MONOGRAFIAS DE LA ACADEMIA DE CIENCIAS Exactas Físicas Químicas y Naturales DE ZARAGOZA

Second Internacional Symposium (Segovia, 1986)

"ON ORTHOGONAL POLYNOMIALS AND THEIR APPLICATIONS"

N.º 1

1988

Depósito legal: Z. 2.298 — 1988

Impreso en:

Sdad. Coop. Art. Gráf. Ltda.
LIBRERIA GENERAL
Pedro Cerbuna, 23
50009-Zaragoza

PROLOGO

La presente Monografía de la Academia de Ciencias de Zaragoza sirve de complemento a la publicación efectuada por la Editorial Springer-Verlag (Berlin, Heidelberg, New York) en su colección "Lecture Notes in Mathematics", con el título "Orthogonal Polynomials and Applications" (Proceedings, XIII + 334 pp., Segovia 1986).

En dichas Actas figuran publicadas 9 conferencias plenarias, 13 comunicaciones y una colección de problemas abiertos. En esta Monografía se incluyen 18 de las comunicaciones restantes.

Todas estas conferencias y comunicaciones fueron presentadas en el "Second International Symposium on Orthogonal Polynomials and their Applications", celebrado en Segovia (España) los días 22 a 27 de Septiembre de 1986, al que asistieron 102 congresistas y del que fue Presidente de Honor el Profesor Luis Vigil Vázquez.

Se celebraron 12 sesiones plenarias de $1^h\ 30^m$, impartidas por 10 Profesores invitados, 1 sesión de problemas abiertos y varias sesiones en las que se presentaron 50 comunicaciones. Las 19 comunicaciones restantes, que no figuran en la publicación de Springer-Verlag, ni en ésta, fueron rechazadas o sus autores no las entregaron para su publicación.

Entre las Instituciones que hicieron posible la celebración del Symposium o la publicación de sus resultados, debemos mencionar las siguientes:

Comisión Asesora de Investigación Científica y Técnica
Consejo Superior de Investigaciones Científicas
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La publicación de las Actas de Springer-Verlag y esta Monografía han sido dedicadas a la memoria del Profesor José Luis Rubio de Francia (+).

El Comité organizador del Congreso estaba compuesto por los siguientes Profesores:

Manuel Alfaro
Jesús Dehesa
Francisco Marcellán
José L. Rubio de Francia
Jaime Vinuesa.

La Academia de Ciencias de Zaragoza se congratula de haber podido contribuir a la publicación de un cierto número de comunicaciones del Symposium.

Rafael Cid Palacios
Editor

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ON EXTENSIONS OF FINITE SEQUENCES OF ORTHOGONAL POLYNOMIALS

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Abstract

It is known that if $\{P_h(z)\}_{h=o}^{n-1}$ is a finite sequence of orthogonal polynomials on the real line or on the unit circle, then, there exists a unique (except for an arbitrary constant factor) polynomial $P_n(z)$ having some prescribed zeros and which is orthogonal to the above sequence. Here, we prove an extension of this theorem.

0. INTRODUCTION

Let $\{P_n(z)\}_{n=o}^{\infty}$ be a sequence of orthogonal polynomials on an algebraic curve γ in $\mathbb{C}$. In particular, $\gamma = T = \{z : |z| = 1\}$ or $\gamma = \mathbb{R}$ will be considered (in the latter case, the notation $\{p_n(x)\}_{n=o}^{\infty}$ will be used). For every n, $P_n^*(z) = z^n\overline{P}_n(\frac{1}{z})$ and the reproducing kernel function K_n is defined by

$$K_n(z,y) = \sum_{h=o}^{n} \overline{P_h(y)}P_h(z) \qquad y \in \mathbb{C} .$$

We will denote by π_n the linear subspace of $\mathbb{C}[z]$ (or $\mathbb{R}[x]$, it depends on the case considered) of the polynomials of degree smaller than $n+1$; m_n will be the matrix of moments and $\Delta_n = \det m_n$.

For orthogonal polynomials on T it was proved in [3] that given a finite family $\{P_h(z)\}_{h=o}^{n-1}$ and $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, there exists a unique (except for an arbitrary positive constant factor) polynomial $P_n(z)$ such that $\{P_h(z)\}_{h=o}^{n}$ is an orthogonal system on T and $P_n(\alpha) = 0$ $\forall n \geq 1$. We will prove that this result can be extended to the case $|\alpha| = 1$.

For real orthogonal polynomials, if the inner product defined in π_{n-1} is known, a similar result will be furthermore proved; in this case, α may be any real number different from the zeros of the (n-1)-th polynomial ; this restriction is not surprising because of the separation of the zeros of consecutive orthogonal polynomials on $\mathbb{R}$.

Both cases are related to a special kind of extensions which will be called Stieltjes extensions (S-extensions, for short). We begin by making a short summary about them.

I. ORTHOGONAL POLYNOMIALS ON A FINITE SET OF POINTS

Let $\{\alpha_i\}_{i=1}^n$ be complex numbers non-zero and different pairwise, provided of the respective positive weights $\{p_i\}_{i=1}^n$. We define in π_{n-1}, vectorial subspace of $\mathbb{C}[z]$, an inner product by means of the Riemann-Stieltjes sums

$$\langle P(z),Q(z)\rangle = \sum_{h=1}^{n} p_h P(\alpha_h)\overline{Q(\alpha_h)} \qquad P,Q \in \pi_{n-1}$$

In particular,

$$c_{ij} = \langle z^i,z^j\rangle = \sum_{h=1}^{n} p_h \alpha_h^i \bar{\alpha}_h^j \quad (0 \leq i,j \leq n-1)$$

and we have a positive definite hermitian matrix $m_{n-1} = (c_{ij})_{i,j=o}^{n-1}$ from which we can obtain $\{P_h(z)\}_{h=o}^{n-1}$ in terms of determinants; this polynomials are an orthogonal basis for π_{n-1}. It is shown that

$$\langle K_{n-1}(z,\alpha_i),K_{n-1}(z,\alpha_j)\rangle = p_i^{-1}\delta_{ij} \qquad (1 \leq i,j \leq n)$$

and $\{K_{n-1}(z,\alpha_i)\}_{i=1}^n$ turns out to be another orthogonal basis for π_{n-1}. The converse is true, so we have:

PROPOSITION 1. Given n complex numbers $\{\alpha_i\}_{i=1}^n$ non zero and different pairwise, and a positive definite hermitian matrix $m_{n-1} = (c_{ij})_{i,j=o}^{n-1}$, the system $\{K_{n-1}(z,\alpha_i)\}_{i=1}^n$ is an orthogonal basis for π_{n-1} if, and only if,

$$c_{ij} = \sum_{h=1}^{n} p_h \alpha_h^i \bar{\alpha}_h^j \qquad (0 \leq i,j \leq n-1)$$

where

$$p_h = K_{n-1}(\alpha_h,\alpha_h)^{-1} > 0$$

If we define

$$c_{nj} = \sum_{i=1}^{n} p_i \alpha_i^n \bar{\alpha}_i^j \qquad j=0,1,2,\ldots,n-1$$

we can construct a polynomial with degree n, $P_n(z)$, in the usual form, whose zeros turn out to be $\{\alpha_i\}_{i=1}^n$. If we define c_{nn} by means of the corresponding Riemann-Stieltjes sum

$$c_{nn}^{(o} = \sum_{i=1}^{n} p_i \alpha_i^n \bar{\alpha}_i^n$$

there results a singular matrix m_n , that is to say

$$\Delta_h \neq 0 \quad h = 0,1,\dots,n-1 \quad ; \quad \Delta_n = 0$$

By defining

$$c_{nn} = c_{nn}^{(0} + e_n \quad , \quad e_n > 0$$

it results $\Delta_n = e_n \Delta_{n-1} > 0$

We have, so, an extension of m_{n-1} and, consequently, an inner product defined on π_n . Besides, $\{P_n(z)\}_{h=o}^{n}$ is an orthogonal basis for π_n .

<u>DEFINITION</u>. We will name S-extension to any extension of m_{n-1} defined by the above form, with $e_n \geq 0$. Such an extension will be said terminal if $e_n = 0$.

It is easy to verify that

$$\frac{P_n(z)}{(z-\alpha_i)P_n'(\alpha_i)} = p_i \, K_{n-1}(z,\alpha_i) \qquad i = 1,2,\dots,n$$

and so,

<u>PROPOSITION 2</u>. Given the matrix of moments m_{n-1} and n complex numbers $\{\alpha_i\}_{i=1}^{n}$ non-zero and different pairwise , $P_n(z) = \Delta_{n-1} \sum_{i=1}^{n} (z-\alpha_i)$ is the nth-orthogonal polynomial in the S-extension of m_{n-1} to m_n , if, and only if, the functional coefficients of the Lagrange interpolation formula of $P_n(z)$ constitute an orthogonal basis for π_{n-1} with regard to the inner product defined in it by m_{n-1} .

NOTE: A detailed study of the S-extensions can be seen in [7] . For orthogonal polynomials on $\mathbb{R}$ and $\mathbb{T}$, [1] and [2] .

II. STIELTJES-EXTENSIONS FOR ORTHOGONAL POLYNOMIALS ON ALGEBRAIC CURVES

Let γ be an algebraic curve of degree h and equation:

$$\sum_{p,q=o}^{h} a_{pq} \, z^p \bar{z}^q = 0 \quad , \qquad (a_{pq}) = \overline{a_{qp}})$$

i) A positive definite hermitian matrix $m_{n-1} = (c_{ij})_{i,j=o}^{n-1}$ verifying the linear conditions

$$\sum_{p,q=o}^{h} a_{pq} \; c_{i+p,j+q} = 0 \qquad i,j = 0,1,\dots n-1$$

is said to be a matrix of order n relative to γ

ii) The polynomials $\{P_h(z)\}_{h=o}^{n-1}$ defined from m_{n-1} in the usual form, are called orthogonal on γ

iii) Any extension of m_{n-1} that is itself relative to γ is a γ-extension of m_{n-1}.

One, naturally, wonders if there exist γ-extensions of Stieltjes. Concerning this, in [9] it is proved that the zeros of the polynomial $P_n(z)$ corresponding to a γ-extension of Stieltjes are simple and belonging to γ. Conversely, given m_n, if its minor m_{n-1} verifies the condition of PROPOSITION 1, $P_n(z)$ has simple zeros on γ, $\{\alpha_i\}_{i=1}^{n}$, which provided of the respective weigths $\{K_{n-1}(\alpha_i,\alpha_i)^{-1}\}_{i=1}^{n}$ determine a S-extension m_n.

If m_n is a S-extension and $\{\alpha_i\}_{i=1}^{n}$ are the n zeros of $P_n(z)$, the elements $\{c_{ho}\}_{h=o}^{n-1}$ are sufficient to determine the remaining elements of m_n : in fact, the numbers $\{p_i\}_{i=1}^{n}$ are obtained as the solutions of the system

$$\left.\begin{array}{l} \sum_{i=1}^{n} p_i \alpha_i^h = c_{ho} \\ \\ h=0,1,\dots,n-1 \end{array}\right\}$$

Consequently, not any matrix relative to a curve γ corresponds to a S-extension.

III. THE CASES $\gamma = \mathbb{R}$ AND $\gamma = \mathbb{T}$. EXTENSIONS WITH A FIXED ZERO

Let $\gamma = \mathbb{R}$ and let $\{p_h(x)\}_{h=o}^{n}$ be orthogonal polynomials on the real line. If $\{\alpha_i\}_{i=1}^{n}$ are the zeros of $p_n(x)$ and in the Christoffel-Darboux formula

$$K_{n-1}(x;y) = \frac{k_{n-1}}{k_n} \cdot \frac{p_{n-1}(y)p_n(x)-p_n(y)p_{n-1}(x)}{x-y}$$

([6], p. 43, form. 3.2.3).

we put $x = \alpha_i$, $y = \alpha_j$, it results $K_{n-1}(\alpha_i,\alpha_j) = 0 \quad \forall i \neq j$, being $K_{n-1}(\alpha_i,\alpha_i) > 0 \quad \forall i$. Hence, in orthogonal polynomials on the real line any γ-extension is a S-extension.

Let $\gamma = \mathbb{T}$ and let $\{P_h(z)\}_{h=o}^{n}$ be orthogonal polynomials on the unit circle. If $P_n(z)$ corresponds to a S-extension, it has n zeros $\{\alpha_i\}_{i=1}^n$ different pairwise, with $|\alpha_i| = 1 \ \forall i$. So, $\det m_n = 0$. Conversely, if $P_n(z)$ has its zeros of modulus 1 , they are just the n zeros of $P_n^*(z)$ and the Christoffel-Darboux formula

$$K_{n-1}(z,y) = \frac{\overline{P_n^*(y)}\,P_n^*(z) - \overline{P_n(y)}\,P_n(z)}{1-\bar{y}z}$$

([6] , p. 293 , form. 11.4.5),

by putting $z = \alpha_i$, $y = \alpha_j$, yields $K_{n-1}(\alpha_i,\alpha_j) = 0$, $i \neq j$ so that, we have a S-extension. In short, if $\gamma = T$ then, a γ-extension is a S-extension if and only if it is a terminal extension.

We will prove now the results mentioned at the beginning.

PROPOSITION 3. Let us consider an inner product in π_{n-1} , defined by a $n \times n$ Hankel matrix , and its associated orthogonal polynomial system $\{p_h(x)\}_{h=o}^{n-1}$. For every $\alpha \in \mathbb{R}$ with $p_{n-1}(\alpha) \neq 0$, there exists a polynomial of degree n , $p_n(x)$, (unique except for an arbitrary positive constant factor) such that $\{p_h(x)\}_{h=o}^{n}$ is an orthogonal system on $\mathbb{R}$ and $p_n(\alpha) = 0$.

Proof. Let $m_{n-1} = (s_{i+j})_{i,j=o}^{n-1}$ the Hankel matrix. By using the expression in terms of determinant for $p_n(x)$, the condition $p_n(\alpha)=0$ can be written

$$\begin{vmatrix} s_o & s_1 & \dots & s_{n-1} & s_n \\ s_1 & s_2 & \dots & s_n & s_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-2} & 0 \\ 1 & \alpha & \dots & \alpha^{n-1} & \alpha^n \end{vmatrix} - s_{2n-1}p_{n-1}(\alpha) = 0 \qquad (1)$$

Because the above determinant depends exclusively on α and on the moments in π_{n-1} , we represent it by $\Delta(\alpha)$. Relation (1) implies

$$s_{2n-1} = \Delta(\alpha) \left[p_{n-1}(\alpha)\right]^{-1} \qquad (2)$$

and s_{2n-1} is determined provided that $p_{n-1}(\alpha) \neq 0$. So, the determinant $p_n(x)$ is defined . #

Nevertheless, to construct m_n we need to know s_{2n} . The relation $\Delta_n = \Delta_n(s_{2n}) = 0$ can be interpreted as an equation in s_{2n} whose solution s^t_{2n} corresponds to a S-extension. Finally, the positive definite extensions are obtained for values

$$s_{2n} = s^t_{2n} + e_n \qquad , \qquad e_n > 0 \quad .$$

We want to note that, according to PROPOSITION 3, the n-th polynomial $p_n(x)$ is obtained before the character of the extension has been decided. So, the zeros of $p_n(z)$ are the same whatever this chatacter is; this situation is very different from the one of orthogonal polynomials on T , and it is, probably, unique in the theory of orthogonal polynomials on algebraic curves.

NOTE: PROPOSITION 3 completes the result about determination of polynomials on the real line by some of their zeros that can be seen in [5] and [10] .

PROPOSITION 4. Given the finite family $\{P_h(z)\}_{h=o}^{n-1}$ of orthogonal polynomials on T and $\alpha \in T$, there exists a polynomial of degree n , $P_n(z)$, (unique except for an arbitrary positive constant factor) such that $\{P_h(z)\}_{h=o}^{n}$ is an orthogonal system on T and $P_n(\alpha) = 0$. The extension of m_{n-1} to m_n so obtained is a S-extension.

Proof. In this case the matrix of moments is a Toeplitz matrix, $m_{n-1} = (c_h)_{h=-n-1}^{n-1}$ with $c_{-h} = \overline{c_h}$.

Any Toeplitz extension is determined by c_n . The S-extensions correspond to values c^t_n such that the associated matrix, m_n , is singular. Every c^t_n belongs to the circle with centre c and radius e_{n-1} ([2] , p. 43 , form.(6.3)). So, for every $\varphi \in [0,2\pi)$, we obtain a S-extension determined by $c^{\varphi}_n = c+e_{n-1}e^{i\varphi}$, with n-th polynomial

$$P_n(z;\varphi) = \begin{vmatrix} c_o & c_1 & \cdots & c_{n-1} & c+e_{n-1}e^{i\varphi} \\ c_{-1} & c_o & \cdots & c_{n-2} & c_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{-n+1} & c_{-n+2} & \cdots & c_o & c_1 \\ 1 & z & & z^{n-1} & z^n \end{vmatrix}$$

which, by a routine algebraic calculus ([2], p. 45. form. (6.9)), can be expressed

$$P_n(z;\varphi) = e_{n+1}\, zP_{n-1}(z) - e_{n-1} e^{i\varphi} P^*_{n-1}(z)$$

Since there must be $P_n(\alpha,\varphi) = 0$ for $\alpha = e^{ia} \in T$, the extension is completely determined by being

$$e^{i\varphi} = e^{ia} \frac{P_{n-1}(e^{ia})}{P^*_{n-1}(e^{ia})} \tag{3}$$

Explicitly,

$$\varphi = 2 \sum_{j=1}^{n-1} \arg (e^{ia} - \beta_j) - (n-2)a$$

where $\{\beta_j\}_{j=1}^{n-1}$ are the zeros of $P_{n-1}(z)$. #

Relations (2) and (3) define both of them functions $s_{2n-1} = s_{2n-1}(\alpha)$ and $\varphi = \varphi(a)$. Their study furnish a simple geometric interpretation of the known property about separation of zeros of n-th polynomials associated to S-extensions on $\mathbb{R}$ or T ([1], [2]).

IV. THE GENERAL PROBLEM OF EXISTENCE OF S-EXTENSIONS

Such as we have explained in I, the existence of S-extensions can be related to the problem of Lagrange interpolation. The Christoffel-Darboux formulae clarify completely the question for orthogonal polynomials on $\mathbb{R}$ or on T. It seems natural to wonder what happens in the remaining cases.

We have Christoffel-Darboux formulae for others algebraic curves, specifically for harmonic hyperbolas [8] and for cassinian curves and lemniscates of Bernouilli, particularly, [4]. In this one, the method employed in III, seems unuseful on account of the nature of the formulae; in the former, the application of this method requires some additional conditions; it is not surprision because, except for the cases $\mathbb{R}$ and T, not any extension is a S-extension, but there are S-extensions.

When one does not have Christoffel-Darboux formulae, another method must be used. Since the following relation holds

$$K_{n-1}(z,y) = \sum_{i,j=0}^{n-1} \mu_{ij}\, z^i \bar{y}^j$$

where $(\mu_{ij})_{i,j=o}^{n-1} = \left[(c_{ij})_{i,j=o}^{n-1}\right]^{-1}$, one could start by determining the values y, for which the corresponding equations $K_{n-1}(z,y) = 0$ have different pairwise zeros $z_1, z_2, \ldots, z_{n-1}$.

Then, one must choose the sets of different elements $(z_1, z_2, \ldots, z_{n+1}, y)$ such that every possible pair of elements of every one of them satisfies $K_{n-1}(z,y) = 0$. Finally, that a S-extension has been really obtained must be shown.

This technique, simple in theory, does not seem easy to put into practice, at least, in general.

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INVERSE PROBLEMS, MOMENT THEORY AND THE MAXIMUM ENTROPY FORMALISM

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Abstract

The reduced Hausdorff moment problem is approached as a special case of a generalized underdetermined inverse problem. The Maximum Entropy (ME) Formalism for these sort of problems is used: We choose between all the weight functions compatible with the constraints imposed by their firs N+1 moments, the one which maximizes the entropy functional of the weight function.

The properties of the approximations to the weight function obtained by this method are studied and a theoretical and numerical comparative study is also done with the more familiar methods as Orthogonal Polynomial expansions and Stieltjes-Chebyshev approximations to the weight function.

1. Introduction

The aim of this paper is not to apply Moment Theory or properties of Orthogonal Polynomials (OP) to physical problems, but conversely, to use a very general physical principle, the Maximum Entropy Principle, in order to obtain sequences of solutions, $\chi_E^{(N)}(x)$, to the reduced Hausdorff moment problem.

In section 2. we briefly review the reduced moment problem and some of the methods to solve it. In section 3. we introduce the ME method and finally in section 4. some numerical results and conclusions are presented and discussed.

2. The reduced Moment Problem. Methods.

Many physical quantities of great interest are related to integral transforms of nonnegative functions.

$$\langle F\rangle \equiv \int_b^a F(x)\ \chi(x)dx\ ,\ \chi(x)\geqslant 0\ ,\ x\in[a,b] \qquad (2.1)$$

The estimation of these quantities using only a limited number of parameters is an old and well known problem. In its usual form the known parameters consist of the moments of the weight $\chi(x)$

$$\mu_m = \langle x^m \rangle = \int_b^a x^m \chi(x)\,dx,\ m=0,1,2,\ldots \tag{2.2}$$

Moment theory provides rigorous bounds on the average of F, $\langle F \rangle$, by using the properties of the OP system associated to the weight. [1] .Sometimes this is not sufficient but information on the actual weight, $\chi(x)$, is wanted.

Even in the theoretical case where moments of all orders exist and are known, the complete set does not necessarily uniquely determine the weight [2,3] .The practical (physical) situation is one where only a few moments are available, either from experimental measurements or from theoretical calculations, so a unique reconstruction of $\chi(x)$ is impossible in view of this limited information.

Nevertheless there are approximation procedures constructing sequences of functions $\chi^{(N)}(x)$ such that their moments are the known ones

$$\int_b^a \chi(x)\, x^m\, dx = \mu_m,\ m=0,1,2,\ldots,N \tag{2.3}$$

and which eventually converge to the true weight when N tends to infinity or such that we have average convergence,

$$\langle F \rangle = \lim_{N\to\infty} \int_b^a F(x)\, \chi^{(N)}(x)\, dx \tag{2.4}$$

Perhaps the two best known methods for this sort of inverse problems are the Orthogonal expansion method and the Stieltjes-Chebyshev technique related to Pade Approximants (PA) and continued fractions.

In the OP expansion method or the Reference weight method, [4] ,we expand the unknown weight in this form :

$$\chi(x) = W(x) \sum_{n=o}^{\infty} a_n P_n(x) \tag{2.5}$$

and we look for sequences of approximations $\chi_{OP}^{(N)}(x)$ such that

$$\chi_{OP}^{(N)}(x) = W(x) \sum_{n=o}^{N} a_n P_n(x) \tag{2.6}$$

where $W(x)$ is a known weight and $P_n(x)$ are OP with respect to $W(x)$. Then we also have the usual orthogonality property:

$$\int_b^a P_n(x)\, P_m(x)\, W(x)\, dx = \delta_{nm} \tag{2.7}$$

We can calculate the coefficients a_n

$$a_n = \int_b^a P_n(x)\, \chi(x)\, dx \quad ,n=o,1,2,\ldots,N \tag{2.8}$$

solving a (N+1)x(N+1) system of linear equations.

The main advantage of the method is that it is easy to solve and also that the approximations are continuous functions.On the other hand some of the difficulties are that we have to choose properly W(x) and to study the convergence of equation (2.5),which depends on the abive choice.It is usual to have many oscillating approximations ans sometimes we loose positivity in the interations.

In the Stieltjes-Chebyshev method we attempt to approximate the nonnegative function $\chi(x)$ by finite sums of delta functions

$$\chi_S^{(N)}(x)\ dx = d\Psi_S^{(N)}(x) = \sum_{i=1}^{n} f_i^{(n)}\delta(x-\varepsilon_i^{(n)})\ dx \qquad (2.9)$$

where N+1=2n is the number of known moments and

$$[n-1/n] = \sum_{i=1}^{n} \frac{f_i^{(n)}}{1+z\varepsilon_i^{(n)}} \quad , \varepsilon_i^{(n)} \in [0,1] \ , \ f_i^{(n)} > 0 \qquad (2.10)$$

is the Pade Approximant to the Stieltjes function $H(z)=\int_0^1 \chi(x)/(1+xz)dx$
The relationship between the moments and the parameters $f_i^{(n)}$,$\varepsilon_i^{(n)}$ being, [2,5] :

$$\mu_k = \sum_{i=1}^{n} (\varepsilon_i^{(n)})^k \ f_i^{(n)} \quad , \quad k = 0,1,2,\dots,2n-1=N \qquad (2.11)$$

This method yields a steps distribution approximation $\chi_S^{(n)}$ to $\chi(x)$:

$$\begin{aligned} \Psi_S^{(n)} &= 0 \quad , \quad 0< x< \varepsilon_1^{(n)} \\ \Psi_S^{(n)} &= \sum_{p=1}^{j} f_p^{(n)} \quad , \quad \varepsilon_j^{(n)} < x< \varepsilon_{j+1}^{(n)} \\ \Psi_S^{(n)} &= \sum_{p=1}^{n} f_p^{(n)} \quad , \quad \varepsilon_n^{(n)} < x \quad , \ \varepsilon_i^{(n)} < \varepsilon_j^{(n)} \text{ for } i< j \end{aligned} \qquad (2.12)$$

Such distributions satisfiy the Chebyshev inequalities:

$$\Psi_S^{(n)}(\varepsilon_i^{(n)}-0) \leq \Psi_S^{(m)}(\varepsilon_i^{(n)}-0) \leq \Psi(\varepsilon_i^{(n)}) \leq \Psi_S^{(m)}(\varepsilon_i^{(n)}+0) \leq \Psi_S^{(n)}(\varepsilon_i^{(n)}+0) \ , \ m > n \qquad (2.13)$$

The main advantage is that we get rigorous upper and lower bounds on the distribution $\chi(x)$,and also on the averages of F, $\langle F \rangle$ .The main difficulty is that we have a discontinuous approximation to de distribution and a smoothing procedure is needed in order to have continuous approximations to the weight $\chi(x)$.

