi-equivalent) are due to Dragilev [Dr].

* * *

Let X be an infinite-dimensional Fréchet space with a basis. We say that: X has the QE property if all the bases of X are quasi-equivalent to each other; X has the CS property if every complemented subspace of X has a basis; the basis (x_n) of X has the CBS property if every basis (y_n) of a complemented subspace of X is quasi-equivalent to a subsequence of (x_n); X has the CBS property if every basis of X does it.

The labels QE, CS and CBS stand for "quasi-equivalence", "complemented subspace" and "complemented basic sequence".

There are three fundamental problems concerning the uniqueness of bases:

QE. Does every nuclear Fréchet space with a basis have the QE property ?

CS. Does every nuclear Fréchet space with a basis have the CS property ?

CBS. Does every basis of a nuclear Fréchet space have the CBS property ?

One can also ask these questions for concrete spaces, concrete bases or for complemented subspaces with certain special properties.

Mitiagin and Djakov have observed that using the Cantor-Bernstein mapping to the sets of indices of the bases one gets:

7.2 If the two base (x_n) and (y_n) of the space X have the CBS property then they are quasi-equivalent.

Hence it is natural to ask:

CBS1. Assume that X has a basis with the CBS. Does every basis of X have this property ?

The problems are rather difficult. During the last 25 years only partial answers have been obtained. In particular, Mitiagin has proved that power series spaces of type 0 have the CS property. Any new partial result, as well as inventing a new technique of handling the problems would be valuable.

For detailed information on these and other problems and an extensive bibliography we refer to the survey [A-Z].

Ad § 6.

The first example of a nuclear Fréchet space without basis, apparently under a psychological influence of Enflo's negative answer to the basis problem for Banach spaces, was given by Mitiagin and Zobin [MZ] in 1974.

The space E presented here, although without bases, is obviously the direct sum of its two-dimensional subspaces, in particular it has the bounded approximation property (i.e., the identity operator is the point-wise sum of a series of finite rank operators). The first example of a nuclear Fréchet space without the b.a.p. was given by Dubinsky [Du], and his construction was essentially simplified by Vogt [V]. The Vogt's example is approximately on the same level of complexity as our space E. Another interesting technique of getting nuclear spaces without bases is given by Moscatelli [M].

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