In principle we only have bounds at the points related to the poles of the PA but by using the sequence of quasi-orthogonal polynomials we can arbitrarily vary the position of one of the poles of the approximants.

3.The Maximum Entropy Formalism

The non uniqueness of the solution of the reduced moment problem forces the search of other alternative methods in order to compare the different solutions. In view of the main difficulties of the previous methods:the lack of positivity in the first and the discontinuity in the second one,we are going to try and get approximations which automatically have positivity and continuity.

The ME method is based on a very general principle which is the foundation of the Statistical Mechanics and has recently had a large number of succesful applications in other inverse problems including image reconstruction,data analysis and information theory. [6,7,8] .

For many years it has been recognized that entropy acts as a kind of measure in the space of probability distributions,in such a way that those distributions of high entropy are in some sense favoured over others.Nature prefers distributions of maximum entropy,so distributions of higher entropy are more likely than others.

We can state briefly the principle in this way [6] :

When we make inferences based on incomplete information we should draw them from that probability distribution that has the maximum entropy permitted by the information we actually have.

The incomplete information we have now is the set of N+1 first moments of a function and the ME principle says that we have to choose between all the weight functions compatible with the constraints imposed by their first N+1 moments the one which maximizes the entropy functional of the weight:

$$S(\chi)=-\int_0^1 \chi(x)\ \ln\chi(x)\ dx + \sum_{n=0}^{N} \lambda_n (\mu_n - \int_0^1 \chi(x)\ x^n\ dx) \qquad (3,1)$$

The ME choice is the least biased choice we can do taking into account the information we do have.We are going to see how this general physical principle leads to sequences of approximations which have many interesting and concrete properties.

To calculate the ME solution to our problem,we have to solve this Lagrange multiplier problem:Find the maximum of $S(\chi)$ permitted by the constraints.

Functional variation with respect to the unknown $\chi(x)$ gives this expresion for the ME solutions:

$$\chi_{E}^{(N)} = \exp(-\sum_{n=0}^{N} \lambda_n\ x^n) \qquad (3,2)$$

suplemented by the conditions

$$\mu_n = \int_0^1 x^n \quad \chi_E^{(N)}(x)\, dx \qquad , n=0,1,\dots,N \tag{3,3}$$

We can see how these approximations automatically incorporate positivity and continuity.In order to get maximum entropy solutions we have to solve a non linear systemof N+1 equations with N+1 unknown Lagrange multipliers. This system can not be solved analyticaly except for N=1.

After normalization we have the following relation between λ_0 and the remaining Lagrange multipliers:

$$\exp(-\lambda_o) = \int_o^1 dx \quad \exp(-\sum_{n=1}^{N} \lambda_n x^n) \equiv Z \tag{3,4}$$

Therefore we have to solve this system of N equations ;

$$\langle x^n \rangle = \mu_n \quad , n=1,2\dots,N \tag{3,5}$$

where

$$\langle x^k \rangle = \int_o^1 dx\, x^k \exp(-\sum_{n=1}^{N} \lambda_n x^n) \,/\, Z \tag{3,6}$$

Now we introduce a potential function $U(\lambda_1,\lambda_2, \quad ,\lambda_N)$ [8] whose stationary points are also the solutions of ME.

$$U = \ln Z + \sum_{n=1}^{N} \mu_n \lambda_n \, , \qquad 0 = \partial U/\partial \lambda = \langle x^n \rangle = \mu_n, n=1,2,\dots N \tag{3,7}$$

There are some properties concerning the solutions of ME and the potential $U(\lambda_1, \lambda_2, \quad , \lambda_N)$.

First it can be proved that the potential U is everywhere convex.This means that if a ststionary point is found it must be a unique absolute minimum.Conversely covexity alone does not guarantee the existence of a minimum.The existence of ME solution depeds on the sequence of known moments,as can easily see in the anallytic case N=1.

$$Z = \int_o^1 dx \exp(-\lambda_1 x) = (1-\exp(-\lambda_1)/\lambda_1 \; , \; U(\lambda_1) = \ln((1-\exp(-\lambda_1))/\lambda_1) + \lambda_1 \mu_1 \tag{3,8}$$

$U(\lambda_1)$ is a convex function but posseses a minimum at some finite λ_1 only if $\mu_1 < 1 = \mu_0$.

The conditions the sequence of moments $\{\mu_n\}_{n \geq 0}$ must satisfy in order to guarantee the existence of a ME solution are given by the following theorem [9].

T1.- A necessary and sufficient condition that the potential U should have a unique minimum at some finite values of lambdas, for any N, is that the moment sequence $\{\mu_n\}_{n\geq 0}$ should be a totally monotonic sequence:

$$\Delta^k \mu_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \mu_{n+m} > 0 \quad ; n,k=0,1,2.. \Leftrightarrow \{\mu_n\}_{n\geq 0} \in TM \tag{3,9}$$

This theorem guarantees the existence of a ME solution $\chi_E^{(N)}$ for any N. The solution is nonegative, absolutely continuous and satisfies the reduced moment problem.

There is also a theorem on convergence [9]:

T2.- A ME sequence $\chi_E^{(N)}$ with the above general properties converges in the following sense:

$$\lim_{N\to\infty} \int_0^1 \chi_E^{(N)} F(x)\, dx = \int_0^1 \chi(x)\, F(x)\, dx \equiv \langle F \rangle \tag{3,10}$$

where F(x) is some continuous function.

4.- Numerical results, comparisons and conclusions.

We have used a clasical Newton minimation procedure to find the parameters $\lambda_1, \lambda_2, \quad, \lambda_N$ where the potential U is minimum, that is to say, where there is a ME solution to the reduced moment problem.

Figure 1 shows the ME sequences obtained by using 1,2,3 and 4 moments of a polynomial weight. It can be seen how the ME approximations cut the true weight so many times as moments we use.

Figure 2 shows the first ME approximations to a more complicated polynomial. In fact the fifth approximation is already a very good one, taking into account the small number the parameters needed.

As PA are exact approximations to rational functions, i.e. when the weight is a finite sum of delta functions, the ME method is also exact when we have the moments of an exponential of polynomials. Figure 3 shows the first three approximations to an exponential of polynomial of degree 3. In fact the fourth appxoximation is just the exact weight. If we insist and try higher polynomials the new lambdas are always zero.

Figure 4 shows a comparison between a Stieltjes and a ME approximation in a physical case where we do not know the true weight function but its first eight moments. The unknown weight is related to the imaginary part of the $K^- p$ scattering amplitude which is not accesible experimentally. [10] .

The Stieltjes histograms for the distribution function constructed by means of the poles and residues of the PA can be seen.Also the interlazing properties of the zeros of the denominators ,which are OP with respect the weight, are apparent.

We can also see how the ME distributions keep inside the bounds imposed by the Chebyshev inequalities.On the right hand side we have the corresponding weight functions obtained from the slopes of the segments joining the mid points of the discontinuities of the Stieltjes distributions and the ME solutions.

And finally in figure 5 we have a comparison between a Stieltjes-Chebyshev approximation obtained by using quasi-orthogonal polynomials and our ME weight.Despite of using very different methods they are quite similar.

The conclusion is we nweed various alternative methods of inverting the reduced moment problem owing to the non uniqueness of the solution.In this way we can compare the differnt solutions.Each method has advantages and dificulties but sometimes they are complementary as in the case of Stieltjes-Chebyshev and Maximum Entropy extrapolations.The first method provides rigorous bounds on the distribution function and the second one chooses between all the possible distributions,compatible with the moments we have ,the one most reasonable in a certain sense:the least biased distribution on the base of the information we actually have.

Acknowledgements

I thank Dr.A.Cruz for many useful and stimulating discussions.Iacknowledge the financial support of CAICYT.

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Figure Captions

Figure 1.-Maximum Entropy approximations obtained with 1,2,3 and 4 moments to the polynomial weight $\chi(x)=2x$

Figure 2.-Maximum Entropy distributions(above) and weights(below) to the weight function $\chi(x)=x^2+2x^3-3x^4$,obtained with 2,3,4 moments.The fifth approximation is already very near to the weight.

Figure 3.-Maximum Entropy weights for an exponential of a polynomial. The fourth approximation is exactly the weight function: Maximum Entropy approximations are exact for this kind of functions.

Figure 4.-Comparison between a Stieltjes and a Maximum Entropy approximations obtained with 2,4,6 and 8 moments.The Maximum Entropy distributions keep inside the bounds imposed by the Chebyshev inequalities.

Figure 5.Comparison between a Stieltjes-Chebyshev approximation using quasiorthogonal Polynomials and a Maximum Entropy approximation.

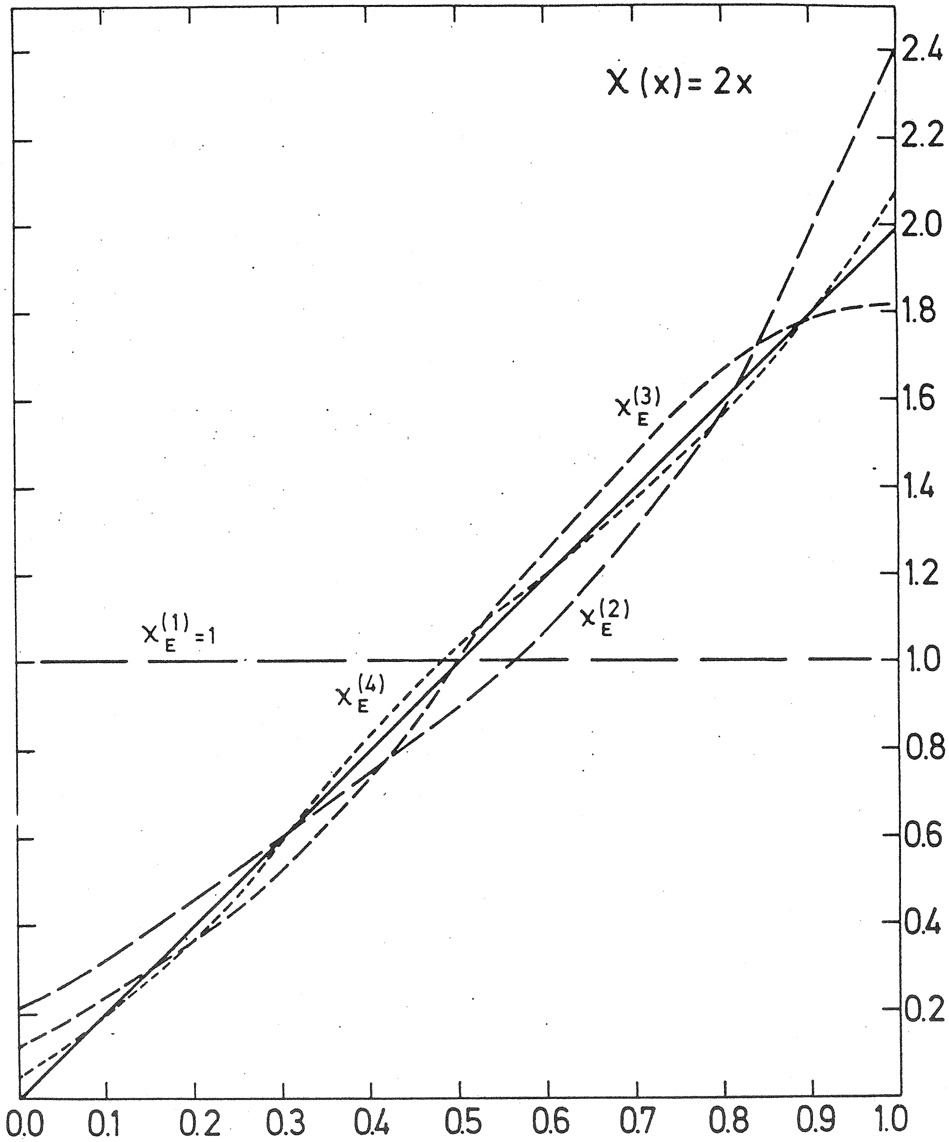
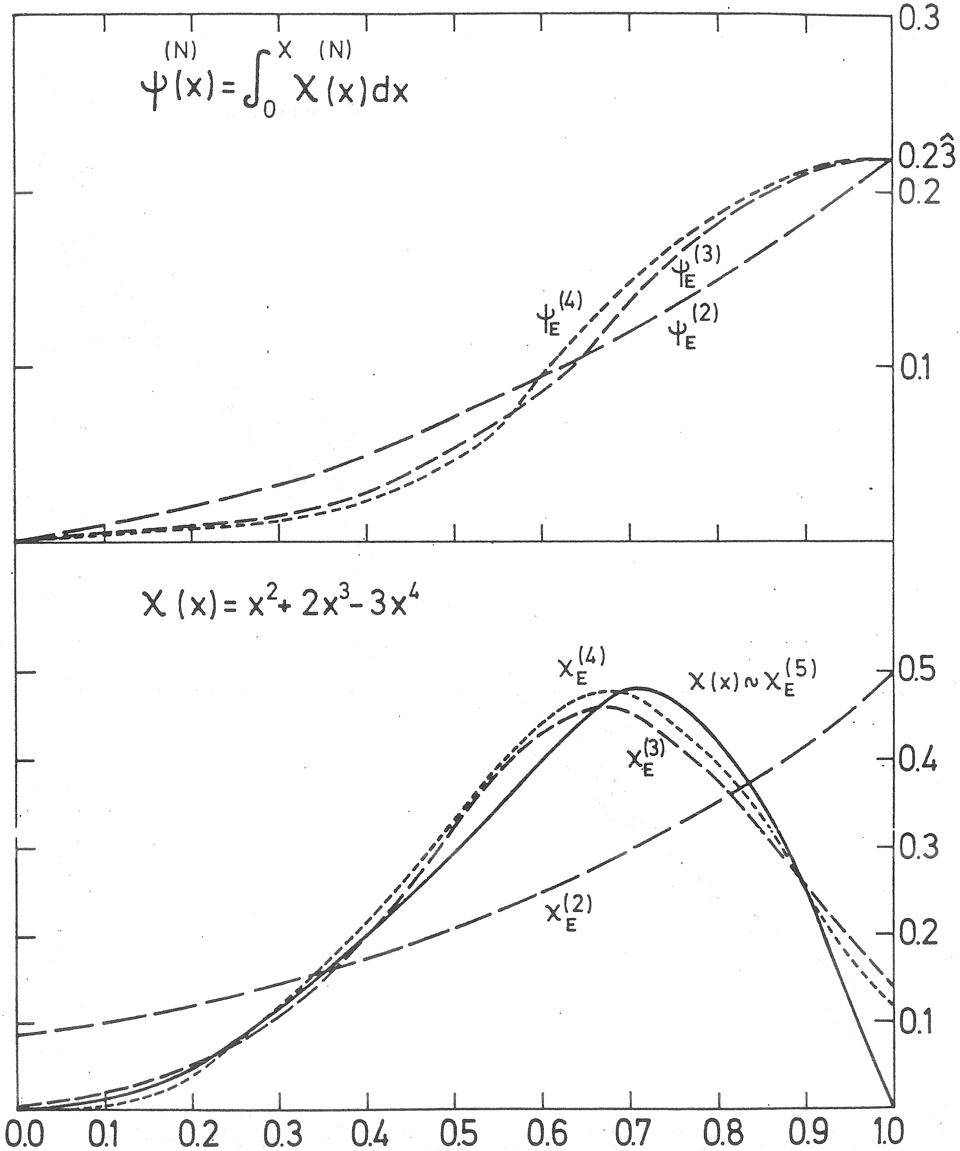
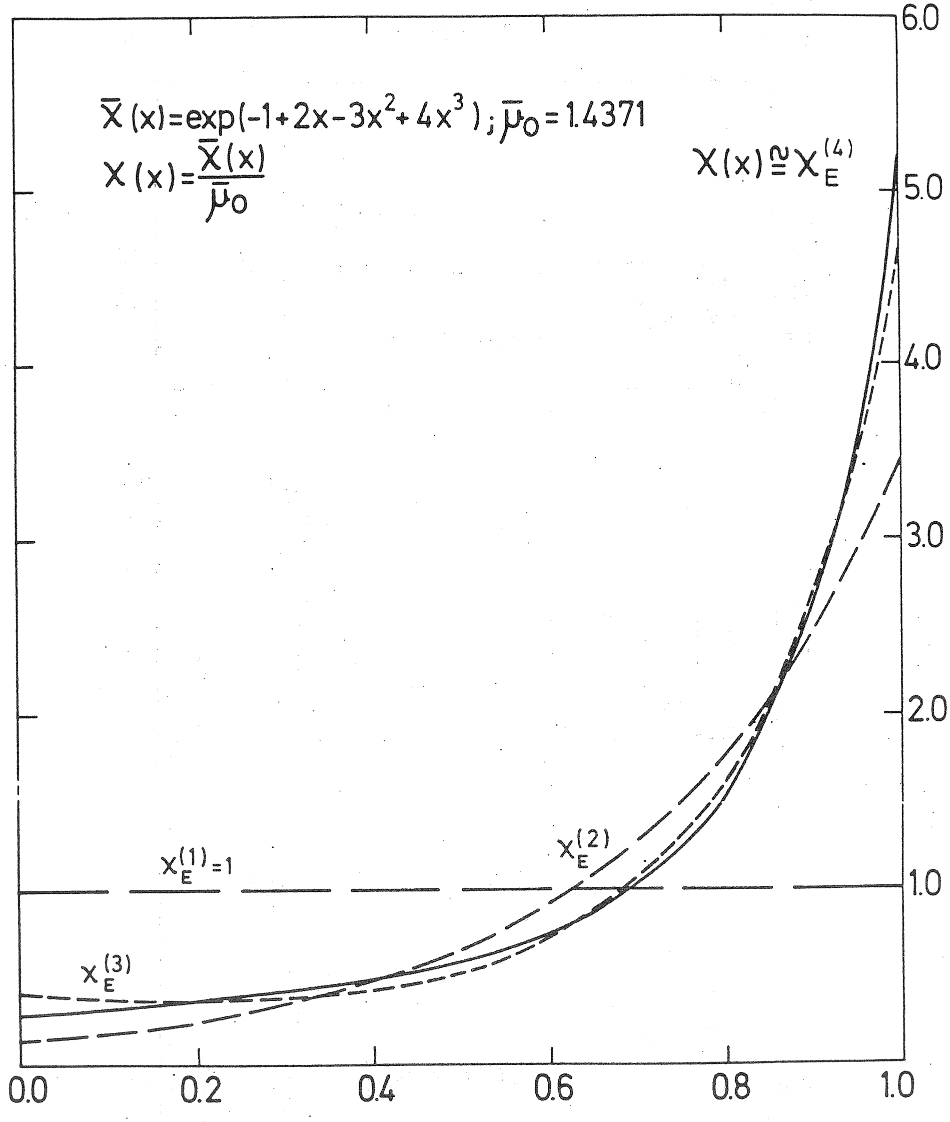

X (x)= 2x
$X_E^{(3)}$
$X_E^{(2)}$
$X_E^{(1)}=1$
$X_E^{(4)}$
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

FIGURE 1



$\psi^{(N)}(x) = \int_0^x \chi^{(N)}(x)\,dx$
$\psi_E^{(4)}$
$\psi_E^{(3)}$
$\psi_E^{(2)}$
0.3
0.2$\hat{3}$
0.2
0.1
$\chi(x) = x^2 + 2x^3 - 3x^4$
$\chi_E^{(4)}$
$\chi(x) \sim \chi_E^{(5)}$
$\chi_E^{(3)}$
$\chi_E^{(2)}$
0.5
0.4
0.3
0.2
0.1
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

FIGURE 2



$\bar{X}(x)=\exp(-1+2x-3x^2+4x^3)$; $\bar{\mu}_0=1.4371$
$X(x)=\frac{\bar{X}(x)}{\bar{\mu}_0}$
$X(x)\cong X_E^{(4)}$
$X_E^{(1)}=1$
$X_E^{(2)}$
$X_E^{(3)}$
0.0
0.2
0.4
0.6
0.8
1.0
2.0
3.0
4.0
5.0
6.0

FIGURE 3

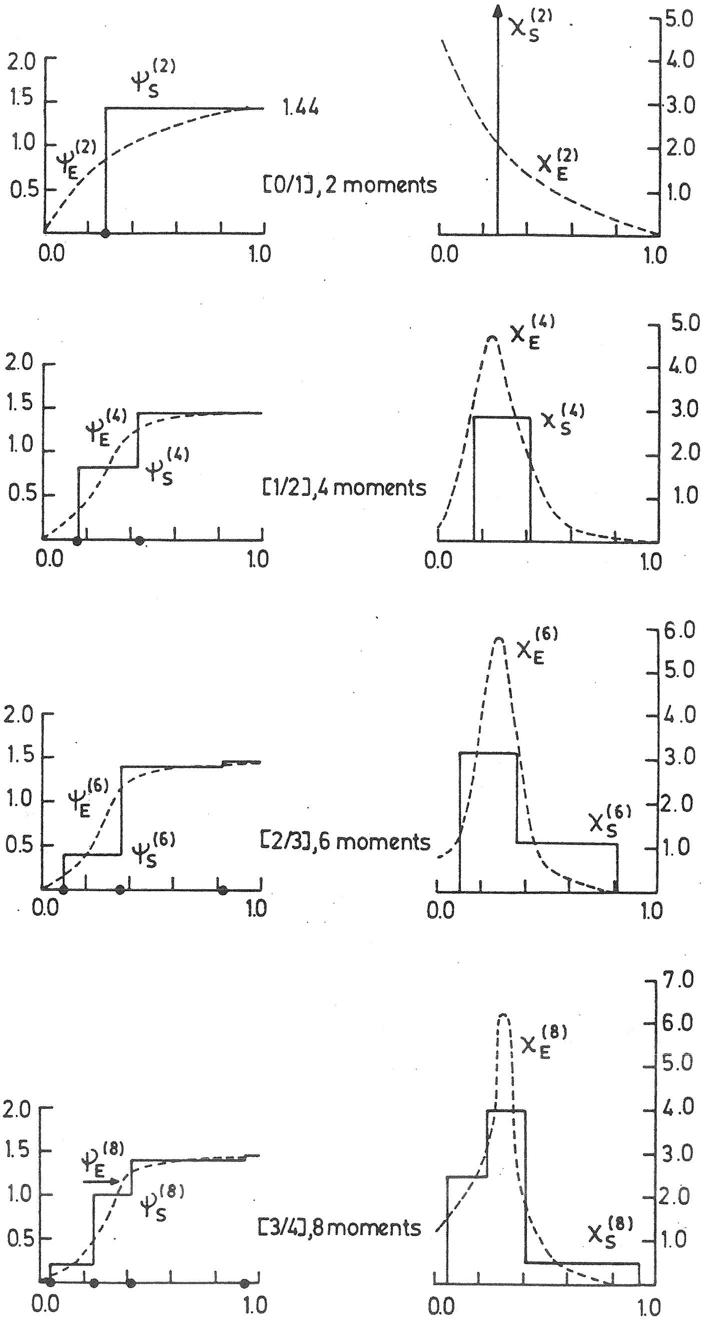
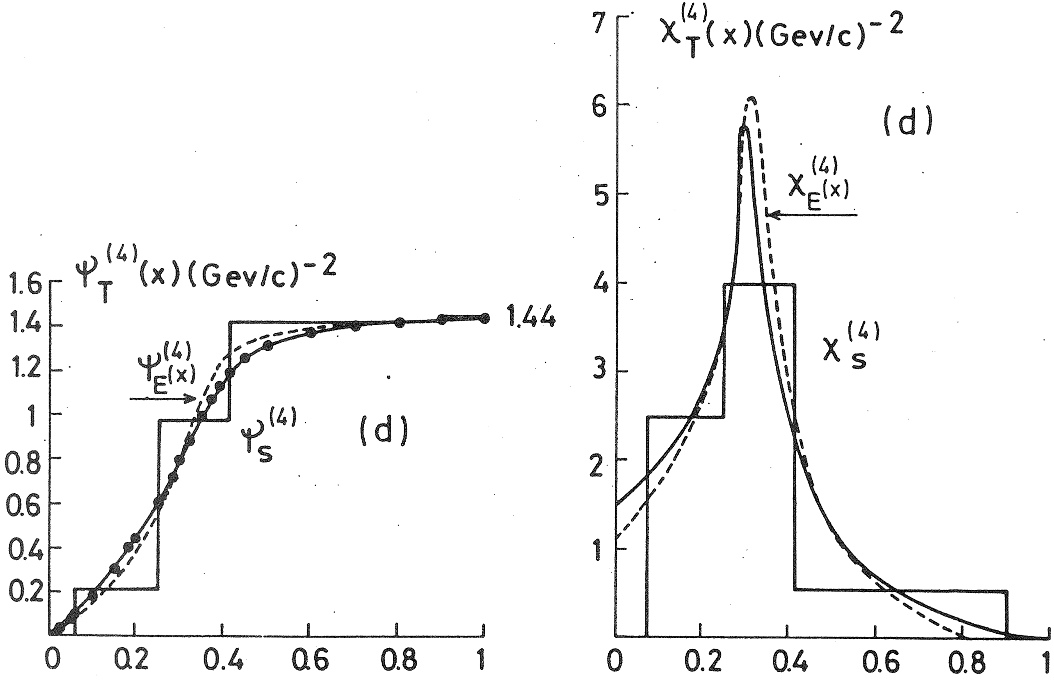

$\psi_S^{(2)}$
1.44
$\psi_E^{(2)}$
[0/1], 2 moments
$\chi_S^{(2)}$
$\chi_E^{(2)}$
$\psi_E^{(4)}$
$\psi_S^{(4)}$
[1/2], 4 moments
$\chi_E^{(4)}$
$\chi_S^{(4)}$
$\psi_E^{(6)}$
$\psi_S^{(6)}$
[2/3], 6 moments
$\chi_E^{(6)}$
$\chi_S^{(6)}$
$\psi_E^{(8)}$
$\psi_S^{(8)}$
[3/4], 8 moments
$\chi_E^{(8)}$
$\chi_S^{(8)}$
0.0
0.5
1.0
1.5
2.0
3.0
4.0
5.0
6.0
7.0

FIGURE 4



ψ_T^(4)(x)(Gev/c)^-2
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
1.44
ψ_E^(4)(x)
ψ_s^(4)
(d)
0
0.2
0.4
0.6
0.8
1
χ_T^(4)(x)(Gev/c)^-2
7
6
5
4
3
2
1
(d)
χ_E^(4)(x)
χ_s^(4)
0
0.2
0.4
0.6
0.8
1

FIGURE 5

HARDY-POLLACZEK POLYNOMIALS AND Γ AND ζ FUNCTIONS

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I. INTRODUCTION

Some of the results presented here were published recently [1]. The others are new. We will start with some historical remarks.

In 1910, Pidduck [2] used the polynomials $\mu_n(z)$ generated by

$$\frac{(1+t)^z}{(1-t)^{z+1}} = \sum_{n=o}^{\infty} \mu_n(z)t^n \tag{1}$$

In 1940, Hardy [3] studied the orthogonal set $P_n(x)$

$$P_n(x) = (-i)^n \mu_n(-\tfrac{1}{2} + ix) \tag{2}$$

$$\int_{-\infty}^{+\infty} \frac{P_n(x)P_m(x)}{\cosh \pi x} dx = \delta_{nm} \tag{3}$$

Taking into account the relations

$$\Gamma(\tfrac{1}{2} + it)\Gamma(\tfrac{1}{2} - it) = \frac{\pi}{\cosh \pi t} \tag{4}$$

we can write the integral (3) as the Hermitian scalar product in $\mathcal{L}_2$ of the functions $\psi_n(t)$ and $\psi_m(t)$ with

$$\psi_n(t) = \pi^{-1/2}\,\Gamma(\tfrac{1}{2} + it)\,\pi_n(-\tfrac{1}{2} + it)\ . \tag{5}$$

The polynomials $P_n(x)$ are a particular case of those introduced in 1950 by Pollaczek [4].

We suggest referring to the polynomials $P_n(x)$ as the <u>Hardy-Pollaczek polynomials</u>. We note that the $M_n(x) = n!P_n(\frac{x}{2})$ are called Meixner [5-6] polynomials and have integral coefficients. The moments of the polynomials μ_n, P_n, M_n are related in a simple way to the Bernoulli and Euler numbers.

In 1982, the Authors (H.B. and M.B.) were led to the following polynomials in x, y, z:

$$\mu_n(x,y,z) = \frac{2^n}{n!} \frac{\Gamma(z+1)}{\Gamma(z+1-n)} \, {}_2F_1(-n,-n-y,z+1-n \; ; \frac{1-x}{2}) . \qquad (6)$$

More precisely, the values taken by those polynomials for $y,z \in \mathbb{N}$ appear as matrix entries for elements of boson algebra in a harmonic oscillator basis [7-8] . The generating function of the $\mu_n(x,y,z)$ is

$$\frac{[1+(1+x)t]^z}{[1-(1-x)t]^{y+z+1}} = \sum_{n=o}^{\infty} \mu_n(x,y,z)t^n \qquad (7)$$

The Pollaczek and Pidduck polynomials correspond to the following particular cases:

$$P_n^{(\lambda)}(\xi) = (-i)^n \mu_n(0,2\lambda-1,-\lambda+i\xi) \qquad (8)$$

$$\mu_n(z) = \mu_n(0,0,z) . \qquad (9)$$

The Pidduck polynomials take integral values for $z \in \mathbb{N}$ with the nice symmetry property

$$\mu_n(\ell) = \mu_\ell(n) = P_\ell^{(n-\ell,0)}(3) \qquad (10)$$

where $P_\ell^{(\alpha,\beta)}$ stand for the Jacobi polynomials. The $\mu_n(\ell)$ are known in Combinatorics as Delannoy numbers [9-10] : $\mu_n(\ell)$ is the number of ways to go in $\mathbb{N}^2$ from (0,0) to (n,l) by steps of (1,0) , (0,1) or (1,1) (a consequence of Eq.(14) below).

II. SOME OF THE MAIN PROPERTIES OF THE H-P POLYNOMIALS

a) Generating functions

$$\sum_{n=o}^{\infty} \mu_n(z)t^n = \frac{(1+t)^z}{(1-t)^{z+1}} \qquad (11)$$

$$\sum_{n=o}^{\infty} \mu_n(z) \frac{t^n}{n!} = e^t \, M(-z,1,-2t) \qquad (12)$$

(M: confluent hypergeometric function).

b) Recurrence relations

$$(n+1)\mu_{n+1}(z) = (2z+1)\mu_n(z) + n\mu_{n-1}(z) \qquad (13)$$

$$\mu_{n+1}(z+1) = \mu_{n+1}(z) + \mu_n(z+1) + \mu_n(z) \qquad (14)$$

c) Contour integrals

$$\mu_n(z) = \frac{1}{2\pi i} \int_1 s^z \frac{(s+1)^n}{(s-1)^{n+1}} ds \qquad (15)$$

$$\mu_n(-\frac{1}{2}+i\xi) = \frac{1}{2\pi i} \int_0 e^{2i\theta\xi} \frac{(\coth\theta)^n}{\sinh\theta} d\theta , \qquad (16)$$

the integrals being around $s = 1$ and $\theta = 0$, respectively.

d) Relations with Laguerre polynomials

$$(-)^n \Gamma(z+1)\mu_n(z) = \int_0^\infty e^{-t} L_n(2t)\, t^{z-1} dt . \qquad (17)$$

III. THE Γ FUNCTIONS AND H-P POLYNOMIALS

The link between the Γ function and the Riemann ζ function and H-P polynomials can be given a group theoretical interpretation via a representation of $SL(2,R)$ investigated by Itzykson [11] . We shall not present these arguments here, however. We proceed in another way.

First, we introduce the Laplace transform

$$\sigma_n(p) = \int_0^\infty e^{-pt} \frac{(t-1)^n}{(t+1)^{n+1}} dt \qquad (18)$$

From

$$\frac{(t-1)^n}{(t+1)^{n+1}} = \int_0^\infty e^{-(t+1)u} L_n(2u)\, du \qquad (19)$$

we get

$$\sigma_n(p) = \int_0^\infty \frac{e^{-u} L_n(2u)}{p+u} du \qquad (20)$$

From recurrence relation of Laguerre polynomials, it is easy to see that

$$\sigma_n(p) - L_n(-2p)\sigma_0(p) \qquad (21)$$

is a polynomial of degree $n-1$. This means that $\sigma_n(p)$ is rationally related with $\sigma_0(p)$. Moreover,

$$S(p,x) = \sum_{n=0}^\infty \sigma_n(p)\, x^n = \frac{1}{1-x}\, \sigma_0 \left(p \frac{1+x}{1-x}\right) . \qquad (22)$$

Now, from

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds$$

and

$$\left(\frac{s}{p}\right)^{z-1} = 2 \sum_{n=o}^{\infty} \mu_n(z-1) \frac{(\frac{s}{p}-1)^n}{(\frac{s}{p}+1)^{n+1}}$$

if follows that

$$\frac{\Gamma(z)}{p^z} = 2 \sum_{n=o}^{\infty} \sigma_n(p)\mu_n(z-1). \tag{23}$$

The properties

$$(-1)^n\sigma_n(p) > 0 \qquad \text{for} \qquad p \geqq 1 \tag{24}$$

$$|\sigma_n(p+1)| < |\sigma_n(p)| \qquad \text{for} \qquad p \geqq 1 \tag{25}$$

make the following series convergent for $p \geqq 1$

$$\gamma_n(p) = \sigma_n(p) + \sigma_n(p+1) + \sigma_n(p+2) + \ldots \tag{26}$$

Another expression can be obtained from the orthogonality property for $\sigma_n(p)$, namely

$$\sigma_n(p) = \frac{(-1)^n}{2} \int_{-\infty}^{+\infty} \frac{\Gamma(\frac{1}{2}+it)\mu_n(-\frac{1}{2}+it)}{\cosh\pi t} p^{-\frac{1}{2}-it} dt \tag{27}$$

IV. THE RIEMANN ZETA FUNCTION

Three expansions have been given in reference [1] for the ζ function. Let us add the following one

$$\Gamma(z)\zeta(z) = 2 \sum_{n=o}^{\infty} \gamma_n\mu_n(z-1) \tag{28}$$

where

$$\gamma_n = \int_o^{\infty} \frac{1}{e^s-1} \frac{(s-1)^n}{(s+1)^{n+1}} ds . \tag{29}$$

We see that γ_n is nothing else than the value taken by the function $\gamma_n(p)$ of (26) for $p = 1$. It follows that

$$\gamma_n = \sum_{k=1}^{\infty} \int_o^{\infty} \frac{e^{-u}L_n(2u)}{k+u} d\mu . \tag{30}$$

Moreover, From (27):

$$\gamma_n = \frac{(-1)^n}{2} \int_{-\infty}^{+\infty} \frac{\Gamma(\frac{1}{2}+it)\mu_n(-\frac{1}{2}+it)}{\cosh\pi t} \zeta(\frac{1}{2}+it)dt \tag{31}$$

All these equations, and especially (28), suggest the use of expansions of $\zeta(z)$ (or related functions) in polynomials $\mu_n(z-1)$, since the zeros of the μ_n's are all on the line $\mathrm{Re}\, z = -1/2$. However, the expansion (28) does not use the symmetry property of the μ_n's , namely

$$\mu_n(-z-1) = (-)^n \mu_n(z) \tag{32}$$

It is then more natural to use, instead of $\zeta(z)$, Riemann's function $\zeta(z)$ which has the required symmetry

$$\xi(z) = \frac{z(z-1)}{2} \pi^{-z/2} \Gamma(\tfrac{z}{2}) \zeta(z) \tag{33}$$

$$\xi(z) = \xi(1-z) \ . \tag{34}$$

It is not very difficult to prove that

$$\xi(z) = \sum_{n=o}^{\infty} \beta_n \mu_{2n}(z-1) \tag{35}$$

with

$$\beta_n = 4 \int_1^{\infty} \frac{(t-1)^{2n}}{(t+1)^{2n+1}} \frac{d}{dt} (t^2 \psi'(t))\, dt \tag{36}$$

where

$$\psi(x) = 1/2 + \sum_{n=1}^{\infty} e^{-\pi n^2 x^2} \tag{37}$$

Note that the expansion (35) seems very promising for the proof of Riemanns's conjecture (ξ has all its zeros on the line $\mathrm{Re}\, z = 1/2$). The fact that the β_n are all positive and rapidly decreasing is encouraging. A natural idea is to investigate the properties of expansions of type (35) . We present some results in the next section. But first we give some properties about Mellin transforms and H-P polynomials.

The functions

$$\psi(t) \ , \qquad \frac{d}{dt}\left[t^2 \psi'(t)\right] \ , \qquad \frac{(t-1)^{2n}}{(t+1)^{2n+1}} \tag{38}$$

satisfy the common property

$$f(\tfrac{1}{x}) = x\, f(x) \tag{39}$$

Any function f satisfying (39) has a Mellin transform $\tilde{f}$ such that

$$\tilde{f}((1/2)+ix) = \int_{i}^{\infty} f(t)\cos(x\log t)\frac{dt}{\sqrt{t}} = \sum_{n=o}^{\infty} a_n i^n P_n(x) \qquad (40)$$

with $$a_n = 2\int_{o}^{\infty} \frac{(t-1)^n}{(t+1)^{n+1}} f(t)\,dt$$

(the proof is easy).

V. THE RIEMANN CONJECTURE?

Here we present three propositions which help us to prove that ξ has no real zero in the critical strip, a very weak result since it is a well-known property, but the singularity of the method (using the expansion (35)) seems to be promising.

Consider a polynomial $R_{2n}(x)$ of degree 2n of the type

$$R_{2n}(x) = \sum_{\ell=o}^{n} a_\ell \mu_{2\ell}(-\frac{1}{2}+ix) \qquad (41)$$

Define

$$F_{2n}(\tanh u) = \sum_{\ell=o}^{n} a_\ell (\tanh u)^{2\ell}. \qquad (42)$$

It is easy to prove that

$$\frac{F_{2n}(\tanh u)}{\cos h\, u} = \int_{-\infty}^{+\infty} \frac{R_{2n}(x)}{\cosh \pi x} e^{2ixu}\, dx \qquad (43)$$

and the Fourier inverse

$$\frac{R_{2n}(x)}{\cosh \pi x} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{F_{2n}(\tanh u)}{\cosh u} e^{-2ixu}\, du \quad .$$

From here, using arguments similar to these in work of Polya [12], follow the three propositions:

<u>Prop. 1</u>. The number of zeros of $R_{2n}(x)$ in the interval $]-\frac{i}{2},\frac{i}{2}[$ is not greater than the number of real zeros of $R_{2n}(\tanh u)$.

<u>Prop. 2</u>. The number of real zeros of $R_{2n}(x)$ is at least equal to the number of zeros of $F_{2n}(\tanh u)$ in $]-\frac{i}{2},\frac{i}{2}[$.

<u>Prop. 3</u>. If the a_ℓ satisfy $0 < a_{\ell+1} < a_\ell$, $F_n(\tanh u)$ has no zero in the strip $|\mathrm{Im}\, u| \leq \frac{\pi}{4}$ and R_{2n} has no zero in $-]\frac{i}{2},\frac{i}{2}[$.

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UNE CARACTERISATION DES POLYNÔMES ORTHOGONAUX SEMI-CLASSIQUES

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Abstract

Let L be a regular linear functional and $\{P_n\}_{n\geq 0}$ the orthogonalpolynomials associated with L. Let define: $Q_n(x)=P_n'(x)+P_n(x)$. We characterize L, in the case when there exists a fixed nonnegative integer s such that $\{Q_n\}_{n\geq 0}$ is a quasi-orthogonal sequence, of order s, with respect to a linear functional H.

1. INTRODUCTION

Les polynômes classiques (Jacobi, Bessel, Laguerre, Hermite) peuvent être caractérisés comme étant des suites orthogonales dont la suite des dérivées est aussi orthogonale [1], [2]. Récemment MARONI ([3], [4]) a présenté une théorie générale des polynômes semi-classiques c'est à-dire des polynômes orthogonaux dont la suite des dérivées est quasi-orthogonale d'un certain ordre; en ce sens, les polynômes semi-classiques généralisent immédiatement les polynômes classiques.

L'objet de cet article est de caractériser les suites $\{P_n\}_{n\geq 0}$ tels que la suite $\{Q_n\}_{n\geq 0}$ définie par: $Q_n(x) = P_n'(x) + P_n(x)$ soit quasi-orthogonale d'un certain ordre.

$\{Q_n\}_{n\geq 0}$ se présente comme une perturbation de P_n' et de P_n.

En imposant une hypothèse aussi peu contraignante que la faible orthogonalité sur la suite $\{Q_n\}_{n\geq 0}$ on démontre que la suite $\{P_n\}_{n\geq 0}$ est semi-classique. Ce point de vue permet de réorganiser, autrement, les suites semi-classiques. Introduisons certaines définitions qu'on utilisera dans la suite:

Définition 1.1 [1] La suite libre $\{B_n\}_{n\geq 0}$ est dite faiblement orthogonale d'index (p,q) par rapport à L, s'il existe un couple d'entiers $p,q\geq 1$ tels que:

$$L(B_{p-1}) \neq 0\,, \qquad L(B_n) = 0 \quad \text{si} \quad n \geq p$$

$$L(x\,B_{q-1}(x)) \neq 0\,, \quad L(x\,B_n(x)) = 0 \quad \text{si} \quad n \geq q\,.$$

Définition 1.2 ([3], [5], [6]). La suite libre $\{B_n\}_{n\geq 0}$ est dite quasi orthogonale d'ordre s relativement à L si elle vérifie:

$$L(x^m B_n(x)) = 0 \quad , \quad 0 \leq m \leq n-(s+1) \quad , \quad n \geq s+1$$

il existe $\tau \geq s$ tel que: $L(x^{\tau-s}B(x)) \neq 0$.

Définition 1.3 ([3]). La suite libre $\{B_n\}_{n\geq 0}$ est dite strictement quasi-orthogonale d'ordre s relativement a L si elle vérifie:

$$L(x^m B_n(x)) = 0 \quad , \quad 0 \leq m \leq n-(s+1) \quad , \quad s \geq s+1$$
$$\forall\, n \geq s \ , \quad L(x^{n-s}\, B_n(x)) \neq 0.$$

2. L'OPERATEUR $\mathcal{D}$

Considérons l'opérateur $\mathcal{D}: \mathcal{P} \longrightarrow \mathcal{P}$ défini par:

$$\mathcal{D} = D + id \tag{2.1}$$

ou $\mathcal{P} = \mathbb{C}[X]$ et $D = d/dx$.

Il est aisé de voir que $\mathcal{D}$ vérifie les résultats donnés par:

Lemme 2.1 Soit $\alpha,\beta \in \mathbb{C}$ et $P,R \in \mathcal{P}$

$$\mathcal{D}(\alpha P+\beta R) = \alpha\, \mathcal{D}(P) + \beta\mathcal{D}(R) \tag{2.2}$$
$$\mathcal{D}(PR) = P\, \mathcal{D}(R) + R\, \mathcal{D}(P) - PR \tag{2.3}$$
$$\mathcal{D}(x) = 1+x \tag{2.4}$$

Soit maintenant $P_n(x)$ un polynôme normalisé, on notera:

$$Q_n = \mathcal{D}(P_n) \quad \text{pour} \quad n \geq 0 \ . \tag{2.5}$$

Considérons une suite normalisée $\{P_n\}_{n\geq 0}$ régulierèment orthogonale par rapport à L . Introduisons la récurrence d'ordre deux vérifiée par la suite $\{P_n\}_{n\geq o}$

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad \text{où} \quad \gamma_n \neq 0,\ n \geq 0$$
$$P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0 \tag{2.6}$$

On a de (2.6) en faisant $n \rightarrow n-1$ et tenant compte du lemme 2.1 :

$$P_n(x) = Q_{n+1}(x) - (x - \beta_n)\, Q_n(x) + \gamma_n\, Q_{n-1}(x) \quad , \quad n \geq 0 \tag{2.7}$$

avec la convention suivante: tout polynôme d'indice négatif sera considéré comme nul. On va caractériser certaines suites régulierèment orthogonales $\{P_n\}_{n\geq 0}$ à l'aide d'une propieté de la suite $\{Q_n\}_{n\geq 0}$

3. LES $\mathcal{D}$-POLYNOMES ORTHOGONAUX SEMI-CLASSIQUES

On se place dans les conditions décrites a la fin du paragraphe 2.

Théoreme 3.1 "Les propositions suivantes son équivalentes:

(P_1) il existe une forme linéaire H telle que la suite $\{Q_n\}_{n\geq 0}$ soit faiblement orthogonale d'index (p,q) par rapport a H, $p,q \geq 1$.

(P_2) les formes linéaires L et H vérifient les conditions:

a) il existe un polynôme Ψ unique de degré $p-1$ tel que:

$$H(\mathcal{D}(P)) = L(\Psi P) \quad , \quad P \in \mathcal{P} \tag{3.1}$$

b) il existe un polynôme Λ unique de degré $q-1$ tel que:

$$H(x\mathcal{D}(P)(x)) = L(\Lambda P), \; P \in \mathcal{P} \tag{3.2}$$

Définissant l'entier $s \geq 0$ par $s+1 = \max(p,q-1)$, il existe un entier r $(0 \leq r \leq s+1)$ et un polynôme Φ unique de degré $s+1-r$ tels que:

$$H(P) = L(\Phi P) \quad , \quad P \in \mathcal{P} \tag{3.3}$$

de plus on a: $\Phi(x) = x\Psi(x) - \Lambda(x)$.

(P_3) il existe s et r deux entiers, $s \geq 0$, $0 \leq r \leq s+1$ et une forme linéaire H tels que: la suite $\{P_n\}_{n\geq 0}$ est strictement quasi-orthogonale d'ordre $(s+1-r)$ par rapport à H. La suite $\{Q_n\}_{n\geq 0}$ est quasi-orthogonale d'ordre s par rapport à H.

(P_{3bis}) il existe une forme linéaire H telle que la suite $\{Q_n\}_{n\geq 0}$ soit quasi-orthogonale d'ordre s par rapport à H.

(P_4) il existe $s \geq 0$ et $0 \leq t \leq s+1$ entiers et un polynôme Φ de degré t tels que:

$$\Phi(x)\, Q_n(x) = \sum_{\nu=n-s}^{n+t} \theta_{n,\nu}\, P_\nu(x) \qquad n \geq s \tag{3.4}$$

$$\exists\; \tau \geq 0 \qquad \text{tel que} \qquad \theta_{\tau,\tau-s} \neq 0 \tag{3.5}$$

(P_{4bis}) il existe $s,t \geq 0$ entiers et un polynôme Φ de degré t tels que (3.4) et (3.5) soient vérifiés.

(P_5) il existe un polynôme Ψ de degré $(p-1)$ et un polynôme Λ de degré $(q-1)$ tels que:

$$L(\Psi P) = L([x\Psi(x) - \Lambda(x)]\; \mathcal{D}(P)(x)) \quad , \quad P \in \mathcal{P} \tag{3.6}$$"

Théoreme 3.2 $\{P_n\}_{n\geq 0}$ suite régulierèment orthogonale par rapport à L. Soit $Q_n = \mathcal{D}(P_n)$, $n \geq 0$; $\mathcal{D} = D + id_{\mathcal{P}}$ et $\{Q_n\}_{n\geq 0}$ faiblement orthogonale d'index (p,q), $p,q \geq 1$, par rapport à H. Alors: $\{P_n\}_{n\geq 0}$ est une $\mathcal{D}$-suite orthogonale semi-classique de classe s [4] , où $s + 1 = \max(p, q-1)$.

4. SUITES SEMI-CLASSIQUES

Comme on l'a signalé dans l'introduction $\mathcal{D} = D + id_{\mathcal{P}}$ est un opérateur qui peut s'interpréter comme une double perturbation soit de l'opérateur D , soit de l'opérateur $id_{\mathcal{P}}$. Dans ce paragraphe on va montrer qu'effectivement, sous les hypothèses du théorème 3.2 $\{P_n\}_{n\geq 0}$ est une suite semi-classique tout court.

Théorème 4.1 "Soit $\{P_n\}_{n\geq 0}$ régulièrement orthogonale par rapport à $\mathcal{L}$ et $\{\mathcal{D}(P_n)\}_{n\geq 0}$ faiblement orthogonale d'index (p,q), $p,q \geq 1$, relativement a $\mathcal{H}$. Alors:

$$\forall P \in \mathcal{P} \qquad \mathcal{H}(D(P)) = \mathcal{L}(\tilde{\Psi}P) \tag{4.1}$$

$$\mathcal{H}(xD(P)(x)) = \mathcal{L}(\tilde{\Lambda}P) \tag{4.2}$$

$$\mathcal{H}(P) = \mathcal{L}(\tilde{\Phi}P). \tag{4.3}$$

Avec,

$$\tilde{\Psi} = \Psi - \Phi \tag{4.4}$$

$$\tilde{\Lambda}(x) = \Lambda(x) - x\Phi(x) \tag{4.5}$$

$$\tilde{\Phi} = \Phi = x\Psi(x) - \Lambda(x), \tag{4.6}$$

où Ψ , Λ et Φ sont les polynômes énoncés dans le théorème 3.1.

De plus on a:

$$\mathcal{L}(\tilde{\Phi}P) = \mathcal{L}([x\tilde{\Psi}(x) - \tilde{\Lambda}(x)]P'(x))."$$

La démonstration de ce théorème est immédiate; en prenant (3.1), (3.2) et (3.3) et en explicitant l'expression de $\mathcal{D} = D + id_{\mathcal{P}}$.

Théoreme 4.2 Soit $\{P_n\}_{n\geq 0}$ régulierèment orthogonale par rapport à $\mathcal{L}$, $\{Q_n\}_{n\geq 0}$ faiblement orthogonale d'index (p,q) relativement à $\mathcal{H}$ et $\{\tilde{Q}_n\}_{n\geq 0}$ définie par $(n+1)\tilde{Q}_n = P'_{n+1}$, $n \geq 0$. Alors:

si $p \neq q - 1$, $\{\tilde{Q}_n\}_{n\geq 0}$ est strictement quasi-orthogonale d'ordre s et donc $\{P_n\}_{n\geq 0}$ est une suite semi-classique de classe s.

si $p = q - 1$, degré $\Phi = s + 1 - r$, $0 \leq r \leq s + 1$

* $r = 0$, $\{\tilde{Q}_n\}_{n\geq 0}$ est strictement quasi-orthogonale d'ordre s et donc $\{P_n\}_{n\geq 0}$ est une suite semi-classique de classe s.

* $r \geq 2$, $\{Q_n\}_{n\geq 0}$ est strictement quasi-orthogonale d'ordre $s-1$ et donc $\{P_n\}_{n\geq 0}$ est une suite semi-classique de classe $s-1$.

* $r = 1$,

- degré $\tilde{\Psi} = s$, degré $\tilde{\Lambda} = s + 1$, $\{\tilde{Q}_n\}_{n\geq 0}$ est strictement quasi-orthogonale d'ordre (s-1) et donc $\{P_n\}_{n\geq 0}$ est une suite semi-classique de classe (s-1).
- degré $\tilde{\Psi} \leq s - 1$, degré $\tilde{\Lambda} = s$, $\{\tilde{Q}_n\}_{n\geq 0}$ est quasi-orthogonale d'ordre (s-2) et donc $\{P_n\}_{n\geq 0}$ est de classe (s-2) .
- degré $\tilde{\Psi} = s - 1$, degré $\tilde{\Lambda} \leq s$.

5. MOMENTS ET FORMES DES SUITES SEMI-CLASSIQUES

En vertu du théorème 4.1 , toutes les formes L régulières associées aux suites semi-classiques sont données par:

$$L(\tilde{\Psi}P) = L([x\,\tilde{\Psi}(x) - \tilde{\Lambda}(x)]\,P(x)\,) , \qquad P \in \mathcal{P} \qquad (5.1)$$

où les polynômes $\tilde{\Psi}$ et $\tilde{\Lambda}$ sont donnés par les expressions (4.4) et (4.5), le degré de $\tilde{\Psi}$ et $\tilde{\Lambda}$ dépend de la position de Φ par rapport à (q-1) . Ainsi on a :

A - $1 \leq p < q - 1$, $s+1 = q-1$

$$\tilde{\Psi}(x) = \sum_{k=p+1}^{s+1} \lambda_k x^k + (\lambda_p - \psi_{p-1})x^p + \sum_{k=1}^{p-1} (\lambda_k + \psi_k - \psi_{k-1})\, x^k + (\psi_o + \lambda_o) \qquad (5.2)$$

$$\Lambda(x) = \lambda_{s+1}\, x^{s+2} + \sum_{k=p+2}^{s+1} (\lambda_k + \lambda_{k-1})x^k + \sum_{k=2}^{p+1} (\lambda_k + \lambda_{k-1} - \psi_{k-2})x^k + (\lambda + \lambda_o)x + \lambda_o \qquad (5.3)$$

$$\Phi(x) = \sum_{k=p+1}^{s+1} \lambda_k\, x^k + \sum_{k=1}^{p} (\psi_{k-1}\ \lambda_k)\, x^k - \lambda_o \qquad (5.4)$$

ou λ_k , ψ_k sont, respectivement, les coefficients de Λ et Ψ . Si l'on pose $\mu_n = L(x^n)$, $n \geq 0$, (5.1) devient:

$$\sum_{k=p+1}^{s+1} \lambda_k(\mu_{n+k} + n\, \mu_{n+k-1}) + \sum_{k=1}^{p} (\lambda_k - \psi_{k-1})\, (\mu_{n+k} - n\, \mu_{n+k-1}) + \sum_{k=o}^{p-1} \psi_k\, \mu_{n+k} +$$

$$\lambda_o(\mu_n - n\, \mu_{n-1}) = o \quad , \qquad n = o .$$

c'est une relation de recurrence a (s+3) termes.

B - $o \leq q - 1 < p$, $s+1 = p$

$$\tilde{\Psi}(x) = \psi_s\, x^{s+1} + \sum_{k=f}^{s} (\psi_k - \psi_{k-1})x^k + \sum_{k=1}^{q-1} (\psi_k - \psi_{k-1} + \lambda_h)x^k + (\psi_o + \lambda_o) \qquad (5.5)$$

$$\tilde{\Lambda}(x) = - \sum_{k=g+1}^{s+2} \psi_{k-2}\, x^k + (\lambda_{q-1} - \psi_{q-2})\, x^q + \sum_{k=2}^{q-1} (\lambda_k + \lambda_{k-1} - \psi_{k-2})x^k + (\lambda_1 + \lambda_o)x + \lambda_o . \qquad (5.6)$$

$$\Phi(x) = \sum_{k=q}^{s+1} \psi_{k-1}\, x^k + \sum_{k=1}^{q-1} (\psi_{k-1} - \lambda_u)\, x^k - \lambda_o . \qquad (5.7)$$

En faisant $P(x) : x^n$ dans (5.1) on a:

$$- \psi_s(\mu_{n+s+1} + n\, \mu_{n+s}) - \sum_{k=q}^{s} \psi_{k-1} (\mu_{n+k} + n\, \mu_{n+k-1}) + \sum_{k=1}^{q-1} (\lambda_k - \psi_{k-1})$$

$$(\mu_{n+k} + n\, \mu_{n+k-1}) + \sum_{k=o}^{s} \psi_k \, \mu_{n+k} + \lambda_o(\mu_n + n\, \mu_{n-1}) = 0 \qquad , n \geq 0$$

encore une fois on a une relation de recurrence a (s+3) termes.

C - <u>p = q-1</u> , s+1 = p = q-1

$$\tilde{\Psi}(x) = (\lambda_{s+1} - \psi_s)x^{s+1} + \sum_{k=1}^{s} (\psi_k - \psi_{k-1} + \lambda_k)\, x^k + (\lambda_o + \psi_o) \qquad (5.8)$$

$$\tilde{\Lambda}(x) = (\lambda_{s+1} - \psi_s)x^{s+2} + \sum_{k=2}^{s+1} (\lambda_k + \lambda_{k-1} - \psi_{k-2})x^k + (\lambda_1 + \lambda_o)x + \lambda_o \qquad (5.9)$$

$$\Phi(x) = \sum_{k=1}^{s+1} (\psi_{k-1} - \lambda_k)x^k - \lambda_o \qquad (5.10)$$

ainsi en remplaçant $P(x)$ par x^n dans (5.1) on obtient:

$$(\lambda_{s+1} - \psi_s) (\mu_{n+s+1} + n\, \mu_{n+s}) + \sum_{k=1}^{s} (\lambda_k - \psi_{k-1}) (\mu_{n+k} + n\mu_{n+k-1}) +$$

$$\sum_{k=o}^{s} \psi_k \, \mu_{n+k} + \lambda_o(\mu_n + n\, \mu_{n-1}) = o \qquad , n \geq o$$

C'est une relation de recurrence à (s+3) ou à (s+2) termes selon que r=0 ou r=1 .

On remarque que les trois relations obtenues sont des équations aux différences dont les coefficients sont des fonctions affines en n et donc relativement faciles à resoudre, ainsi on obtiendrait les moments qui déterminent la forme linéaire associée. On ne traitera pas ici la résolution de ces équations mais on va chercher une représentation intégrale (si elle existe) de la forme L selon l'expression:

$$L(P) = \int_C Z(x)\, P(x)\, dx \qquad (5.11)$$

où C un chemin, éventuellement pris dans le champ complexe, Z une fonction assez réguliére, seront précisés plus loin.

On a donc pour (5.1) d'aprés (5.11)

$$\int_C Z(x)\, \tilde{\Psi}(x)\, P(x)\, dx + \int_C [\Phi(x) Z(x)] P(x) dx = Z(x) \Phi(x) P(x) \Big|_C$$

si la fonction Z et le chemin C sont tels que:

$$(\Phi Z)' + \tilde{\Psi} Z = 0 \qquad (5.12)$$

$$\Phi\, Z\, P \Big|_C = 0 \qquad (5.13)$$

ils fournissent, au moins formellement, une solution de l'équation (5.1) par l'intérmédiaire de (5.11).

Ecrivons (5.12) autrement, en se rappelant que $\tilde{\Psi}$, Ψ et Φ sont liés par $\tilde{\Psi} = \Psi - \Phi$. Ainsi on divisant (5.12) par Φ on obtient:

$$\frac{(\Phi Z)'}{\Phi Z} = 1 - \frac{\Psi}{\Phi} . \qquad (5.14)$$

où degré $\Psi = p-1$, $p \geq 1$, degré $\Phi = s+1-r$, $0 \leq r \leq s+1$, cette équation géneralise l'équation différentielle de K.Pearson ([7], [8]). On va traiter le cas où Ψ et Λ sont des polynômes à coefficients réels.

1er cas: $p \neq q - 1$

Conformement à ce qui est donné en (5.4) et (5.7) Ie degré de Φ est strictement plus grand que celui de Ψ .
si $p \neq q - 1$ on a: degré $\Phi = s+1$, Φ peut s'écrire alors:

$$\Phi(x) = K \prod_{i \in I_1} (x-\rho_i) \prod_{i \in I_2} (x-\eta_i)^{h_i} \prod_{i \in I_3} (x-\delta_i)(x-\bar{\delta}_i)$$

ou ρ_i sont des racines réelles simples, η_i sont des racines réelles de multiplicité h_i et δ_i sont des racines complexes -on a supposé qu'elles sont simples- avec $|I_1| + \sum_{i \in I_2} h_i + 2|I_3| = s + 1$.

Ainsi on a:

$$\frac{\Psi(x)}{\Phi(x)} = \sum_{i \in I_1} \frac{a_i}{x-\rho_i} + \sum_{i \in I_2} \frac{b_i}{x-\eta_i} + \sum_{i \in I_2} \sum_{j=2}^{h_i} \frac{c_{i,j}}{(x-\eta_i)^j} +$$

$$+ \sum_{i \in I_3} \left(\frac{d_i}{x-\delta_i} + \frac{\bar{d}_i}{x-\bar{\delta}_i} \right) ,$$

où $a_i = \text{Résidu}(\Psi/\Phi, \rho_i)$, $b_i = \text{Résidu}(\Psi/\Phi, \eta_i)$, $d_i = \text{Résidu}(\Psi/\Phi, \delta_i)$

et donc (5.14) a pour solution:

$$Z(x) = \tilde{K} e^x \prod_{i \in I_1} (x - \rho_i)^{-1-ui} \prod_{i \in I_2} (x- \eta_i)^{-bi-hi} \prod_{j=2}^{hi} \exp \left\{ \frac{(i-j)\, C_{i,j}}{(x - \eta_i)^{j-1}} \right\}$$

$$\times \prod_{i \in I_3} (x^2 - 2x\, \mathcal{R}e(\delta_i) + |\delta_i|^2)^{-1-\text{Re}(di)} \exp \left\{ \frac{\text{Re}(\delta_i) - \text{Re}(\delta_i \delta_i)}{\text{Jm}(\delta_i)} \text{Arc tg} \right.$$

$$\left. \left(\frac{x - \text{Re}(\delta_i)}{\text{Jm}(\delta_i)} \right) \right\} \qquad (5.15)$$

($\tilde{K}$ est une constante).

Le choix de C dépend des positions de ρ_i, η_i et des signes de $1+a_i$, b_i+h_i, $c_{i,j}$. Supposons par exemple qu'il existe $i_o \in I_1$ tel que $\rho_{io} < \rho_i$, $i \neq i_o$, $i \in I_1$ et $1+a_{io} < 0$, alors $C = (-\infty, \rho_{io}]$ convient et (5.13) est bien vérifiée.

On remarque que Z donnée par (5.15) géneralise à la fois les fonctions poids associées aux polynômes de Laguerre et Jacobi.

<u>2° cas $p = q - 1$</u>

Dans ce cas le degré de Φ peut devenir plus petit que celui de Ψ, c'est-à-dire degré $\Phi = s+1-r$, $0 \leq r \leq s+1$. Pour $r = 0$ on trouve pour Z la même expression que celle donnée en (5.14). Si $r \geq 1$ on a:

$$\Psi/\Phi = A + (B/\Phi) \quad \text{avec degré } A = r-1, \quad \text{degré } B < \text{degré } \Phi$$

plus précisement:

$$\frac{\Psi}{\Phi} = A + \sum_{i\in J_1} \frac{\alpha_i}{x-\rho_i} + \sum_{i\in J_2} \frac{\beta_i}{x-\eta_i} + \sum_{i\in J_2} \sum_{j=2}^{h_i} \frac{\gamma_{i,j}}{(x-\eta_i)^{j-1}} +$$

$$+ \sum_{i\in J_3} \frac{2x\,\mathrm{Re}\nu_i - 2\mathrm{Re}\nu_i\delta_i}{x^2 - 2x\mathrm{Re}\delta_i + |\delta_i|^2}$$

avec $\alpha_i = \text{Résidu}(B/\Phi, \rho_i)$, $\beta_i = \text{Résidu}(B/\Phi, \eta_i)$, $\nu_i = \text{Résidu}(B/\Phi, \delta_i)$ et $|J_1| + \sum_{i\in J_2} h_i + 2|J_3| = s + 1 - r$.

Soit $R(x) = \int [1 - A(x)]\, dx$, $R(x)$ est un polynôme de degré r.

$$Z(x) = \tilde{K}\, e^{R(x)} \prod_{i\in J_1} (x-\rho_i)^{-1-\alpha_i} \prod_{i\in J_2} (x-\eta_i)^{-\beta_i-h_i} \prod_{j=2}^{h_i} \exp\left\{\frac{(1-j)\gamma_{i,j}}{(x-\eta_i)^{j-1}}\right\}$$

(5.16)

$$\times \prod_{i\in J_3} (x^2-2x\mathcal{R}e(\delta_i)+|\delta_i|^2)^{-1-\mathcal{R}e(\nu_i)} \exp\left\{2.\frac{\mathcal{R}e(\delta_i)-\mathcal{R}e(\nu_i\delta_i)}{\mathcal{J}m(\delta_i)} \mathrm{Arctg}\left(\frac{x-\mathcal{R}e(\delta_i)}{\mathcal{J}m(\delta_i)}\right)\right\}$$

Les expressions (5.15) et (5.16) peuvent être considérées comme une généralisation des fonctions poids étudiées par AMUNDSEN, DAMGAARD ([10]) et NEVAI, BONAN et LUBINSKY ([11]). Aussi RONVEAUX ([12]) donne un cas particulier de (5.16).

6. <u>ETUDE D'UN CAS PARTICULIER</u>

$$(p,q) = (1,2) \qquad \text{autrement} \quad s = 0$$

On a: $\Psi(x) = \psi_0$, $\Lambda(x) = \lambda_1 x + \lambda_0$, $\Phi(x) = (\psi_0-\lambda_1)x - \lambda_0$

(on suppose $\psi_0 \neq \lambda_1$, c'est-a-dire degré $\Phi = 1$).

Ainsi

$$Z(x) = \frac{k}{\psi_0-\lambda_1} \; e^x \left(x - \frac{\lambda_0}{\psi_0-\lambda_1} \right)^{(2\psi_0-\lambda_1)/(\lambda_1-\psi_0)} , \quad k \in \mathbb{R} ,$$

si $\psi_0/(\lambda_1-\psi_0) > 0$, alors on choisit comme chemin d'intégration $(-\infty, \lambda_0/(\psi_0-\lambda_1)]$:

$$L(P) = \frac{k}{\psi_0-\lambda_1} \int_{-\infty}^{\frac{\lambda_0}{\psi_0-\lambda_1}} e^x \left(x - \frac{\lambda_0}{\psi_0-\lambda_1}\right)^{\frac{2\psi_0-\lambda_1}{\lambda_1-\psi_0}} P(x)\, dx$$

ainsi on voit que L est la forme linéaire associée aux polynômes de Laguerre. Pour retrouver l'expression classique de L , on va procéder à un changement de la variable; on pose: $u = -(x - \lambda_0/(\psi_0-\lambda_1))$ et la forme devient

$$L(P) = \tilde{k} \int_0^{+\infty} e^{-u} u^{\alpha} P(u)\, du$$

ou $\tilde{k} = (-1)^{\alpha} e^{\lambda_0/(\psi_0-\lambda_1)} \dfrac{k}{\psi_0-\lambda_1}$ avec $\alpha = \dfrac{2\psi_0-\lambda_1}{\lambda_1-\psi_0}$,

et donc $\{P_n\}_{n\geq 0}$ est constituée des polynômes de Laguerre en u :

$$P_n(x) = P_n^*(u) = L_n^{\alpha}(u) .$$

De (4.3) et (4.5) on tire:

$$H(P) = (\Phi P) = (-1)^{\alpha+1} e^{\lambda_0/(\psi_0-\lambda)} k \int_0^{+\infty} e^{-u} u^{\alpha+1} P(u)\, du ,$$

on voit que les Q_n sont aussi de "Laguerre" d'un ordre supérieur à celui des $\{P_n\}_{n\geq 0}$ d'une unité.

Finalement, $P_n' = Q_n - P_n$ se traduit par:

$$\frac{dL_n^{\alpha}(x)}{du} = - L_n^{\alpha+1} + L_n^{\alpha}(u) .$$

ainsi on voit que la derivée d'un polynome de Laguerre est égale à la différence de deux polynômes de Laguerre de même degré et dont l'ordre varie d'une unité. Cette relation peut être aisément retrouvée à partir de relations connues ([9]).

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ORTHOGONAL POLYNOMIALS AND GEOMETRIC CONVERGENCE OF PADE-TYPE APPROXIMANTS

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I. Introduction

In this article we are concerned with Padé-type Approximants [1] both in one and two points (PTA's and 2PTA's).

The problem of a (precisely defined) optimal velocity of convergence is considered for sequences of these approximants and the answer is found in terms of the Tchebycheff polynomials of the region where the function is approximated over.

Our results refer to regular regions, i.e. those with a Green's function $G(x,y)$ with pole at infinity [2].

In the sequel, we call "uniformly distributed points" on a curve Γ to those equidistant on Γ with respect to the metric $d\mu = \frac{\partial G}{\partial n}|dt|$ where G represents the Green's funtion of $\mathrm{Ext}(\Gamma)$ and $\underline{n}$ is the exterior normal to Γ [2].

The following result will be useful later.

Lemma 1 ([2])

Let E be an open connected of $\bar{\mathbb{C}}$. Let ∂E be an analytic Jordan curve and assume that $K = \bar{\mathbb{C}} \setminus E$ is regular. Let $G(z)$ represent the Green's function of K with pole at infinity and $\{\beta_j\}$ are points uniformly distributed on ∂E. In these conditions, by taking

$$Q_m(z) = \prod_{j=1}^{m} (z-\beta_j), \text{ one has}$$

1) $\overline{\lim_{m\to\infty}} |Q_m(z)|^{1/m} = \mathrm{cap}(E), \ \forall z \in E$ and

2) $\overline{\lim_{m\to\infty}} |Q_m(z)|^{1/m} = e^{G(z)}\,\mathrm{cap}(E); \ \forall z \in K.$

II. One Point Padé - Type Approximants

Let f(z) be an analytic funtion defined on $D\cup\partial D$, D is a region of $\mathbb{C}$ containing the origin. Hence f admits a Taylor expansion of the form

$$f(z) = \sum_{j=0}^{\infty} c_j z^j$$

We shall consider sequences of PTA's $(m-1/m)_f(z) = P_m(z)/Q_m(z) = R_m(z)$, such that $Q_m(z) \neq 0$, $\forall z\in D$.

Definition

A sequence P_m/Q_m of PTA's to f(z) has "optimal velocity of convergence" when $\overline{\lim_{m\to\infty}} |\bar{Q}_m(z)|^{-1/m} . \sup_{t\in\Gamma}|\bar{Q}_m(t)|^{1/m}$ (an upper bound of the contour integral error formula) attains its minimum value for $\bar{Q}_m = Q_m$.

Baumel et al. [3] essentially used this concept in a problem of approximation in presence of branch points.

For sequences of this kind, it can be proved the following

Theorem 1

Let D be a Jordan region with boundary $\partial D = C$ and Green's function G. The optimal velocity of convergence of $\{R_m(z)\}$ is attained when their poles are uniformly distributed on C.

Proof

The contour error formula for these approximants is

$$f(z) - R_m(z) = \frac{z^{m+1}}{2\pi i Q_m(z)} \int_\Gamma \frac{f(t)Q_m(t)dt}{t^{m+1}(t-z)}$$

By taking limits and using Lemma 1, one has

$$\overline{\lim_{m\to\infty}} |f(z) - P_m(z)/Q_m(z)|^{1/m} \leq \frac{|z|}{d} \frac{1}{|\phi(z)|} \quad \text{where}$$

$d = \min_{t\in\Gamma}\{|t|\}$ and $\phi(z) = e^{G+iH}$ (H the armonic conjugate of G in D)

On the other hand, taking a different sequence of approximants $\bar{P}_m(z)/\bar{Q}_m(z)$, yields

$$\overline{\lim_{m\to\infty}} |f(z) - \bar{P}_m(z)/\bar{Q}_m(z)|^{1/m} \leq \frac{|z|}{d} \lim \frac{\sup_{t\in\Gamma}|\bar{Q}_m(t)|^{1/m}}{|\bar{\Phi}_m(z)|^{1/m}}$$

since

$$\overline{\lim_{m\to\infty}} \left| \sup_{t\in\Gamma} |Q_m(t)| / |Q_m(z)| \right|^{1/m} \leq \frac{1}{|\phi(z)|},$$

we get the desired result.

The next Theorem makes use of the properties of the conformal mapping in order to select the poles of the approximants of Theorem 1. Our approach is simply to follow the images of the Tchebycheff knots in a map from $[-1,1]$ to $U = \{z: |z| = 1\}$ and then to the boundary of the region under consideration.

Theorem 2

Let C be a Jordan curve in the z-plane and f(z) the conformal map carrying the interior of C onto the unit disk in the w-plane and C onto U.

Let $G_2(w)$ be differentiable on U and such that $\int_\gamma \frac{\partial G_2}{\partial n_2}|dt_2|$ depends only on the length of $\gamma \subset U$. In these conditions $\int_\delta \frac{\partial G_1}{\partial n_1}|dt_1|$ depends only on the length of $\delta \subset C$, where $G_1 = G_2 \circ f$, and n_1 and n_2 are the normals to C and U respectively.

Proof

The analytic function f(z) maps the curve C in the z-plane (z = x+iy) onto the curve U in the w-plane (w=u+iv). Let w_o be any point on U. Define the function

$$h(w) = \int_{\Gamma_1} - \int_{\Gamma_2} \left(\frac{\partial G_2}{\partial n_2}\right) |dt_2|$$

Γ_1 and Γ_2 are the arcs of C from to w_o to w and from w_o to $\tilde{w}$ with $\operatorname{Arg}(w-w_o) = \operatorname{Arg}(w_o-\tilde{w})$. Under these conditions, $h \equiv 0$.

Let $z = z(r)$ and $w = w(s)$ be parametric representations of respectively C and U.

The function $H(z) = h(f(z))$ is given by the integration on the arcs c_1 and c_2 (preimages of Γ_1 and Γ_2) of the function $\frac{\partial G_1}{\partial n_1}|dt_1|$. To justify the last statement, we make use of the property of f being conformal. Then the function corresponding to $G_1(z)$ through the map $f: z \to w$ is $G_2(w)$ and $\operatorname{grad}(G_1) = \operatorname{grad}(G_2).\bar{f}'$. Where grad(.) denotes gradient of (.).

But $\frac{\partial G_1}{\partial n_1} = \operatorname{grad}(G_1).\vec{\alpha}$ and $\frac{\partial G_2}{\partial n_2} = \operatorname{grad}(G_2).\vec{\beta}$

Since f is conformal, $\vec{\alpha} = \vec{\beta}$

Moreover, $|dt_1| = z'(r)|dr| = \frac{dz}{dr}|dr|$, and

$$|dt_2| = w'(s)|ds| = \frac{dw}{ds}|ds|.$$

Hence $|dt_1| = |dt_2|/\bar{f}'$. It follows that

$$H(z) = \int_{c_1} - \int_{c_2} \frac{\partial G_1}{\partial n_1}|dt_1| = 0.$$

which is equivalent to the result to be proved.

Corollary

Let D be a Jordan region with boundary C. Then, there exists a function f(x) from [-1,1] to C such that the images of the zeros of the Tchebycheff polynomials are uniformly distributed on C.

This result constitutes a practical choice of the poles in the optimal sequence of PTA's in Theorem 1.

II.1 Uncompletely known function on the unit circle

A slightly more complicate situation is the following one, where the coefficients of the power series of a function are not exactly known.

It is well known that the values of a function on a circle contained in its region of holomorphy determine the function on the whole disk. We are interested in the following problem: "Let $U = \{x: |x|=1\}$ and assume that an analytic function f(x) on $U \cup \text{Int}(U)$ can be known (e.g., by measurements) only on U. We seek a rational approximation to f(x) for Int(U)". This kind of problems is frequently found in practice [4], and if, as usually happens, one is interested in the behaviour of f(x) near the origin, PTA's are candidates for a solution.

Formally, (m-1,m)-PTA's are required interpolating at the the points $(x_j, f(x_j)) \in U \times f(U)$ $(j=1,2,\ldots,m)$ optimally, i.e. with asymptotic velocity of convergence.

Existence and uniqueness of such approximants (for a given set of x_j's) are derived from the conditions

$$R_m(x_j) = f(x_j) \;;\; j = 1, \ldots, m \tag{1}$$

where $R_m(x) = \sum_{j=o}^{m-1} a_j x^j \Big/ \sum_{j=o}^{m} b_j x^j \equiv P_{m-1}(x) \,/\, Q_m(x)$

Note that the coefficients of the power series $f(x) = \sum_{j=o}^{\infty} c_j x^j$ are not known and the approximate coefficients used to define the approximants are

$$c_n^{(m)} = \frac{1}{2\pi i} \int_U \frac{f(t)}{t^{n+1}}\, dt \simeq \frac{1}{n} \sum_{j=1}^{m} \frac{f(x_j)}{x_j^{n+1}} \qquad (n = 0, 1, \ldots, m-1)$$

The coefficients of the numerator are

$$a_l = \sum_{n=o}^{m-l-1} c_n^{(m)} b_{l+n+1}$$

Since, the error in the estimation of the c_n's corresponds to the trapezoidal formula, we have

$$\lim_{m\to\infty} c_n^{(m)} = c_n \quad (n = 0,1,\ldots,m-1)$$

It can be proved that the error formula for this class of rational interpolation is ([2] p. 169)

$$f(x) - R_m(x) = \frac{1}{2\pi i Q_m(x)} \int_{C'} \frac{W_m(x) Q_m(t) f(t)}{W_m(t)(t-x)} dt; \; x \in D$$

where C' represents an integration path close to U and containing $\bar{U}$ in its interior. Also $W_m(x) = \prod_{j=1}^{m} (x-x_j)$

The next result is used in Theorem 3

Lemma 2

Let the sequence $\{R_m(x)\}$ be defined by (1). Then there is a $n \in \mathbb{N}$ such that $R_m(x)$ has no poles on $\bar{U}$ for $m > n$.

Theorem 3

The optimal asymptotic velocity of (geometric) convergence ofthe approximants $\{R_m\}$ is attained when the interpolating knots are equidistant with respect to arc length on U.

II.2 Another special case: an interval of R

We now study the case of an interval I of R; for simplicity the interval [-1,1] will be considered. Here Int(I) = ϕ, then Theorem 1 can give no answer about the location of the poles of the approximants and one has to resort to interpolating PTA's. Our main result is

Theorem 4

Let f(x) be analytic on I = [-1,1]. The optimal velocity of convergence of the approximants $\{R_m\}$ given by (1) is attained when the interpolating knots are the zeros of the m-th polynomial $T_m(x)$.

Proof (Sketch)

It is enough to see that the images of the zeros of $T_m(x)$ through the conformal mapping from $[-1,1]$ to $U = \{z: |z|=1\}$ are uniformly distributed points on U.

Remark

This result can be extended to any bounded interval by using translated Tchebycheff polynomials.

Example

In the computation of the function $f(x) = \mathrm{arctg}(x)$ with the McLaurin series in $|x| \leq 1$, and an accuracy of eight figures, a number k of terms is required, such that

$$\frac{(-1)^{2k-1}}{2k+1} < 0.5.10^{-8}$$

i.e., $k \geq 10^8$. The interpolating ATP $R_k(x)$ with $k = 13$, gives also 8 exact figures.

III. Two-Point Pade-Type Approximants

Let us assume now that $f(z)$ represents an analytic function in a region or set of regions $D = \bigcup D_j$, containing the origin and the infinity, where the D_j's are limited by Jordan curve (Γ_j), nonintersecting mutually.

Let $f_o = \sum_{j=0}^{\infty} c_j z^j$ and $f_\infty = \sum_{j=1}^{\infty} c^*_{-j} z^{-j}$ be the Taylor and Laurent expansions of $f(z)$ on neighbourhoods of the origin and infinity.

Now we determine sequences of 2PTA's with respect to (f_o, f_∞) of the form $(k(m)/m)_{(f_o,f_\infty)}(z) = P_m(z) / Q_m(z) = R_m(z)$, where $Q_m(z) \neq 0$ in D, $(0 \leq k(m) \leq m)$ and $\lim_{m\to\infty} k(m)/m = 1/2$, with asymptotically "optimal" convergence to $f(z)$ in D. For this purppose, it will be useful, to recall the following expression of the error for 2PTA's [5]

$$f(z) - R_m(z) = \frac{1}{2\pi i}\int_\Gamma \frac{z^{k(m)} Q_m(t)}{t^{k(m)} Q_m(z)} \frac{f(t)}{t-z} dt \qquad (2)$$

$$z \in D_j, \quad 1 \leq j \leq p$$

First, we shall consider the case where D is defined by $D = \mathrm{Ext}(\Gamma)$ Γ being a closed Jordan curve such that $0 \notin \mathrm{Int}(\Gamma)$.

Furthermore, we assume that D is connected and regular. Making use of |2|, we can establish

Theorem 5

Let $Q_m(z) = \prod_{j=1}^{m} (z-\beta_j)$, where the points $\{\beta_j\}$ are uniformly distributed on the boundary Γ. Then, the sequence $\{R_m(z)\}$ with denominators $\{Q_m\}$ satisfies

$$\overline{\lim_{m\to\infty}} \left| f(z) - R_m(z) \right|^{1/m} \leq \left| \frac{z}{d} \right|^{1/2} ; \quad \forall z \in D$$

where $d = \min_{t\in\Gamma}\{|t|\}$

Usually Γ represents a circle; then geometric convergence is guaranteed in certain neighbourhoods of $z = 0$ and $z = \infty$.

The next result can be proved making use of Lemma 1 and (2) and the fact that, in this case $\phi(z) = (z-a)/r$.

Theorem 6

If f is an analytic function in $D = \bar{C}\{z: |z-a| < r\}$ with $|a|>r$, then the sequence $\{(k(m)/m)\}(0 \leq k(m) \leq m)$ with uniformly distributed poles on $|z-a| = r$, converges geometrically to $f(z)$ in

$$\{z : |z|<\rho\} \cup \{z : |z|>\rho'\}$$

where $\rho < |a|-r$ and $\rho' > (|a|+r)^2/(|a|-r)$

Remark

When r is sufficiently small, ρ and ρ' are close to $|a|$, and this allows to extend at will the domain of geometrical convergence. This is the case of the function $f(z) = \exp(1/(z-1))$, which is analytic in $\bar{\mathbb{C}}\backslash\{1\}$. Hence, taking a circle with center $z = 1$ and radius ε (ε close to zero), we get geometric convergence on the whole extended complex plane except on an arbitrarily narrow annulus ($\bar{\mathbb{C}}\backslash\{1-\varepsilon < |z| < 1+\varepsilon\}$).

Let us prove now that the uniform distribution of the poles of the approximants on the boundary of the domain yields optimal asymptotic degree of convergence. More precisely,

Theorem 7

Let $D = \text{Ext}(\Gamma)$, where Γ is a closed Jordan curve, such that $0\notin\text{Int}(\Gamma)$ and D is regular. Then, optimal velocity of convergence is attained when the poles are uniformly distributed on Γ.

Proof

Let $\bar{Q}_m$ be an arbitrary polynomial of degree m, such that $\bar{Q}_m(z)$ does not vanish $\forall z \in D$. Then

$$\overline{\lim_{m\to\infty}} \frac{|\bar{Q}_m(z)|}{|Q_m(z)|^{1/m}} \leq \sup_{t\in\Gamma} \overline{\lim_{m\to\infty}} \frac{|Q_m(t)|}{|Q_m(z)|^{1/m}} = \sup_{t\in\Gamma} \overline{\lim_{m\to\infty}} \frac{|\bar{Q}_m(t)|}{\mathrm{cap}(\mathrm{Int}(\Gamma))}$$

Therefore, by using Lemma 1

$$\sup_{t\in\Gamma} \overline{\lim_{m\to\infty}} |\bar{Q}_m(t)| \geq \overline{\lim_{m\to\infty}} |Q_m(z)|/|\phi(z)| \qquad (3)$$

where $\phi = e^{G+iH}$, G is the Green's function of D and H is conjugate to G in D.

So, if R_m and $\bar{R}_m$ are the (k(m)/m) approximants with denominators Q_m and $\bar{Q}_m$ respectively, one has

$$\overline{\lim_{m\to\infty}} |f(z)-R_m(z)|^{1/m} = \left(\frac{|z|}{d}\right)^{1/2} \frac{1}{|\phi(z)|}, \quad \forall z \in D.$$

where $d = \min_{t\in\Gamma}\{|t|\}$.

Similarly, by (3)

$$\overline{\lim_{m\to\infty}} |f(z) - \bar{R}_m(z)|^{1/m} = \overline{\lim_{m\to\infty}} \frac{|z|^{1/2}}{|\bar{Q}_m(z)|^{1/m}} \geq$$

$$\geq \left(\frac{|z|}{d}\right)^{1/2} \frac{1}{|\phi(z)|}$$

Hence, the proof is concluded.

Finally, let's study the case of an unconnected domain of the form $D = \mathrm{Int}(\Gamma_1) \cup \mathrm{Ext}(\Gamma_2)$, such that $0 \in \mathrm{Int}(\Gamma_1) \subset \mathrm{Int}(\Gamma_2)$.

Let us suppose that the poles are uniformly distributed on Γ_1 and Γ_2, that is

$$Q_m(z) = \prod_{j=1}^{k(m)} (z-\beta_j) \prod_{j=1}^{l(m)} (z-\beta_j'); \; l(m) = m - k(m) \qquad (4)$$

where the $\{\beta_j\}$ are on Γ_1 and the $\{\beta_j'\}$ are on Γ_2.

In these conditions, by using again Lemma 1, one has

i) If $z \in \mathrm{Int}(\Gamma_1)$, $\overline{\lim_{m\to\infty}} |f(z)-R_m(z)|^{1/m} \leq \left(\frac{|z|}{d_1}\right)^{1/2}$ (5)

where $d_1 = \min_{t\in\Gamma_1}\{|t|\}$.

ii) If $z \in Ext(\Gamma)$,

$$\overline{\lim_{m\to\infty}} |f(z)-R_m(z)|^{1/m} \leq \frac{(|z|/d_2)^{1/2} \sup_{t\,\Gamma_2} |\phi_1(t)|^{1/2}}{|\phi_1(t)\phi_2(t)|^{1/2}} \qquad (6)$$

where $d_2 = \min_{t\in\Gamma_2}\{|t|\}$ and $\phi_j = \exp(G_j+iH_j)$; $j = 1,2$.

In the usual case of circular domains the geometric convergence is assured by the following

Theorem 8

Let Γ_1 and Γ_2 be circles centered at the origin with radii r and R respectively $(r < R)$; the sequence $R_m(z) = P_m(z)/Q_m(z)$, where Q_m is given by (4), converges geometrically in the domain $Int(\Gamma_1) \cup Ext(\Gamma_2)$.

Proof

In this case $d_1 = r$ and $d_2 = R$. Hence, from (5) one has

$$\overline{\lim_{m\to\infty}} |f(z)-R_m(z)|^{1/m} \leq \sqrt{|z|/r} < 1 \text{ for } z\in Int(\Gamma_1)$$

On the other hand, if $z\in Ext(\Gamma_2)$, then $|z| > R$, therefore, as $\phi_1(z) = z/r$ and $\phi_2(z) = z/R$, from (6) we get

$$\overline{\lim_{m\to\infty}} |f(z)-R_m(z)|^{1/m} \leq \sqrt{R/|z|} < 1,$$

and the proof is completed.

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CONVERGENCE OF SOME MATRIX PADE-TYPE APPROXIMANTS TO THE EXPONENTIAL

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Abstract.

In this paper, we study the uniform and geometric convergence on compact sets of the complex plane of matrix (k-1/k) Padé-type approximants (Brezinski,1980; Draux, 1983) to the exponential such that their scalar analogous have as a single pole the inverse of a root of the k-th Laguerre polynomial of order zero.

The interest of these results lies in the fact that, in general, a sequence of approximants of the above mentioned class does not necessarily converge, not even punctually to the exponential (Nørsett, 1978).

We also give a condition for the pole of the (k/k) approximants to the exponential whose single poles are the values of certain roots of the derivative of the k+1-th Laguerre polynomial of order zero.

1. Introduction.

The Padé-type approximation was introduced by Brezinski [1] and has been object of many papers in the last years, among them the generalization to the matricial case [3] and to non-conmutative algebras [2].

It is well known that given a formal power series in one complex va-

variable z

$$f(z) = \sum_{i=0}^{\infty} C_i z^i \qquad (1)$$

where the C_i's are $n\times n$-matrices, it is possible to define P_m and $\bar{P}_m$ polynomials in z of degree m whose matrix coefficients are associated to another polynomial Q_k of degree k, such that

$$f(z)\, Q_k(z) = P_m(z) + O(z^{m+1}) \qquad z\to 0 \qquad (2)$$

$$Q_k(z)\, f(z) = \bar{P}_m(z) + O(z^{m+1}) \qquad z\to 0 \qquad (3)$$

Thus, if Q_k is invertible, one can define the so called matrix (m/k) Padé-type approximants (MPTA) at the right (and respectively at the left) associated to f and Q_k as:

$$(m/k)_f(z) \equiv P_m(z)\, Q_k^{-1}(z) = f(z) + O(z^{m+1}) \qquad z\to 0 \qquad (4)$$

$$\overline{(m/k)}_f(z) \equiv Q_k^{-1}(z)\, \bar{P}_m(z) = f(z) + O(z^{m+1}) \qquad z\to 0 \qquad (5)$$

Since one has to compute Q_k^{-1}, it seems necessary to select a Q_k such that the number of operations is minimized and the resulting approximant verifies certain properties of order, stability, and convergence.

In this paper we study the uniform and geometric convergence of a MPTA to the exponential which is obtained after certain choice of Q_k. The punctual convergence of the corresponding scalar case has been studied by Nørsett [4].

Thus, let the matrix exponential be defined by the power series

$$\exp(-Az) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} A^i z^i \;, \quad A\in\mathbb{C}^{n\times n},\; z\in\mathbb{C} \qquad (6)$$

Let

$$Q_k(z) = (\, I + \gamma A z\,)^k \qquad (7)$$

where γ is a real number. We denote by

$$(m/k)^{(\gamma)}_{\exp(-Az)}$$

the MPTA associated to exp(-Az) and Q_k (in this case the right and the left approximants are identics).

It is well known that for a fixed γ, the $(m/k)^{(\gamma)}_{\exp(-Az)}$ converge punctually to exp(-Az) and uniformly for Re $z > 0$.

Further, the results of Nørsett [4] reveal that if we choose γ to maximize the order of the approximant, that is, if

$$\gamma = \gamma_{k,\nu}$$

in Q_k is the inverse of the ν-th root of the Laguerre polynomial L_k of degree k and order zero ($L_k(1/\gamma_{k,\nu})=0$, $\nu=1,\ldots,k$), then the sequence of scalar approximants

$$\left\{ (k-1/k)^{(\gamma_{k,\nu})}_{\exp(-z)} \right\}_{k=1}^{\infty} \tag{8}$$

does not necessarily converge, not even punctually to the exponential.

More precisely, for a fixed ν, the corresponding sequence <u>does not converge</u> to the exponential.

The same remark can be make for the matrix approximant, since the eigenvalues do not converge.

Therefore, we will determine a "good" choice of $\nu = \nu(k)$ such that the sequence

$$\left\{ (k-1/k)^{(\gamma_{k,\nu(k)})}_{\exp(-Az)} \right\}_{k=1}^{\infty}$$

converges uniformly and geometrically to exp(-Az) on every compact of the complex plane.

The derivation of these properties is based on the following result [3]:

"Let $f(t) = \sum_{i=0}^{\infty} c_i t^i$, $c_i, t \in \mathbb{C}$ be an analytic function in a domain D of the complex plane which contains the origin. Further, let A be a $n\times n$-matrix and $t\geq 0$, such that the eigenvalues of At are in D. Let Γ' be a simple closed path in D which contains in its interior the eigenvalues of At and the origin. Let Γ represent the image of Γ' by the map $z \to 1/z$. If

$$f_A(t) = \sum_{i=0}^{\infty} c_i A^i t^i \quad ,$$

then

$$f_A(t)-(k-1/k)_{f_A}(t) = \frac{t^k}{2\pi i}\left[\int_\Gamma z^{-1} f_A(z^{-1})\, \tilde{Q}_k(z)\, (1-zt)^{-1}\, dz\right] Q_k^{-1}(t) \quad (9)$$

$$f_A(t)-\overline{(k-1/k)}_{f_A}(t) = \frac{t^k}{2\pi i}\, Q_k^{-1}(t) \int_\Gamma z^{-1} f_A(z^{-1})\, \tilde{Q}_k(z)\, (1-zt)^{-1}\, dz \quad (10)$$

where

$$\tilde{Q}_k(z) = z^k\, Q_k(z^{-1}) \text{ ''}.$$

We new summarize, our results for the uniform and geometric convergence of MPTA's of the class

$$\left\{ (k-1/k)_{\exp(-Az)}^{(\gamma_{k,\nu(k)})} \right\}_{k=1}^{\infty}$$

2. Uniform convergence.

Theorem 1. Let A be a $n\times n$-matrix. Let $1/\gamma_{k,\nu}$ be the ν-th root of L_k, and $\{\gamma_{k,\nu(k)}\}_{k=1}^{\infty}$ a sequence such that there is a $k_o>0$ so that $\gamma_{k,\nu(k)} \leq c/k$ for any $k\geq k_o$, where c is a positive constant. In these conditions, the sequence of MPTA's

$$\left\{ (k-1/k)_{\exp(-Az)}^{(\gamma_{k,\nu(k)})} \right\}_{k=1}^{\infty}$$

converges uniformly to $\exp(-Az)$ in every compact of the complex plane.

Proof.

$$Q_k(x) = (\, I + \gamma_{k,\nu(k)}\, A\, x\,)^k$$

$$\tilde{Q}_k(x) = (\, x\, I + \gamma_{k,\nu(k)}\, A\,)^k$$

From (9) and (10):

$$\exp(-Az)-(k-1/k)^{(\gamma_{k,\nu(k)})}_{\exp(-Az)}(z) = \frac{z^k}{2\pi i} Q_k^{-1}(z) \int_\Gamma (x^{-1}\exp(-Ax^{-1}\tilde{Q}_k(x))(1-xz)^{-1}dx ,$$

where $\Gamma = \{t \in \mathbb{C} \ / \ |t| = 1/r\}$, $|z| < r$.

We get the following inequalities ($\|.\|$ is here any matrix norm):

$$\|\tilde{Q}_k(x)\| \leq \|x I + \gamma_{k,\nu(k)} A\|^k \leq (1/r + \|A\| c/k)^k \leq$$

$$\leq \exp(\|A\| cr) / r^k , \quad \forall k \geq k_o$$

$$|1 - xz| \geq 1 - |z|/r$$

$$\|\exp(-Ax^{-1})\| \leq \exp(\|A\| r)$$

$$\|Q_k(x)\| \leq \|I + \gamma_{k,\nu(k)} Az\|^k \geq |1 - c\|A\| |z| /k|^k , \forall k \geq k_o$$

Further, there is a $k_1 > 0$ such that

$$|1 - c\|A\| |z| /k|^k \geq \exp(-c\|A\| |z|) , \quad \forall k > k_1$$

Hence, if $k' = \max\{k_o , k_1\}$ then

$$\|\exp(-Az) - (k-1/k)^{(\gamma_{k,\nu(k)})}_{\exp(-Az)}\| \leq \exp(\|A\|[(c+1)r + |z|]) (|z|/r)^k / (1 - |z|r)$$

for any $k > k'$.

If z is any point of a compact K of the complex plane and $|z| = r_o < r$, then

$$\sup_{z \in K} \|\exp(-Az) - (k-1/k)^{(\gamma_{k,\nu(k)})}_{\exp(-Az)}\| \leq$$

$$\leq \exp(\|A\|[(c+1)r + r_o]) (r_o/r)^k / (1 - r_o/r) \quad (11)$$

By fixing r_o and r, the uniform convergence is obtained.

3. Geometric convergence.

Theorem 2. For every compact K of the complex plane, there is an integer k' (depending on K), such that the sequence

$$\left\{ (k-1/k)^{(\gamma_{k},\nu(k))}_{\exp(-Az)} \right\}_{k=k'}^{\infty}$$

converges geometrically on K.

Proof.

Let $g(r,r_o)$ be the right hand of (11)

$$g(r,r_o) = \exp(\|A\|[(c+1)r+r_o])\ (r_o/r)^k/(1-r_o/r)$$

with the variable $x=r/r_o>1$, one has

$$g(r,r_o) \equiv F(x) = \exp(\|A\|r_o[(c+1)x+1])/[(x-1)x^{k-1}] \quad , \quad x\in]1,+\infty[$$

The minimum of this function is obtained when

$$\|A\|(c+1)r_o x^2 - [\|A\|(c+1)r_o+k]\ x + k-1 = 0 \qquad (12)$$

There are two positive roots of (12), since their sum and product are both positive.

Since

$$\lim_{x\to+\infty} F'(x) = +\infty \quad \text{and} \quad \lim_{x\to 1^+} F'(x) = -\infty ,$$

the unique root greater than 1 is the minimum value of F(x), i.e.,

$$x_1 = [\|A\|(c+1)r_o + k + \sqrt{\Delta}\ ]/[2\|A\|(c+1)r_o]$$

Let

$$x' = [\|A\|(c+1)r_o + k]/[\|A\|(c+1)r_o] = 1 + k/[\|A\|(c+1)r_o] \in]1,+\infty[$$

then we have

$$F(x') = [(c+1)\|A\|r_o/k]^k\exp(\|A\|r_o+k)\ \{1+[(c+1)\|A\|r_o/k][1+O(1/k)]\}$$

and, hence

$$\lim_{k\to\infty} |F(x')|^{1/k} = 0$$

Therefore,

$$\varepsilon_k = \sup_{z\in K} \| \exp(-Az) - (k-1/k)_{\exp(-Az)}^{(\gamma_{k,\nu(k)})}(z) \| \leq \inf_{x\in]1,\infty[} |F(x)| \leq |F(x')|$$

Thus

$$\lim_{k\to\infty} \varepsilon_k^{1/k} = 0 ,$$

hence, there is a $k_2=k_2(K)$ such that $|\varepsilon_k|^{1/k} \leq q$ for any $q<1$ and $k \geq \max\{k_o, k_1, k_2\}$, where k_o, k_1 are given by Theorem 1.

Finally, the geometric convergence on the compact K is visualized by setting $k' = \max\{k_o, k_1, k_2\}$.

Note that the integer k_2 and, consequently k', are increasing functions of r_o, because for sufficiently large values of k one has

$$|\varepsilon_k|^{1/k} \simeq [(c+1)\|A\| r_o e]/k$$

and given a $q<1$ then it is enough to take $k \geq (c+1)\|A\| r_o e/q$ to attain geometric convergence of ratio q.

<u>An example</u>.

As example, in the following tables, we show the error bounds ε_k associated to k, $\gamma_{k,\nu(k)}$ and $1/x=r_o/r$ ($r_o=0.5$).

This bound is an increasing function of r.

k	ν(k)	1/x	Error bound
1	1	.125	0.12D+03
2	1	.021	0.77D+02
3	2	.011	0.12D+02
4	2	.078	0.73D+02
5	3	.081	0.93D+00
6	3	.065	0.16D+01
7	4	.062	0.37D-01
8	4	.053	0.37D-01
9	5	.050	0.86D-03
10	5	.044	0.66D-03

Table 1

k	ν(k)	1/x	Error bound
1	1	.421	0.17D+02
2	1	.392	0.70D+03
3	2	.255	0.34D+01
4	2	.236	0.51D+01
5	3	.169	0.22D+00
6	3	.159	0.16D+00
7	4	.125	0.67D÷02
8	4	.119	0.32D-02
9	5	.098	0.11D-03
10	5	.095	0.42D-04

Table 2

Similar results have seen obtained for the MPTA's $(k/k)^{(\gamma_{k,\nu(k)})}_{\exp(-Az)}$ where $1/\gamma_{k,\nu(k)}$ is the ν-th root of L'_{k+1} .

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ON THE BEHAVIOUR OF THE ZEROS OF SOME S-ORTHOGONAL POLYNOMIALS

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ABSTRACT

Monotonicity properties of the zeros of some n-orthogonal polynomials, as well as comparison theorems, are given.

1. INTRODUCTION

The class of s-orthogonal polynomials is of considerable interest for a number of applications, like, e.g., boundary problems for o.d.e. [1] hipergaussian or quasi-gaussian quadrature rules with multiple nodes [2,3,4,5]. Polynomials s-orthogonal in [a,b] with respect to a given weight $w(x)$, are defined by means of the following conditions

$$\int_a^b w(x)x^k \,[P_{m,s}(x)]^{2s+1}dx = 0 \qquad k=0,1,\dots,m-1$$

where m denotes the degree of the polynomial $P_{m,s}(x)$ and $s \in N$.

In spite of the importance of these polynomials, only a few of their properties have been established so far [7 : 11]. In relation to the order of convergence of the above mentioned quadrature rules, the following question is of particular interest. If we denote by x_i $(i = 1,\dots,m)$ the gaussian nodes (i.e., the zeros of suitable s-orthogonal polynomials), do the following relations (where C does not depend on m):

$$(1.1) \qquad |x_{m,i+1} - x_{m,i}| < C/m \quad , \qquad i = 1,\dots,m-1$$

hold? An affirmative answer has been found in [12] for the case $s=0$ and in [13] for some forms of the weight function, when $s \geq 0$.

In this paper, we shall consider the behaviour of the zeros of the polynomials $\{P_{m,s}(x)\}$ that are s-orthogonal in $[-1,1]$ when the weight function has certain forms to be specified later. In Section 2, some comparison theorems for these zeros are given. In Sections 3 and 4 the behaviour, with respect to s, of the zeros of the $\{P_{m,s}(x)\}$ is studied for the cases $m = 2$ and $m = 3$ and for the following weight functions

$$w_1(x) = 1-x^2 \quad , \quad w_2(x) = (1-x^2)^2 \quad ; \quad x \in [-1,1] .$$

The results obtained in this paper give a first answer to a question asked in [14] that the shall discuss in Section 2. A summary of results is as follows. Monotonicity properties with respect to s are shown to hold for the zeros of the sets $\{P_{2,s}^{(i)}(x)\}$, $\{P_{3,s}^{(i)}(x)\}$ $(i = 1,2)$ when the weight function is $w_i(x)$ $(i=1,2)$. It is further proved that the positive zeroes of $\{P_{2,s}^{(i)}(x)\}$ and $\{P_{3,s}^{(i)}(x)\}$, $s \in N$, $(i = 1,2)$ form increasing sequences. In the theorems needed to prove these properties, upper and lower bounds for the zeros are also found. Finally, pertinent numerical tables are given in Section 5.

II. SUMMARY OF PREVIOUS RESULTS

Let us consider a typical quasi-gaussian quadrature rule [14] of the form

$$(2.1)\qquad \int_{-1}^{1} p(x)f(x)\,dx = \sum_{J=1}^{l}\sum_{k=o}^{\alpha_J-1} B_{kJ}f^{(k)}(Y_J) + $$
$$+ \sum_{i=1}^{m}\sum_{h=o}^{2s} A_{hi}f^{(h)}(x_i)+R_m(f)$$

Here, $y \notin (-1,1)$ are pre-assigned multiple nodes and x_i are gaussian multiple nodes. In the study of the convergence of the above rule, the distance between the gaussian nodes plays an important role. It is well known [5] that the maximum degree of precision $\nu = \sum_{J=1}^{l} \alpha_J + m(2s+2)-1$ is obtained when the nodes x_i , $(i=1,2,\ldots,m)$ coincide with the zeros of the polynomials that are s-orthogonal with respect to the weight function

$$(2.2)\qquad w(x) = p(x) \cdot \prod_{J=1}^{l} (x-y_J)^{\alpha_J}$$

It has been proved in [14] that the remainder $R_m(f)$ in (2.1) has the property

$$(2.3)\qquad R_m(f) = [o(1/m)]^{2s} \qquad , \qquad f(x) \in C^{2s}\,[y_1,y_2]$$

provided that (1.1) is met. In the same paper, upper and lower bounds are established for the zeros $x_{m,i}$ of the polynomials $\{P_{m,s}(x)\}$, s-orthogonal with respect to the weight (2.2). In particular, assuming that the nodes y_J , $j=1,l$ are symmetrical with respect to the origin and that $p(x) = 1$, the following theorem is proved.

<u>Theorem 1</u>. "The positive zeros $x_{m,i}$ of $P_{m,s}(x)$, satisfy the

inequalities

$$(2.4)\qquad X_{m,i}^{(s)} < x_{m,i} < Y_{m,i}^{(s)}$$

where $X_{m,i}^{(s)}$ and $Y_{mi}^{(s)}$ are the positive zeros of the polynomials that are s-orthogonal in $[-1,1]$ with respect to the weights

$$(1-x^2)^P \quad , \qquad P = \sum_{j=1}^{l} \alpha_j$$

$$(1-x^2)^{\alpha_1}$$

respectively".

It is known [8] that the positive zeros of the polynomials s-orthogonal in $[-1,1]$ with respect to the weight functions $(1-x^2)^{\beta}$, $\beta > -1$, are decreasing functions of β. Taking into account this property, as well as the fact that the Tchebicheff polynomials of the first kind are s-orthogonal with respect to $(1-x^2)^{-1/2}$, the following theorem is easily established:

<u>Theorem 2</u>. "The positive zeros of $P_{m,s}(x)$ satisfy the inequalities

$$(2.5)\qquad X_{m,i}^{(s)} < x_{mi} < T_{m,i}$$

where the $T_{m,i}$'s are the positive zeros of the Tchebicheff polynomials of the first kind".

Furthermore, if

$$(2.6)\qquad s > P - 1/2$$

then the following theorem ensues from (2.5) :

<u>Theorem 3</u>. "The positive zeros of $P_{ms}(x)$ satisfy the inequalities

$$(2.7)\qquad V_{mi} < X_{mi}^{(s)} < x_{mi}$$

where the $V_{m,i}$'s are the positive zeros of the Tchebicheff polynomials of the second kind".

The proof of (2.7) follows from (2.4) taking into account the property of the Tchebicheff polynomials of the second kind of being s-orthogonal with respect to the weight function $(1-x^2)^{s+1/2}$.

Suppose now that condition (2.6) is not met. In order to deduce from (2.5) a further lower bound for the $x_{m,i}$'s involving the zeros of orthogonal (instead of s-orthogonal) polynomials, it is to be inquired whether the zeros $x_{mi}^{(s)}$ behave as increasing functions of s .

Finally we note that (2.3) can be considered as a convergence result, pertaining to (2.2) for fixed m and increasing s .

It is in this context, that the problems treated in the Sections that follow, acquire their meaning.

III. BEHAVIOUR OF THE ZEROS OF THE POLYNOMIALS $\{P_{2s}^{(1)}(x)\}$, and $\{P_{3,s}^{(1)}(x)\}$, s-ORTHOGONAL W.R.T. THE WEIGHT $w(x)$.

For the polynomials $\{P_{ms}^{(i)}(x)\}$, of degree m , s-orthogonal in $[-1,1]$ w.r.t. the weight

(3.1) $$w_1(x) = 1-x^2$$

the following conditions hold

(3.2) $$\int_{-1}^{1} (1-x^2)x^k[P_{ms}^{(1)}(x)]^{2s+1}\,dx = 0 , \qquad k=0,1,...,m-1$$

where $s \in N$.

Because of the symmetry of the interval $[-1,1]$ and of the weight $w_1(x)$, it is known that the positive and negative zeros of $P_{m,s}^{(1)}(x)$ are symmetrical with respect to the origin, whereas $P_{m,s}^{(1)}(0) = 0$, if m is odd . So, if the coefficient of x^m , in $P_{ms}^{(1)}(x)$ is thought to be equal 1 , we may write

(3.3) $$P_{m,s}(x) = x^{m-2\nu} . \prod_{k=1}^{\nu} (x^2-\alpha_{m,s,k})$$

where $\nu = [m/2]$ and $\pm\sqrt{\alpha_{m,s,k}}$ are the zeros of $P_{m,s}^{(1)}(x)$ which are different from origin.

When $m = 2$ or $m = 3$, it follows from (3.2) and (3.3) that the location of the only positive zero of $P_{2s}^{(1)}(x)$ and $P_{3s}^{(1)}(x)$ (respectively) is led to the solution of the equations, respectively

(3.4) $$\int_{o}^{1} (1-x^2)(\alpha_{2s}-x^2)^{2s+1}\,dx = 0$$

(3.5) $$\int_{o}^{1} (1-x^2)(\alpha_{3s}-x^2)^{2s+1}x^{2s+2}\,dx = 0 ,$$

in which it is intended that $\alpha_{2s} = \beta_{2s}^2$, $\alpha_{3s} = \beta_{3s}^2$. Before going on with the treatment of this subject, it is convenient to give here some recurrence relation for the function $G_{a,b}(t)$, defined by means of:

$$(3.6) \qquad G_{a,b}(t) = \int_0^1 x^a(t-x^2)^b dx \quad , \qquad a,b \in N .$$

It is easy to recognize a connection between $G_{a,b}(t)$ and the integrals in (3.4) and (3.5).

The mentioned recurrence relations are the following:

$$(3.7) \qquad G_{a,b}(t) = \frac{(a-1).t.G_{a-2,b}(t)+(-1)^b(1-t)^{b+1}}{a+2b+1}$$

$$(3.8) \qquad G_{a,b}(t) = \frac{2btG_{a,b-1}(t)+(-1)^b.(1-t)^b}{a+2b+1}$$

The relation (3.7) is deduced from (3.6), assuming $d(t-x^2)^{b-1}$ as differential factor and integrating by part; whereas (3.8) is derived from (3.6) by integrating by part with dx as differential factor. Now we may proceed to prove the theorem 4 .

Theorem 4. The sequence $\{\alpha_{2s}\}$, $s \in N$, of the solutions of equations (3.4) is increasing as s increases ; moreover for the positive zeros β_{2s} of the polynomials $P_{2s}^{(1)}(x)$, s-orthogonal w.r.t the weight (3.1), the following bounds

$$1/\sqrt{5} < \beta_{2s} < \sqrt{(4s+1)/(8s+5)} \ , \qquad s \in N^+$$

hold.

Proof. It is immediate to prove that (3.4) has, for a fixed $s \in N$, only one solution $\alpha_{2s} \in (0,1)$. Indeed, the function

$$(3.9) \qquad f_{1s}(t) = \int_0^1 (1-x^2)(t-x^2)^{2s+1} dx$$

is increasing as t increases, $t \in [0,1]$; moreover $f_{1s}(0) < 0$, $f_{1s}(1) > 0$. Then, we may derive a recurrence relation between $f_{1s}(t)$ and $f_{1,s-1}(t)$, $s \geq 1$, making use of (3.7) and (3.8) . Putting $n=2s+1$, from (3.9) and (3.6) , we get

$$f_{1s}(t) = G_{o,n}(t)-G_{2,n}(t)$$

and this becames, by (3.7),

$$(3.10) \qquad f_{1s}(t) = G_{o,n}(t).(1-t/(2n+3))+(1-t)^{n+1}/(2n+3)$$

Consequently, for $s \geq 1$, we may write

$$(3.11) \qquad G_{o,n-2}(t)(1-t/(2n-1)) = f_{1,s-1}(t)-(1-t)^{n-1}/(2n-1) .$$

Applying twice (3.8), $G_{o,n}(t)$ may be connected to $G_{o,n-2}(t)$ according to

$$G_{o,n}(t) = 4n(n-1)/(4n^2-1).t^2.G_{o,n-2}(t) + + (1-t)^{n-1}[(4n-1)t-(2n-1)]/(4n^2-1)$$

which may also be written as follows

$$G_{o,n}(t) = C_s t^2 G_{o,n-2}(t) + (1-t)^{n-1}.P_{1s}(t) \tag{3.12}$$

If we put, for the sake of short notation

$$\left\{ \begin{aligned} C_s &= 4(n^2-n)/(4n^2-1) \\ P_{1s}(t) &= [(4n-1)t-(2n-1)]/(4n^2-1) \\ R_{1s}(t) &= 1-t/(2n+3) \end{aligned} \right. \tag{3.13}$$

Now putting (3.12) in (3.10), we obtain

$$f_{1s}(t) = C_s t^2 R_{1s}(t).G_{o,n-2}(t) + + (1-t)^{n-1}[R_{1s}(t).P_{1s}(t) + (1-t)^2/(2n+3)]$$

and from this, by the aid of (3.11), we finally get the following recurrence relation

$$\begin{aligned} R_{1s-1}(t)f_{1s}(t) &= C_s t^2 R_{1s}(t).f_{1,s-1}(t) + \\ &+ (1-t)^{n-1}[-C_s t^2 R_{1s}(t)/(2n-1)+P_{1s}(t).R_{1s}(t)R_{1,s-1}(t)+ \\ &+ (1-t)^2 R_{1,s-1}(t)/(2n+3)] \end{aligned} \tag{3.14}$$

The polynomial in the brackets is only apparently of degree three; in fact, it reduces to

$$-R_{1,s}(t)[t^2(2n+1)-4nt+(2n-1)]/(4n^2-1)+(1-t)^2 R_{1,s-1}(t)/(2n+3)$$

The obvious divisibility by (1-t) of the first term allows us to pass from (3.14) to the following (3.15)

$$f_{1s}(t) = C_s t^2 R_{1s}(t) f_{1s-1}(t)/R_{1s-1}(t)+(1-t)^n Q_{1s}(t)/R_{1,s-1}(t) \tag{3.15}$$

where it is intended that:

$$Q_{1s}(t) = 2[(4n+1)t-(2n-1)]/[(4n^2-1).(2n+3)]$$

Now, let be $\{\gamma_s\}$, $s \in N^+$ the sequence of the zeros of $\Omega_{1s}(t)$; it is easy matter to verify that

$$\gamma_{s-1} < \gamma_s , \qquad s \in N^+ ; \quad \lim_{s\to\infty} \gamma_s = 1/2$$

with $\gamma_o = \alpha_{20} = 0.2$.

Moreover from (3.15), written for $s = 1$, $t = \gamma_1$ one gets

$$(3.16) \qquad f_{11}(\gamma_1) > 0$$

Consequently, and because of the fact that $f_{1s}(t)$ increases for $t \in [0,1]$, $s \in N$, we obtain

$$(3.17) \qquad f_{1s}(\gamma_s) > 0 \qquad , \qquad s \in N^+$$

To the aim to prove (3.17), let us take $s=2$, $t= \gamma_2$; then (3.1) gives

$$f_{12}(\gamma_2) = C_2\gamma_2 \, R_{12}(\gamma_2)\cdot f_{11}(\gamma_2)/R_{11}(\gamma_2)$$

so, in view of (3.13), it is possible deduce that

$$f_{12}(\gamma_2) > 0$$

holds; going on in the same way, we get (3.17). Then, from this, it follows

$$(3.18) \qquad \alpha_{s-1} < \gamma_{s-1} \qquad , \qquad s = 2,3,...$$

so that we reach the thesis

$$\alpha_{s-1} < \alpha_s \qquad , \qquad s \in N^+$$

putting $t = \alpha_{s-1}$ in (3.15) and taking into account that (3.18) implies that:

$$Q_{1s}(\alpha_{s-1}) < 0 .$$

We have already noticed that we may derive, from (3.4) , $\alpha_{2,0}=0,2$; this remark and (3.17) lead to the bounds for β_{2s}, of the thesis.

Concerning the case $m = 3$, still for the weight $w_1(x)$ given in (3.1), it is possible to prove theorem 5 , which deals with the behaviour of the positive zeros of $\{P_{3s}^{(1)}(x)\}$, $s \in N$; in the same time an upper bound for those zeros is supplied.

<u>Theorem 5</u>. The sequence $\{\beta_{3s}\}$, $s \in N$, of the positive zeros of the

polynomials $\{P_{3s}^{(1)}(x)\}$, s-orthogonal in $[-1,1]$ w.r.t the weight $w_1(x)$, increases as s increases ; moreover, for $s \in N^+$,

$$\sqrt{3/7} < \beta_{3s} < \sqrt{\delta_s} \quad , \quad \delta_s = (30s+7-\sqrt{324(s^2+s)+49})/16s$$

hold.

Proof. The zeros β_{3s} are given by

$$\beta_{3s} = \sqrt{\alpha_{3s}} \quad , \quad s \in N$$

where α_{3s} solves the equation

$$(3.19) \quad F_{1,s}(t) = \int_0^1 (1-x^2)(t-x^2)^{2s+1}.x^{2s+2}dx = 0 \quad , \quad s \in N$$

Now (3.7), (3.8) can be used to obtain a recurrence relation for $F_{1,s}(t)$. For the sake of short, let us put

$$(3.20) \quad \begin{cases} J_s(t) = G_{n+1,n}(t) \quad , & n = 2s+1 \\ d_k = 6s+7-2k & k = 0,1,2,3 \\ T_{1s}(t) = 1-(2s+3)t/d_o \; ; \; T_{1,s-1}(t) = 1-(2s+1)t/d_3 & \\ S_{1s} = d_o^{-1} \quad , \quad S_{1,s-1} = d_3^{-1} & \end{cases}$$

Making use of (3.20), (3.7), (3,8), from (3.19) one gets

$$(3.21) \quad F_{1,s}(t) = J_s(t)T_{1,s}(t) + (1-t)^{n+1}.S_{1s}$$

from which we deduce

$$(3.22) \quad J_{s-1}(t).T_{1s-1}(t) = F_{1,s-1}(t)-(1-t)^{n-1}.S_{1,s-1}$$

$J_s(t)$ and $J_{s-1}(t)$ may be connected according the following relation (see (3.7) , (3.8))

$$(3.23) \quad J_s(t) = A_s t^3 J_{s-1}(t)+(1-t)^{n-1}M_{1s}(t)$$

where

$$(3.24) \quad \begin{cases} A_s = 4n^2(n-1)/d_1 d_2 d_3 \\ M_{1,s}(t) = -n[4.(n-1)t^2-5d_3.t+3d_3]/d_1 d_2 d_3 \; . \end{cases}$$

Inserting (3.22) into (3.23) , and the obtained result in (3.21), we reach the following recurrence relation for $F_{1s}(t)$

$$(3.25)\qquad F_{1s}(t)\cdot T_{1s-1}(t) = A_s t^3 F_{1,s-1}(t) T_{1,s}(t) +$$
$$+(1-t)^{2s}\{[-A_s S_{1,s-1} t^3 + M_{1s}(t)\cdot T_{1,s-1}(t)] T_{1s}(t) +$$
$$+ (1-t)^2 S_{1s}\cdot T_{1s-1}(t)\}$$

To the aim of investigating the behaviour of the sequence $\{\alpha_{3s}\}$ of the zeros of $F_{3s}(t)$, $s \in N$, let us analyze the polynomial in the brackets of (3.25).

In view of (3.24), it is easy to verify that the coefficient of the term of highest degree is given by

$$A_s(2s+3)/d_o d_3 - 4s(2s+1)^2(2s+3)/d_o d_1 d_2 d_3^2 = 0$$

Moreover

$$-A_s S_{1s-1} + M_{1,s}(1) T_{1,s-1}(1) = 0$$

so that the above polynomial is divisible by $1-t$ and we may pass from (3.25) to

$$(3.26)\qquad F_{1s}(t) T_{1,s-1}(t) = A_s\cdot t^3\cdot F_{1,s-1}(t) T_{1s}(t) +$$
$$+ (1-t)^{2s+1}\cdot S_{1,s-1} S_{1s} V_{1s}(t)$$

where

$$(3.27)\qquad V_{1s}(t) = [(8-8_n)t^2 + (30_n-16)t - (18_n-12)]/3d_1$$

For every fixed s, $V_{1s}(t)$ has a zero $\delta_s \in (0,1)$, being also

$$(3.28)\qquad F_{1s}(\delta_s) > 0 \quad , \qquad s \in N^+ .$$

In fact, note that from (3.20) it follows

$$\alpha_{3o} = 3/7 < \delta_1$$

and so, putting $s = 1$, $t = \delta_1$ in (3.26), we get

$$F_{11}(\delta_1) > 0$$

for $F_{1s}(t)$ is increasing. As the sequence $\{\delta_s\}$ of the zeros of $V_{1,s}(t)$, $s \in N^+$, namely

$$(3.29)\qquad \delta_s = (15n-8-\sqrt{81n^2-32})/(8n-8)$$

is increasing (as far as $F_{1s}(t)$, $s \in N$), then, from $F_{11}(\delta_2) > 0$

it follows: $F_{12}(\delta_2) > 0$ and so on, to (3.28).
Note that (3.28) implies

$$\alpha_{3,s-1} < \delta_{s-1} < \delta_s ,$$

then, in view of (3.26), it results

$$F_{1s}(\alpha_{3,s-1}) < 0$$

that is the same as saying that $\{\alpha_{3s}\}$ is increasing. Finally, let us remark that from (3.29) and (3.26) it follows that

$$F_{11}(\alpha_{30}) < 0$$

so we may conclude that also the second part of the thesis holds:

$$\sqrt{3/7} < \beta_{3s} < \sqrt{\delta_s} .$$

IV. BEHAVIOUR OF THE ZEROS OF THE POLYNOMIALS $\{P^{(2)}_{2s}(x)\}$, $\{P^{2}_{3s}(x)\}$ s-ORTHOGONAL w.r.t. THE WEIGHT $w_2(x)$.

Concerning the weight

$$w_2(x) = (1-x^2)^2 \tag{4.1}$$

we will be able to demonstrate the following theorem 6, and theorem 7. First some remark. The positive zeros μ_{2s} of $\{P^{(2)}_{2s}(x)\}$, $s \in N$, are related to the zeros λ_{2s} of the equations:

$$f_{2s}(t) = \int_0^1 (1-x^2)^2 (t-x^2)^{2s+1}\, dx = 0 \quad , \qquad s \in N \tag{4.2}$$

by means of: $\mu_{2s} = \sqrt{\lambda_{2s}}$.

So, we are interested in the study of the behaviour of the sequence $\{\lambda_{2s}\}$, $s \in N$; then it is convenient to connect the funcions $f_{2,s}(t)$ and $f_{2,s-1}(t)$, $s \in N^+$. This can be made, writing, first of all, $f_{2s}(t)$ by means of (3.6); having

$$f_{2s}(t) = G_{o,n}(t) - 2G_{2,n}(t) + G_{4,n}(t)$$

the repeated use of (3.7) allows us to establish the following relation (where $n = 2s+1$, as usual)

$$f_{2s}(t) = G_{o,n}(t)\ \frac{3t^2-2(2n+5)t+(2n+3)(2n+5)}{(2n+3)(2n+5)} + \frac{(1-t)^{n+1}(-3t+2n+7)}{(2n+3)(2n+5)}$$

this can be written in a more synthetical form; introducing the notations

$$(4.3)\quad \begin{cases} R_{2s}(t) = [3t^2-2(2n+5)t + (2n+3)(2n+5)]/[(2n+3)(2n+5)] \\ W_{2s(t)} = (-3t+2n+7)/[(2n+3)(2n+5)] \end{cases}$$

we may write

$$(4.4)\qquad f_{2s}(t) = G_{o,n}(t)R_{2s}(t) + W_{2s}(t)$$

and this, taking into account (3.12), gives rise to

$$(4.5)\qquad f_{2s}(t) = C_s t^2 G_{o,n-2}(t)R_{2s}(t) + $$
$$+ (1-t)^{n-1}[P_{1s}(t)R_{2s}(t)+(1-t)^2 W_{2s}(t)].$$

Making use of (4.4) with the index $s-1$, we may derive a relation between $G_{o,n-2}(t)$ and $f_{2,s-1}(t)$, reaching so the recurrence relation

$$(4.6)\qquad R_{2,s-1}(t)f_{2,s}(t) = C_s t^2 R_{2s}(t)f_{2s-1}(t) +$$
$$+(1-t)^{n-1}\ [H_{2,s}(t)\ R_{2s}(t) + (1-t)^2 R_{2,s-1}(t)W_{2s}(t)]$$

where it is intended that $H_{2,s}(t)$ is given by

$$H_{2s}(t) = -C_s t^2 R_{2s}(t)W_{2,s-1}(t) + P_{1s}(t)R_{2,s-1}(t)$$

The polynomial in the brackets of (4.6) is only apparently of degree five; in fact, with some calculation, it is possible to verify that the coefficients of the terms of degreee five and four, equal zero, when taking into account (3.13) and (4.3) . By means of the same relations, we can also verify that the mentioned polynomial is divisible by $(1-t)^2$; as a consequence, from (4.6) we pass to

$$(4.7)\qquad f_{2s}(t)\cdot R_{2,s-1}(t) = C_s t^2 f_{2,s-1}(t)R_{2s}(t) + (1-t)^{n+1}Q_{2s}(t)$$

where

$$(4.8)\qquad Q_{2s}(t) = 8\ [(4n+3)t-(2n-1)]/[(4n^2-1)(2n+3)(2n+5)]$$

For a fixed $s \geq 1$, (4.8) is solved by

$$\Lambda_s = (2n-1)/(4n+3)$$

and the sequence $\{\Lambda_s\}$ is increasing ; moreover

$$\Lambda_1 = 1/3\ ,\qquad \lim_{s\to\infty} \Lambda_s = 1/2$$

Now we are able to prove the following properties:

(4.9) the sequence $\{\lambda_{2s}\}$, $s \in N$ is increasing ,

(4.10) $\lambda_{2s} < \Lambda_s$, $s \in N^+$.

Remark, first of all, that the functions $R_{2s}(t)$, $s \in N$ (4.3) are positive in $[0,1]$, and that the functions $f_{2,s}(t)$ increase as $t \in [0,1]$, then, from (4.7), one gets

$$f_{21}(\Lambda_1) = R_{21}(\Lambda_1)C_1\Lambda_1^2 f_{2o}(\Lambda_1)/R_{2o}(\Lambda_1) > 0$$

for (4.2), written with $s = 0$, implies

$$\lambda_{2o} = 1/7 < \Lambda_1 .$$

This allows us to derive that

$$f_{2s}(\Lambda_s) > 0$$

so that

$$\lambda_{2s} < \Lambda_s \qquad s \in N^+$$

that is to say (4.10) holds; then from (4.7), written with $t = \lambda_{2,s-1}$, we derive also that

$$f_{2s}(\lambda_{2,s-1}) < 0 ,$$

hence (4.9) holds.

Taking into account that $\mu_{2s} = \sqrt{\lambda_{2s}}$, the above conclusions give rise to the following.

<u>Theorem 6</u>. The positive zeros μ_{2s} of the polynomials $\{P_{2,s}^{(2)}(x)\}$, $s \in N$, s-orthogonal in $[-1,1]$ w.r.t. the weight $w_2(x)$, form an increasing sequence, moreover the following bounds from below and from above

$$1/\sqrt{7} < \mu_{2s} < \sqrt{(4s+1)/(8s+7)}$$

hold.

Dealing with the case of the polynomials $\{P_{3s}^{(2)}(x)\}$, of degree three, s-orthogonal in $[-1,1]$ w.r.t. the weight (4.1), we may study the sequence $\{\lambda_{3s}\}$, $s \in N$, of the zeros of equations

(4.11) $$F_{2s}(t) = \int_o^1 (1-x^2)^2(t-x^2)^{2s+1}.x^{2s+2}dx = 0 , \qquad s \in N$$

Also for the functions $F_{2s}(t)$ it is possible derive a recurrence relation, as well as for the functions $f_{2s}(t)$. Starting from (4.11) and recalling (3.6) , we get

$$F_{2s}(t) = G_{n+1,n}(t)-2G_{n+3,n}(t) + G_{n+5,n}(t)$$

and acting on this, by means of (3.7), (3.8), (3.23), we reach

$$(4.12) \qquad F_{2s}(t)T_{2,s-1}(t) = A_s t^3 F_{2,s-1}(t)T_{2s}(t)+(1-t)^{n+1}.V_{2s}(t)$$

where

$$(4.13) \qquad \begin{cases} T_{2,s}(t) = (n+4)(n+2)t^2/d_o d_s-2(n+2)t/d_o+1 \\ V_{2,s}(t) = 8n\left[-4(n-1)t^2+(15n-6)t-(9n-6)\right]/ \prod_{k=o}^{4} d_k \\ d_4 = 3n+6 \end{cases}$$

with A_s , d_k given by (3.20).

In this case, as well as for the weight $w_1(x) = 1-x^2$, $V_{2,s}(t)$ has, for fixed s , a zero $\Delta_s \in (0,1)$, namely

$$(4.14) \qquad \Delta_s = \left(15n-6- \sqrt{81n^2+60(n-1)}\right)/8(n-1)$$

and it is easy to see that

$$\Delta_s < \Delta_{s+1} \quad , \quad s \in N \quad ; \lim_{s\to\infty} \Delta_s = 3/4$$

Because of the fact that $T_{2,s}(t)$ are positive (for a fixed $s \in N$) as $t \in [0,1]$, the same reasoning seen in the above theorem, brings to the conclusion that

$$(4.15) \qquad F_{2s}(\Delta_s) > 0 \quad , \quad s \in N^+$$

when one starts from the remark that (cfr. (4.12))

$$F_{21}(\Delta_1) = A_1\Delta_1^3 F_{2o}(\Delta_1)T_{21}(\Delta_1)/T_{2o}(\Delta_1) > 0 ,$$

for, as it follows from (4.11) and (4.14),

$$\lambda_{3o} = 1/3 < \Delta_1$$

From (4.15) we deduce

$$(4.16) \qquad \lambda_{3s} < \Delta_s \quad , \quad s \in N^+$$

so that, if we put $t = \lambda_{3,s-1}$ in (4.12) we get

$$F_{3;s}(\lambda_{3,s-1}) < 0 \quad ,$$

that is to say that it results:

$$\lambda_{3,s-1} < \lambda_{3s} \quad .$$

Hence, the following theorem holds, in which $\{\mu_{3s}\} = \{\sqrt{\lambda_{3s}}\}$ is the sequence of the positive zeros of the polynomials $\{P^{(2)}_{3s}(x)\}$, $s \in N$, s-orthogonal in $[-1,1]$ w.r.t the weight $w_2(x)$:

Theorem 7. The sequence $\{\mu_{3s}\}$, $s \in N$, increase as s increases, moreover

(4.17) $$1/\sqrt{3} < \mu_{3s} < \sqrt{\Delta_s} \quad , \qquad (\Delta_s \text{ given by } (4.14)).$$

Finally, let us remark that upper bounds in (4.17) as well as the analogue upper bounds given in theorems 4,5,6, improve the result of theorem 2.

Investigation about the weights

$$w(x) = x^{2n} \quad , \qquad n \in N^+ .$$

is in progress.

V. SOME NUMERICAL RESULTS.

Now we give here four tables, pertaining respectively the following cases:

$w_1(x) = 1-x^2$	$m = 2$	$s = 1,2,...,6$
$w_1(x) = 2-x^2$	$m = 3$	$s = 1,2,...,6$
$w_2(x) = (1-x^2)^2$	$m = 2$	$s = 1,2,...,6$
$w_2(x) = (1-x^2)^2$	$m = 3$	$s = 1,2,...,6$

TABLE 1

s = 1	$\beta_{21} = 0.53337686$
s = 2	$\beta_{22} = 0.57388977$
s = 3	$\beta_{23} = 0.59794803$
s = 4	$\beta_{24} = 0.61406030$
s = 5	$\beta_{25} = 0.62563402$
s = 6	$\beta_{26} = 0.63450644$

TABLE 2

s = 1	$\beta_{31} = 0.73767913$
s = 2	$\beta_{32} = 0.77216399$
s = 3	$\beta_{33} = 0.79131391$
s = 4	$\beta_{34} = 0.80359643$
s = 5	$\beta_{35} = 0.81219012$
s = 6	$\beta_{36} = 0.81856423$

TABLE 3

s = 1	μ_{21} = 0.47234141
s = 2	μ_{22} = 0.52119385
s = 3	μ_{23} = 0.55177365
s = 4	μ_{24} = 0.57295105
s = 5	μ_{25} = 0.58858818
s = 6	μ_{26} = 0.60067479

TABLE 4

s = 1	μ_{31} = 0.68016296
s = 2	μ_{32} = 0.72679937
s = 3	μ_{33} = 0.75378323
s = 4	μ_{34} = 0.77150029
s = 5	μ_{35} = 0.78408238
s = 6	μ_{36} = 0.79351261

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A note on the summability of the Riesz means
of multiple Fourier series

By

Shigehiko Kuratsubo

1. Let R^k denote the k-dimensional Euclidean space, T^k the k-dimensional torus (identified with the cube $Q^k = \{ x = (x_1, x_2, \ldots, x_k) \in R^k : -\frac{1}{2} \leqq x_j < \frac{1}{2}$ ($j = 1, 2, \ldots, k$) $\}$) and Z^k the integral lattice of R^k. Through this paper we assume $k \geqq 2$.

Let $f(x)$ be a Lebesgue integrable function on T^k. We denote the Fourier series of $f(x)$

$$f(x) \sim \sum_{m \in Z^k} \hat{f}(m)\, e(mx)$$

with

$$\hat{f}(m) = \int_{T^k} f(x)\, e(-mx)\, dx \qquad (m \in Z^k) ,$$

where $e(t) = e^{2\pi it}$ ($t \in R$) and $mx = m_1x_1 + m_2x_2 + \ldots + m_kx_k$.

We next denote the Riesz means of order α of $f(x)$

$$S_t^{\alpha}(f:x) = \sum_{|m|^2 < t} (1 - \frac{|m|^2}{t})^{\alpha}\, \hat{f}(m)\, e(mx)$$

where $|m| = (m_1{}^2 + m_2{}^2 + \ldots + m_k{}^2)^{\frac{1}{2}}$.

Further for $\alpha \geqq 0$, $t > 0$ and $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k) \in R^k$, we introduce the following functions

$$A_{\alpha}(t:x,y) = \frac{1}{\Gamma(\alpha+1)} \sum_{|m-y|^2 < t} (t - |m-y|^2)^{\alpha}\, e((m-y)x)$$

and

$$P_\alpha(t:x,y) = A_\alpha(t:x,y) - C_{k,\alpha}\ e(-xy)\ \ (x) \quad ,$$

where $\delta(x) = 1$ or 0 according as $x \varepsilon Z^k$ or not and

$$C_{k,\alpha} = \frac{1}{\Gamma(\alpha+1)} \int_{|z|^2<t} (t-|z|^2)^\alpha\, dz = \frac{\pi^{\frac{k}{2}}\, t^{\frac{k}{2}+\alpha}}{\Gamma(\frac{k}{2}+\alpha+1)} \quad .$$

For instance, in the case of $k = 2$, $x = y = 0$ and $\alpha = 0$,

$$P_0(t) = P_0(t:0,0) = \sum_{|m|^2<t} 1 - \pi t \ ,$$

that is, $P_0(t)$ = the difference between the number of lattice points in the open disk centered at the origin and having radius $\sqrt{t}$ and the area of this disk (lattice remainder term). The estimating problem for $P_0(t)$ is called Gauss's circle problem.

The following estimates are well known standard ones. (We note that some better estimates have been known.)

(1) Upper estimate : $P_0(t) = O(t^{\frac{12}{37}+\varepsilon})$ for any $\varepsilon > 0$,

(2) Lower estimate : $P_0(t) = \Omega(t^{\frac{1}{4}} \log^{\frac{1}{4}} t)$ and

(3) What is the best possible estimate ? There is a conjecture that $P_0(t) = O(t^{1/4+\varepsilon})$ for any $\varepsilon > 0$.

(Hardy's conjecture)

(In (2) , $\phi(t) = \Omega(\psi(t))$ implies $\phi(t) \neq o(\psi(t))$.)

Generally, the estimating problems for $P_\alpha(t:x,y)$ are called the lattice point problem with weight in a k-dimensional sphere. (See Fricker [6].)

2. The theme of our paper is on some relations between the pointwise convergence problem of multiple Fourier series and the lattice point problem. (See also Kuratsubo [9].)

The following equality shows a closed relation between two problems.

$$S_t^\alpha(f:x) = \frac{\Gamma(\alpha+1)}{t^\alpha} \int_{T^k} f(x-z)\, P_\alpha(t:z,0)\, dz$$

Next, we shall recall well known results on pointwise convergence of $S_t^\alpha(f:x)$ to $f(x)$: Suppose $1 \leqq p \leqq 2$ and $\alpha \geqq 0$ and put $\alpha_p = \frac{k-1}{2} - \frac{k}{p'}$ where $p' = \frac{p}{p-1}$. Then we have the following

(1) If $\alpha > \alpha_p + \frac{1}{p'}$, then for any function $f(x) \in L^p(T^k)$ $\lim_{t\to\infty} S_t^\alpha(f:x) = f(x)$ almost everywhere. (Stein [11])

(2) If (i) $\alpha=0$ and $\frac{2k}{k+1} \leqq p < 2$, or (ii) $\alpha = \alpha_p$ and $1 \leqq p \leqq \frac{2k}{k+1}$, then there exists a function $f(x) \in L^p(T^k)$ such that $S_t^\alpha(f:x)$ diverges almost everywhere. ((i) Fefferman [5] and Babenko [3], (ii) Babenko [3] and Stein [12])

(3) If $\alpha < \alpha_p$ and $1 \leqq p \leqq \frac{2k}{k+1}$, then there exists a function $f(x) \in L^p(T^k)$ such that $\limsup_{t\to\infty} |S_t^\alpha(f:x)| = \infty$ almost everywhere,

where we can take for such a function $f(x)$, the function $f_\sigma(x)$ such that $\hat{f}_\sigma(m) = \frac{1}{|m|^\sigma}$ ($|m| \geqq 1$) with $\frac{k}{p'} < \sigma < \frac{k-1}{2} - \alpha$.

(Babenko [3]) (See also Alimov, Il'in and Nikishin [2, §5] and Kuratsubo [8].)

Now, what is the case of $\max\{0, \alpha_p\} < \alpha \leqq \alpha_p + \frac{1}{p'}$? It is unknown yet. (See Stein [13, Problem 4].)

We got recently the following result as a corollary of our main theorem on the strong summability. (Kuratsubo [10])

THEOREM *If* $0 \leqq \alpha < \alpha_p$ *and* $1 \leqq p \leqq \frac{2k}{k+1}$, *then for any* $\tau > \frac{1}{p}$ *there exists a function* $f(x) \in L^p(T^k)$ *such that*

$$S_t^{\alpha}(f:x) = \Omega(t^{\frac{1}{2}(\alpha_p - \alpha)} \log^{-\tau} t) \quad \text{as} \quad t \to \infty , \quad \text{everywhere} ,$$

where we can take for such a function $f(x)$, *the function* $f_{p,\tau}(x)$ *such that*

$$\hat{f}_{p,\tau}(m) = \frac{1}{|m|^{\frac{k}{p'}} \log^{\tau} |m|} \qquad (|m| > 1) .$$

This is better than (3) of this section by having an explicit divergent order and being valid not almost everywhere but everywhere. Our main result in this paper is an improvement of this divergent order.

3. Our main theorem is the following

THEOREM *Suppose* $\tau \geqq 0$, $0 \leqq \alpha < \frac{k-1}{2}$, $0 \leqq \sigma$ *and* $\alpha + \sigma \leqq \frac{k-1}{2}$. *Then we have*

$$\sum_{1<|m|^2<t} (1 - \frac{|m|^2}{t})^{\alpha} \frac{1}{|m|^{\sigma} \log^{\tau} |m|} e(mx) = \Omega(t^{\frac{1}{2}(\frac{k-1}{2} - \alpha - \sigma)} (\log t)^{\frac{1}{k}(\frac{k-1}{2} - \alpha) - \tau})$$

everywhere.

Now, let $g_{\sigma,\tau}(x)$ be a function whose Fourier coefficients are given by $\hat{g}_{\sigma,\tau}(m) = \dfrac{1}{|m|^{\sigma} \log^{\tau}|m|}$ ($|m| > 1$) , that is,

$$g_{\sigma,\tau}(x) = \frac{C}{|x|^{k-\sigma} \log^{\tau}\frac{1}{|x|}} + \phi(x) ,$$

where C is a positive constant number and $\phi(x)$ is a function in $C^{\infty}(T^k)$. Then $g_{\sigma,\tau}(x) \varepsilon L^p(T^k)$ if and only if $\sigma > \frac{k}{p'}$, or $\sigma = \frac{k}{p'}$ and $\tau > \frac{1}{p}$. (Wainger [15, Theorem 7, p.39])

Therfore, we have directly the following corollary from our main theorem.

<u>COROLLARY</u> <u>If</u> $0 \leq \alpha < \alpha_p$ <u>and</u> $1 \leq p \leq \frac{2k}{k+1}$, <u>then for any</u> $\tau > \frac{1}{p}$ <u>there exists a function</u> $f(x) \varepsilon L^p(T^k)$ <u>such that</u>

$$S_t^{\alpha}(f:x) = \Omega(t^{\frac{1}{2}(\alpha_p-\alpha)} (\log t)^{\frac{1}{k}(\frac{k-1}{2}-\alpha)-\tau}) \text{ as } t\to\infty , \text{ everywhere,}$$

<u>where we can take for such a function</u> $f(x)$, <u>the function</u> $f_{p,\tau}(x)$ <u>such that</u>

$$\hat{f}_{p,\tau}(m) = \frac{1}{|m|^{\frac{k}{p'}} \log^{\tau}|m|} \quad (|m| > 1).$$

We need two lemmas for the proof of our theorem.

<u>LEMMA</u> 1 <u>For</u> $0 \leq \alpha < \frac{k-1}{2}$, <u>we have</u>

$$\sum_{|m|^2<t} (1-\frac{|m|^2}{t})^{\alpha} e(mx) = \Omega(t^{\frac{1}{2}(\frac{k-1}{2}-\alpha)} (\log t)^{\frac{1}{k}(\frac{k-1}{2}-\alpha)}) \text{ as } t\to\infty ,$$

<u>everywhere</u>.

The proof of Lemma 1 is based on the following equality

$$\frac{1}{\ell!}\int_0^\infty e^{-t}\, t^\ell\, P_\alpha(\xi t:x,y)\,dt = \frac{\xi^{\frac{1}{2}(\frac{k}{2}+\alpha)}}{\pi^\alpha}\sum_{m+x\neq 0}\frac{H_\ell^{(\frac{k}{2}+\alpha)}(\pi^2|m+x|^2\xi)}{|m+x|^{\frac{k}{2}+\alpha}}\, e(my)$$

where $H_\ell^{(\beta)}(u) = u^{\frac{\beta}{2}}\, e^{-u}\, L_\ell^{(\beta)}(u)$ and $L_\ell^{(\beta)}(u)$ is Laguerre polynomial of order β. The proof of lemma was obtained for the case $\alpha = 0$ and $x = 0$ by Szegö [14], for the case $\alpha = 0$ and any x by Berndt [4] and for the general case by Hafner [7].

The next lemma is a generalization of Alimov and Il'in ([1, lemma 2.2]). (Kuratsubo [10])

LEMMA 2 Suppose $s > -1$ and $s = r+\kappa$ where r is an integer and κ satisfies $0 < \kappa \leq 1$. For $\beta > 0$ and $\tau \geq 0$, define a function $b(\lambda)$ as $\lambda^\beta \log^\tau\lambda$, or 0 according as $\lambda \geq e$, or not respectively. Further, for any numerical series $\sum_{n=1}^\infty a_n$, let σ_λ^s, $\overline{\sigma_\lambda^s}$ be $\sum_{n<\lambda}(1-\frac{n}{\lambda})^s a_n$, $\sum_{n<\lambda}(1-\frac{n}{\lambda})^s b(n)\, a_n$ respectively.

Then we have

$$\overline{\sigma_\lambda^s} = b(\lambda)\,\sigma_\lambda^s + (-1)^{r+1}\int_0^1 \frac{t^{r+1}}{(r+1)!}\,\sigma_{t\lambda}^{r+1}\left(\frac{d}{dt}\right)^{r+2}[(b(t\lambda)-b(\lambda))(1-t)^s]\,dt$$

and, for some positive constant C,

$$\int_0^1 \frac{t^{r+1}}{(r+1)!}\left|\left(\frac{d}{dt}\right)^{r+2}[(b(t\lambda)-b(\lambda))(1-t)^s]\right|dt \leq C\, b(\lambda),\quad \lambda > 0.$$

4. The proof of theorem. Assume the existence of x such that

$$\sum_{1<|m|^2<t} \left(1-\frac{|m|^2}{t}\right) \frac{1}{|m|^{\sigma}\log^{\tau}|m|} e(mx) = o\left(t^{\frac{1}{2}\left(\frac{k-1}{2}-\alpha-\sigma\right)} (\log t)^{\frac{1}{k}\left(\frac{k-1}{2}-\alpha\right)-\tau}\right)$$

Next, applying lemma 2 to the case $s = \alpha$, $\beta = \frac{\sigma}{2}$, $\tau = \tau$ and

$$a_n = \sum_{|m|^2=n} \frac{1}{|m|^{\sigma}\log^{\tau}|m|} e(mx) ,$$

then we have

$$\sigma_{\lambda}^{s} = \sum_{1<|m|^2<\lambda} \left(1-\frac{|m|^2}{\lambda}\right)^{\alpha} \frac{1}{|m|^{\sigma}\log^{\tau}|m|} e(mx)$$

and

$$\overline{\sigma_{\lambda}^{s}} = 2^{\tau} \sum_{1<|m|^2<\lambda} \left(1-\frac{|m|^2}{\lambda}\right)^{\alpha} e(mx) .$$

Therefore, we have

$$|\overline{\sigma_{\lambda}^{s}}| \leqq \lambda^{\frac{\sigma}{2}} \log^{\tau}\lambda \; |\sigma_{\lambda}^{s}| + \int_0^1 |\sigma_t^2| \frac{t^{r+2}}{(r+1)!} \left|\left(\frac{d}{dt}\right)^{r+2}[(b(t\lambda)-b(\lambda))(1-t)^{s}]\right| dt$$

$$= \lambda^{\frac{\sigma}{2}} \log^{\tau}\lambda \; o\left(\lambda^{\frac{1}{2}\left(\frac{k-1}{2}-\alpha-\sigma\right)} (\log \lambda)^{\frac{1}{k}\left(\frac{k-1}{2}-\alpha\right)-\tau}\right)$$

$$= o\left(\lambda^{\frac{1}{2}\left(\frac{k-1}{2}-\alpha\right)} (\log \lambda)^{\frac{1}{k}\left(\frac{k-1}{2}-\alpha\right)}\right) .$$

This is inconsistent with lemma 1.

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INTERPOLATION DE LAGRANGE

Alain Lascoux

Nous rappelons que l'interpolation d'une fonction d'une variable par un polynôme est intimement liée au calcul du reste de la division de deux polynômes. Nous donnons l'extension à plusieurs variables dans la Proposition 8 . Les différences divisées se révèlent être le concept approprié. Comme applications immédiates, citons l'interpolation rationnelle, les restes successifs de la division euclidienne, le P.G.C.D. de deux polynômes, les décompositions dans la base canonique des éléments de la cohomologie d'une grasmannienne, etc... Dans tous ces exemples que nous ne développerons pas, il s'agit de calculer l'action du même produit de différences divisées. Les Fonctions de Schur suffiraient à exprimer la solution de tous les problèmes évoqués ci-dessus, car ceux-ci sont symétriques en les points d'interpolation, ou les racines des polynômes en cause; néanmoins, les différences divisées fournisssent un point de vue et des algorithmes différents : plutôt que de calculer dans l'anneau des fonctions symétriques, on utilise l'action du groupe symétrique sur l'anneau des polynômes.

L'auteur ne disposait que d'une machine a traiter le français du XXI^{eme} siècle, ce qui explique la gestion aléatoire des signes diacritiques.

Etant donnée une fonction $f : \mathbb{C} \longrightarrow \mathbb{C}$, "interpoler" f en l'ensemble de "points" $B = \{b, c, \ldots, d\}$ consiste à trouver une autre fonction g , par exemple un polynôme, telle que f et g coincident en B et telle que f-g soit un "reste" négligeable, ce dernier terme ayant un sens précis suivant la catégorie choisie.

Newton et Lagrange interpolent f par le même polynome (de degre $\leq n-1$ si n est le cardinal de B) , trouvant fort heureusement le même reste, mais présentent différemment leurs calculs. Newton ordonne totalement $B = \{b < c < \ldots < d\}$, écrivant

$$(1) \qquad f(x) - f(b) - (f\partial).(x-b) - (f\partial\partial).(x-b)(x-c) - \ldots = \mathrm{Reste}_{/\mathrm{Newton}}$$

les coefficients $(f\partial\partial\ldots)$ étant ses fameuses différences divisées que nous expliciterons plus loin.

Lagrange, quant à lui, conserve la symétrie des éléments de B, et même, divisant par le produit $(x-b)(x-c)\dots(x-d)$, trouve une expression symétrique en $\{x,b,c,\dots,d\}$:

(2) $$f(x)/(x-b)(x-c)\dots(x-d) + f(b)/(b-x)(b-c)\dots(b-d) +\dots+ f(d)/(d-x)(d-b)(d-c)\dots = \text{Reste}_{/\text{Lagrange}} .$$

Cést en fait sur le reste qu'il est intéressant de porter son effort. On pose:

(3) $$\mathbb{D} = \begin{vmatrix} 1 & b & b^2 & \dots & b^{n-1} & f(b) \\ . & . & . & & . & . \\ . & . & . & & . & . \\ 1 & d & d^2 & & d^{n-1} & f(d) \\ 1 & x & x^2 & & x^{n-1} & f(x) \end{vmatrix}$$

Le calcul de Lagrange consiste simplement en la remarque que le membre de gauche de (2) est le développement du déterminant $\mathbb{D}$ suivant sa derniere colonne. Posant Δ = Vandermonde = produit des différences deux a deux, on a en effet :

(2') $$\text{Reste}_{/\text{Lagrange}} = \mathbb{D} / \Delta(x,b,c,\dots,d) .$$

Newton, ayant quant à lui conservé le facteur $(x-b)\dots(x-d)$, obtient le reste:

(1') $$\text{Reste}_{/\text{Newton}} = \mathbb{D} / \Delta(b,c,\dots,d) = \text{Reste}_{/\text{Lagrange}} \cdot (x-b)\dots(x-d) .$$

Euclide échange droite et gauche dans les équations précédentes et prend comme reste de sa fameuse division de f par le polynôme $(x-b)\dots(x-d)$ le polynôme interpolant f :

(4) $$f - \frac{\mathbb{D}}{\Delta(x,b,\dots,d)} (x-b)\dots(x-d) = \text{Reste}_{/\text{Euclide}} = * \, x^{\circ} + ** \, x^{1} +\dots+ *** \, x^{n-1} ,$$

les coefficients $*\dots*$ pouvant se calculer a l'aide des differences divisées ainsi qu'il est expliqué en (10) .

Reprenons le membre de gauche de la formule de Lagrange (2) , en adoptant une écriture qui respecte la symétrie entre les éléments de $A = \{x,b,\dots,d\}$ et en notant $R(a, A-a)$ le produit des différences de $a \in A$ avec tous les autres éléments de A :

(5) $\sum_{a \in A} f(a) / R(a,A-a) = \text{Reste}_{\text{Lagrange}}$

Comme une fonction symétrique des éléments de A-a est une fonction de a ayant pour coefficients des fonctions symétriques en A et que la formule (5) reste valable par extension des scalaires aux fonctions symétriques en A , on peut introduire une fonction symétrique g de n variables et donner au membre de gauche de (5) la forme suivante :

(6) $\sum_{a \in A} g(A-a).f(a) / R(a, A-a)$

On est ainsi conduit, à la suite de Sylvester, à sommer sur tous les sous-ensembles de même cardinal : soient p , q deux entiers positifs, A un ensemble d'intéterminées de cardinal p+q ; pour tout sous-ensemble B de cardinal p , on note R(B, A-B) le produit des différences des éléments de B avec ceux de l'ensemble complémentaire A-B : $R(B, A-B) = \prod_{b \in B\ ,\ c \in A-B} (b-c)$. Soient enfin deux fonctions symétriques f,g de p,q variables respectivement . On peut alors énoncer :

La somme

(7) $\sum_{B \subset A} g(A-B).f(B) / R(B, A-B) = \psi(g, f, A, p, q)$

est une fonction symétrique en A .

La sommation précédente peut être considérée comme un opérateur bilinéaire Gnm (p) x Gnm (q) $\longrightarrow$ Gnm (p+q) sur l'espace des fonctions symétriques Gnm (.) . Cet opérateur a la propriété essentielle de préserver les fonctions de Schur , c'est à dire d'envoyer le produit direct de deux fonctions de Schur sur une fonction de Schur ; cette proprieté se trouve à la base de maintes identités de Cauchy, Jacobi, Sylvester, Borchardt, etc... et s'interprète comme le théoreme de Bott pour la cohomologie des grassmanniennes. Décomposant f et g dans la base des fonctions de Schur, on obtient ainsi une expression de la fonction cherchée $\psi(g,f,A,p,q)$.

Une méthode plus puissante consiste a décomposer cet opérateur en produit de différences divisées. Pour cela, il nous faut ordonner totalement A : $a_1 < a_2 < \ldots < a_{p+q}$. On note σ_i la transposition de a_i et a_{i+1} et h^{σ_i} l'image par σ_i d'une fonction h des variables A .

DEFINITION. La i-ième différence divisée ∂_i est l'opérateur, noté à droite, $h \longrightarrow h\partial_i = (h - h^{\sigma_i}) / (a_i - a_{i+1})$; plus généralement, pour toute coupure de A en $A' = \{a_1 < \ldots < a_p\}$ $A'' = \{a_{p+1} < \ldots < a_{p+q}\}$, la différence divisée $\partial_{A''A'}$ est le produit

$$(\partial_p \partial_{p+1} \ldots \partial_{p+q-1})\ (\partial_{p-1} \ldots \partial_{p+q-2})\ \ldots\ (\partial_1 \ldots \partial_q)\ ,$$

et la différence divisée maximum $\partial_{A\omega}$ est le produit

$$(\partial_1 \partial_2 \ldots \partial_{p+q-1})\ (\partial_1 \ldots \partial_{p+q-2})\ \ldots\ (\partial_1)$$

Proposition 8. Etant donnés deux fonctions symétriques f,g de p, resp q variables, et un ensemble de variables A de cardinal $p+q$, on a l'egalité

$$(8) \qquad \sum_{B\subset A} g(A-B).f(B) \ / \ R(B,A-B) = g(A'').f(A')\partial_{A''A'} \quad .$$

Preuve. Comme $R(B,A-B) = \Delta(A) / \Delta(B).\Delta(A-B)$ et que $\Delta(A)$ est égal a la somme alternée des monômes en A de multidegré $0,1,2,\ldots,p+q-1$, la somme $\sum g(A-B)f(B)/R(B,A-B)$ est éqale à

$$\sum_\mu \left\{ \frac{g(a_{p+1},\ldots,a_{p+q}) f(a_1,\ldots,a_p) \Delta(a_1,\ldots,a_p) \Delta(a_{p+1},\ldots,a_{p+q})}{p!\ q!\ \Delta(a_1,\ldots,a_{p+q})} \right\}^\mu$$

somme sur toutes les permutations μ des éléments de A. Comme par ailleurs, $\Delta(A')\partial_{A'\omega} = p!$, on est réduit à vérifier que $\partial_{A\omega} = \partial_{A'\omega} \cdot \partial_{A''\omega} \cdot \partial_{A''A'}$ est l'opérateur $f \longrightarrow \sum_\mu (f/\Delta)^\mu$. Ceci résulte, si l'on veut, de ce que les deux opérateurs envoient le même monôme sur la même fonction de Schur ; étant linéaires, ils coincident donc [cf. [Dem], [B/G/G] ; la décomposition de l'opérateur $\partial_{A\omega}$ en produit de ∂_i ramène en définitive a vérifier que l'opérateur ∂_i est linéaire en les fonctions de $a_1,\ldots,a_{p+q}$ symétriques en a_i et a_{i+1}, et envoie $1/(1-a_i)$ sur $1/(1-a_i)(1-a_{i+1})$, ce qui est très exactement la formule de Lagrange pour un ensemble de cardinal 2] . #

La sommation (8) a été étudiée par Cauchy et Jacobi pour l'interpolation rationnelle (cf.[Ros]), par Sylvester [Syl], Borchardt, et pour les modernes, outre les géomètres des grassmanniennes et variétés de drapeaux, Milne [Mil] . La Physique moderne reconnait en le cas particulier $f = a^p$ de la formule de Lagrange (5) une proprieté essentielle des groupes unitaires et des niveaux d'énergie atomique [Bied] .

On peut maintenant terminer le calcul d'Euclide, à l'aide des fonctions symétriques élémentaires $\Lambda_k(A)$ définies par [Mac] :

$$\sum z^k \Lambda_k(A) = \pi_{a\in A} (1+za) .$$

Proposition 9. Soient $f(x)$ un polynôme en une variable, $A = \{a_1,a_2,\ldots,a_{n+1}\}$. Alors, modulo le polynôme $\prod_i (x-a_i)$, on a l'égalité

$$(9) \qquad f(x) \equiv (-1)^n \Lambda_n(B).f(a)\,\partial_{Ba}\, x^o + (-1)^{n-1} \Lambda_{n-1}(B)\,.f(a)\,\partial_{Ba}\, x^1 + \ldots - \Lambda_1(B).f(a)\partial_{Ba}\, x^{n-1} + f(a)\,\partial_{Ba}\, x^n ,$$

en notant $a_1 = a$ et $(a_2,\ldots,a_{n+1}) = B$.

Preuve. Les deux membres sont linéaires en f , il suffit de vérifier la formule pour la base $\{x^o,x^1,\ldots,x^n\}$. Dans ce cas, l'on a $\Lambda_{n-i}(B).a^j\partial_{Ba} = (-1)^{n-i}$ ou 0 , $0 \leq i , j \leq n$, selon que $i = j$ ou non; c'est bien l'égalité demandée.

#

Un exemple illustrera l'identité des calculs de Lagrange, Newton, Euclide. Soit f un polynôme d'une variable, $A = \{a,b,c\}$ un ensemble d'interpolation. Alors, modulo $(x-a)(x-b)(x-c)$, on a pour Lagrange,

$$f(x) \equiv f(a) \frac{(x-b)(x-c)}{(a-b)(a-c)} + f(b) \frac{(x-a)(x-c)}{(b-a)(b-c)} + f(c) \frac{(x-a)(x-b)}{(c-a)(c-b)}$$

pour Newton,

$$f(x) \equiv f(a) + f(a)\partial_1 (x-a) + f(a)\partial_1\partial_2 (x-a)(x-b) ,$$

pour Euclide,

$$f(x) \equiv bcf(a)\partial_1\partial_2\, x^o + (b+c)f(a)\partial_1\partial_2\, x^1 + f(a)\partial_1\partial_2\, x^2 .$$

Nous avons déjà mentionné que la somme (8) se rencontre dans l'interpolation d'une fonction d'une variable par une fonction rationnelle (problème résolu par Cauchy et Jacobi et que a donné lieu à tous les beaux développements connus sous le nom d'Approximantes de Padé [Brez]). Soient A un ensemble de cardinal $p+q$, f une fonction d'une variable. On Cherche deux polynômes $\pi_{c\in C}(x-c) = R(x,C)$ et $\pi_{d\in D}(x-d) = R(x,D)$ de degrés respectifs p et $q-1$ tels que les fonctions f et $R(x,C) / R(x,D)$ coincident sur A . Donnons par exemple le numérateur $R(x,C)$, en écrivant $f^{\pi}(B)$ pour le produit $\pi_{b\in B} f(b)$; le dénominateur est fourni para une expression analogue.

Proposition 10. (Cauchy). Le numérateur de l'interpolante rationnelle est à un facteur près

(10) $\sum_{B \subset A} f^{\pi}(B) . R(x, A-b) / R(B, A-B)$

la somme étant étendue a tous les sous-ensembles B de cardinal q.

L'énoncé ci-dessus est stricto-sensu incorrect, il faut ajouter une hypothèse de généricité pour s'assurer que le facteur parasite n'est pas nul et que l'interpolation est possible.

Il est clair que l'expression (10) est un polynôme de degré $\leq p$ et guère difficile de montrer que ce polynôme répond à la question. La Proposition 8 permet de retrouver la forme déterminantale classique [Brez], les différences divisées agissant simplement sur les déterminants en cause. Rosenhain [Ros] montre que le calcul de Cauchy permet d'exprimer le résultant de deux polynômes.

Sylvester a considéré des sommes plus élaborées que (10) ; c'est ainsi qu'il obtient par exemple le P.G.C.D. de deux polynômes.

Proposition 11 (Sylvester). Soient p et q deux entiers positifs tels que $p+q$ est égal au degré du P.G.C.D. de deux polynômes $R(x,A)$ et $R(x,B)$.. Alors le dit P.G.C.D. est égal à

$$\sum_{A' \subset A, B' \subset B} \frac{R(x,A') . R(x,B') . R(A',B') . R(A-A', B-B')}{R(A', A-A') . R(B', B-B')}$$

somme sur tous les sous-ensembles A' de cardinal p et B' de cardinal q.

Nous renvoyons à [Syl. I, p. 58] pour cette affirmation (la démonstration peut bien entendu se faire par une chaîne de différences divisées). Le même auteur généralise l'interpolation de Lagrange au cas de plusieurs polynômes d'une variable dans [Syl , I p. 645-646]. Pour l'interpolation de Newton à plusieurs variables, voir [L&S 2].

Remarque. Tous les calculs précédents peuvent s'interpréter geométriquement. Dans le cas de Lagrange/Euclide, on a affaire à un espace projectif, dont l'anneau de cohomologie est le quotient de l'anneau des polynômes en x par un polynôme unitaire, l'opérateur $\partial_1 . \partial_2 \ldots = \partial_{Ba}$ s'interprétant comme le morphisme de projection sur la base (morphisme de Gysin). Le cas plus général de la Proposition 8 correspond a la grassmannienne relative des quotients de rang p d'un fibré vectorial de rang $p+q$, et l'operateur $\partial \ldots \partial$ est la projection de la cohomologie de la grassmannienne sur sa base. Pour plus de détails sur les différences divisées et leur usage en cohomologie des grassmannienses et variétés de drapeaux, voir [L&S 1] .

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AN ULTRASPHERICAL GENERALIZATION OF KRALL'S JACOBI TYPE POLYNOMIALS

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1. INTRODUCTION

In [3], [4] H.L. Krall introduced orthogonal polynomials for which the weight function is the Jacobi weight function combined with a delta function at the end point(s) of the interval of orthogonality. These polynomials were described in more details by A.M. Krall [2]. Recently Koornwinder [1] studied orthogonal polynomials with weight function $(1-x)^{\alpha}(1+x)^{\beta} + M\delta(x-1) + N\delta(x+1)$. It was shown that these polynomials, which include Krall's Jacobi type polynomials, can be expressed in terms of Jacobi polynomials as

$([a_n x + b_n]\frac{d}{dx} + c_n)P_n^{(\alpha,\beta)}(x)$ for certain coefficients a_n, b_n and c_n. We guessed that a second order linear differential operator working on the Jacobi polynomials might give the polynomials which are orthogonal with respect to a weight function being a linear combination of the Jacobi weight functions, two delta functions and two derivatives of delta functions at the points 1 and -1.

But we realized that in

$$\langle\delta'(x-1),(f(x))^2\rangle = -\langle\delta(x-1),2f(x)f'(x)\rangle = -2f(1)f'(1)$$

the sign depends on the function. Therefore such a linear combination would not lead to an inner product.

Instead we considered the following two inner products:

a) <u>The symmetric case</u>

(1.1) $$\langle f,g\rangle = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}(\Gamma(\alpha+1))^2}\int_{-1}^{1} f(x)g(x)(1-x^2)^{\alpha}\,dx + M[f(1)g(1) + f(-1)g(-1)] + N[f'(1)g'(1) + f'(-1)g'(-1)], \quad \alpha > -1, \quad M \geq 0, \quad N \geq 0.$$

b) <u>The asymmetric case</u>

(1.2) $$\langle f,g\rangle = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx + Pf(1)g(1) + Qf'(1)g'(1), \quad \alpha > -1,\ \beta > -1,\ P \geq 0,\ Q \geq 0.$$

It is obvious that the inner products (1.1) and (1.2) cannot be obtained by a weight function because in both cases $\langle x,x\rangle \neq \langle 1,x^2\rangle$. Since many of the known properties of orthogonal polynomials (such as the three terms recurrence relation, the zeros inside of the interval of integration etc.) depend on the existence of a weight function, we cannot expect the polynomials, orthogonal with respect to these inner products, to satisfy these properties. They constitute an orthogonal system of functions, which are polynomials, but they are not orthogonal polynomials in the ordinary sense.

Here we report on the symmetric case. For the details we refer to [5] and [6].

2. THE POLYNOMIALS

We have shown that the polynomials $S_o^{\alpha,M,N}(x) = -(\alpha,1)\gamma_o^2$, $S_1^{\alpha,M,N}(x) = -(\alpha+1)\gamma_1^2 x$,

(2.1) $$S_n^{\alpha,M,N}(x) = S_n(x) = \left[a_n x^2 \frac{d^2}{dx^2} + b_n x \frac{d}{dx} + c_n\right] R_n^{(\alpha)}(x) \quad , \quad n\geq 2,$$

where $R_n^{(\alpha)}(x) = R_n(x)$ are ultraspherical polynomials normalized $R_n^{(\alpha)}(1) = 1$, are orthogonal with respect to the inner product (1.1) if

$$a_n = -2N\left[\frac{2R''_n(1)}{\alpha+1} M + \gamma_n R'_n(1)\right] \quad ,$$

$$b_n = 4MNR''_n(1) \left[\frac{(n+\alpha)(n+\alpha+1)}{(\alpha+1)(\alpha+3)} - 1\right] + 2(\alpha+1)\gamma_n M +$$

$$+ NR'_n(1)\gamma_n \left[(n-1)(n+2\alpha+2)-2\alpha-2\right]$$

and

$$c_n = \frac{-4MNR'''(1)R'_n(1)}{\alpha+2} - 2(\alpha+1)R'_n(1)\gamma_n M - (\alpha+1)\gamma_n^2 +$$

$$- 2N\gamma_n R'''_n(1) \frac{[(\alpha+2)n^2+(\alpha+2)(2\alpha+1)n+2\alpha+2]}{(n-1)(n+2\alpha+2)} ,$$

where $\gamma_n = \dfrac{n!\Gamma(2\alpha+2)}{\Gamma(n+2\alpha+1)}$, $R'_n(1) = \dfrac{n(n+2\alpha+1)}{2(\alpha+1)}$,

$$R''_n(1) = \frac{n(n-1)(n+2\alpha+1)(n+2\alpha+2)}{4(\alpha+1)(\alpha+2)} ,$$

$$R'''_n(1) = \frac{n(n-1)(n-2)(n+2\alpha+1)(n+2\alpha+2)(n+2\alpha+3)}{8(\alpha+1)(\alpha+2)(\alpha+3)}$$

The coefficients a_n, b_n, c_n have been determined from the orthogonality relation $<S_n, p_m> = 0$, where p_m is a polynomial of degree $m < n$. The differential equation for ultraspherical polynomials was applied frequently in order to rewrite the terms properly. Furthermore from the symmetry of the weight function it follows that S_{2n} is an even function and S_{2n+1} is an odd function.

3. SOME PROPERTIES OF $S_n(x)$

a) The values $S_n(1)$ and $S_n'(1)$

These values can be computed in a much shorter way than by substituting $x = 1$ in the definition above. Also the results are less complicated than the coefficientes b_n and c_n would suggest. We obtain

$$S_n(1) = -(\alpha+1)\gamma_n^2 + 2N\gamma_n R_n'''(1) ,$$

$$S_n'(1) = -(\alpha+1)\gamma_n \left[\frac{2R_n''(1)}{\alpha+1} M + \gamma_n R_n'(1)\right] .$$

b) Zeros

As was mentioned, the polynomials $S_n(x)$ are not ordinary orthogonal polynomials. This can be seen by the fact that $S_n(1)$ may vanish for certain values of N and α . However, $S_n'(1) \neq 0$ for all $n \geqq 1$, which implies that the polynomials S_n are not identically zero.

C) Another representation for $S_n(x)$

The fact that the values of $S_n(1)$ and $S_n'(1)$ are so much easier than the coefficient b_n and c_n inspired the first author to look for another, less complicated representation for the polynomials $S_n(x)$. The result is

$$S_n(x) = \left[\frac{a_n(1-x^2)^2}{4(\alpha+2)(\alpha+3)} \frac{d^4}{dx^4} + d_n(1-x^2) \frac{d^2}{dx^2} + e_n\right] R_n^{(\alpha)}(x)$$

where

$$a_n = -2N \left[\frac{2R_n''(1)}{\alpha+1} M + \gamma_n R_n'(1)\right] ,$$

$$d_n = N\gamma_n \frac{R_n'''(1) R_n'(1)}{R_n''(1)} + \gamma_n M ,$$

$$e_n = S_n(1) = -(\alpha+1)\gamma_n^2 + 2N\gamma_n R_n'''(1) .$$

Especially in the case $N = 0$ (the case treated by Koornwinder) we obtain a simple formula (in Koornwinder's notation)

$$\frac{P_n^{\alpha,\alpha,M,M}(x)}{P_n^{\alpha,\alpha,M,M}(1)} = \frac{-M\Gamma(n+2\alpha+1)}{(\alpha+1)n!\Gamma(2\alpha+2)} (1-x^2) \frac{d^2}{dx^2} R_n^{(\alpha)}(x) + R_n^{(\alpha)}(x)$$

d) Recurrence relation

From the definition of the inner product (1.1) we derive

$$\langle xS_n, S_i\rangle - \langle S_n, xS_i\rangle =$$

$$\begin{cases} 0 & , \text{ if } n+i \text{ is even} , \\ 2N[S_n(1)S_i'(1)-S_n'(1)S_i(1)] & , \text{ if } n+i \text{ is odd} . \end{cases}$$

Hence we can choose ρ_n such that $S_{n+1}(x) - \rho_n xS_n(x)$ is a polynomial of degree at most n. Thus we must write

$$S_{n+1}(x) - \rho_n xS_n(x) = \sum_{i=o}^{n} c_i S_i(x) .$$

Obviously $c_i = 0$ if $n+i$ is even. In particular $c_n = 0$.

For $n+i$ odd and $i \leq n-3$ we have

$$c_i\langle S_i,S_i\rangle = \langle S_{n+1}(x) - \rho_n xS_n(x) , S_i(x)\rangle$$
$$= -\rho_n\langle xS_n(x) , S_i(x)\rangle = -2\rho_n N[S_n(1)S_i'(1)-S_n'(1)S_i(1)] .$$

In general $c_i \neq 0$. This implies that for the polynomials S_n in general no three term recurrence relation exists as is the case for ordinary polynomials.

4. SECOND ORDER DIFFERENTIAL EQUATION

From (2.1), the differential equation for ultraspherical polynomials

(4.1) $$(1-x^2)R_n''(x)-2(\alpha+1)xR_n'(x)+n(n+2\alpha+1)R_n(x) = 0$$

and the derivatives of these two relations we eliminate R_n, R_n'' and R_n''' and obtain

(4.2) $$p_3(x)S_n(x)-p_2(x)(1-x^2)S_n'(x) = q_4(x)R_n'(x) ,$$

where p_3, p_2 and q_4 are polynomials of the degree given by their index and $p_3(1) = 2(\alpha+2)p_2(1) \neq 0$. Furthermore $q_4(1) = 0$ if and only if $S_n(1) = 0$. By differentiating (4.2) we easily derive that in the case $S_n(1) = 0$ we have

$$2(\alpha+3)p_2(1).S_n'(1) = q_4'(1)R_n'(1) .$$

Since $R_n'(1) \neq 0$ and $S_n'(1) \neq 0$ (see 3b) it follows that $q_4'(1) \neq 0$. Thus, q_4 is not identically zero.

We finally obtain a second order linear differential equation of the following form

$$(4.3) \qquad \alpha(x)(1-x^2)S_n''(x) + x\beta(x)S_n'(x) + \gamma(x)S_n(x) = 0 ,$$

where $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are even polynomials of the degree ≤ 8 and $\alpha(x)$ is not identically zero.

5. EXPRESSION AS HYPERGEOMETRIC SERIES

From the formulas for ultraspherical polynomials (Szegö [7] , (4.7.3), (4.7.30) and (4.21.2))

$$R_{2n}^{(\alpha)}(x) = (-1)^n \frac{(1/2)_n}{(\alpha+1)_n} \; {}_2F_1(-n, n+\alpha+1/2; 1/2; x^2) ,$$

$$R_{2n+1}^{(\alpha)}(x) = (-1)^n \frac{(3/2)_n}{(\alpha+1)_n} \; x\, {}_2F_1(-n, n+\alpha+\tfrac{3}{2} ; \tfrac{3}{2} ; x^2) ,$$

$$\frac{d}{dx} R_n^{(\alpha)}(x) = \frac{n(n+2\alpha+1)}{2(\alpha+1)} R_{n-1}^{(\alpha+1)}(x) ,$$

we obtain the following results

$$S_{2n}(x) = (-1)^n \frac{(1/2)_n}{(\alpha+1)_n} \; c_{2n} \; {}_4F_3(-n, n+\alpha+1/2, \lambda+1, \mu+1; 1/2, \lambda, \mu; x^2)$$

where we have written

$$[2k(2k-1)a_{2n}+2kb_{2n}+c_{2n}] = 4a_{2n}(k+\lambda)(k+\mu) = c_{2n}\frac{(\lambda+1)_k}{(\lambda)_k} \; \frac{(\mu+1)_k}{(\mu)_k}$$

and

$$S_{2n+1}(x) = (-1)^n \frac{(3/2)_n}{(\alpha+1)_n} \; c_{2n+1} \; {}_4F_3(-n, n+\alpha+\tfrac{3}{2}, \lambda+\tfrac{3}{2}, \mu+\tfrac{3}{2}; \tfrac{3}{2}, \lambda+\tfrac{1}{2}, \mu+\tfrac{1}{2}; x^2)$$

with

$$[2k(2k+1)a_{2n+1}+(2k+1)b_{2n+1}+c_{2n+1}] = 4a_{2n+1}(k+\lambda+\tfrac{1}{2})(k+\mu+\tfrac{1}{2}) =$$

$$c_{2n+1}\ \frac{(\lambda+\frac{3}{2})_k(\mu+\frac{3}{2})_k}{(\lambda+\frac{1}{2})_k(\mu+\frac{1}{2})_k}$$

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SOME REMARKS ON THE STRUCTURE OF THE ERROR OF TAU METHOD APPROXIMATIONS

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1. INTRODUCTION

In a paper published in 1983 Ortiz and Rivlin [1] showed that if the Chebyshev polynomials of the first kind defined on [a,b] are plotted in the square $S := \{x,y \in \mathbb{R}^2 : a \leq x,y \leq b\}$, a series of characteristic patterns appear, which are generated by their mutual intersections. They also showed that such property is shared by other polynomials, in particular by Legendre polynomials defined on [a,b], if appropriate shape functions or envelopes are applied to them.

More recently, Freilich and Ortiz [2], Onumanyi and Ortiz [3] and Namasivayam and Ortiz [4] - [5] discussed the properties of polynomial approximations of degree n of a given function y(x), implicitly defined by a differential equation, when the solution of such equation is approximated by using the Tau Method (see Lanczos [6] and Ortiz [7]) ; we call these approximate solutions Tau polynomials.

These authors showed in [2] - [5] that for certain classes of ordinary and partial differential equations the order of approximation of Tau polynomials is close to that of best uniform approximations of y(x) by polynomials of the same degree n.

Namasivayam and Ortiz [8] discussed the problem of intersection patterns in the case of normalized error curves of Tau polynomials when the function y(x) is defined by a differential equation with constant coefficients. In this case Tau polynomials are constrained by the requirement that they must satisfy exactly the given supplementary conditions, therefore they must have zero error at the point where the initial condition is given.

These authors showed also that in this case an envelope curve L, which pases through the origin, encapsulates the normalized error curves inside the square S. This result gives some information about the structure of the error of Tay polynomials: their normalized error curves can be decomposed into the product of two factors. One of them is a highly oscillating function and the other is the envelope L, which "modulated" these oscillations. Another interesting feature of Tau polynomials, consistent with the fact that they are close to best uniform approximations of $y(x)$, is the presence of patterns which, although similar to those discussed by Ortiz and Rivlin in [1] for best uniform approximations of zero in $[a,b]$, are now compressed inside the domain defined by the envelope L.

2. ANALYSIS OF A CASE WHERE THE COEFFICIENTS ARE VARIABLE

In this note we show that a similar result is true when $y(x)$ is implicitly defined by a differential equation with variable coefficients; we construct the envelope curve L for the concrete case considered.

Let us consider the differential equation

$$A(x)y'(x) + B(x)y(x) = 0, \quad \text{for} \quad a \leq x \leq b,$$

with $A(x) = 2x$; $B(x) = -1$; $a = 0.5$; $b = 1.0$ and with the supplementary condition $y(1) = 1$.

(1)

Let $T_n(x)$ be the Chebyshev polynomial of the first kind and degree n defined on $-1.0 \leq x \leq 1.0$ and let $T^*_n(x)$ stand for these polynomials when defined on $0.5 \leq x \leq 1.0$.

We consider the <u>Tau Problem</u> associated with (1) and definied by

$$2xy'_n(x) - y_n(x) = \tau_n T^*_n(x), \quad \text{for} \quad 0.5 \leq x \leq 1.0,$$

with the condition $y_n(1) = 1$.

(2)

It follows that the error function $e_n(x) := y_n(x)-y(x)$ satisfies the same differential equation as $y_n(x)$, but with an homogenous initial condition. Therefore,

$$e_n(x) := \tau_n x^{1/2} \int_{4x-3}^{1} [T^*_n(t)/(t+3)^{3/2}]dt .$$

The last integral is given by

$$\{n \sin[n \cos^{-1}(4x-3)] \sin[\cos^{-1}(4x-3)]/[8x^{3/2}(n^2-1)]\}+o(1/n^2),$$

thus,

$$[8(n-1/n)e_n(x)]/\tau_n := \{\sin[n \cos^{-1}(4x-3)] \sin[\cos^{-1}(4x-3)]\}/x + o(1/n^2).$$

Therefore the curves defined by

$$[8(n-1/n)e_n(x)]/\tau_n$$

have the envelope L defined by

$$\pm \sin [\cos^{-1}(4x-3)]/x \ .$$

3. FIGURES

Figures 1-4 display the normalized error curves of Tau polynomials of degrees $n = 6(1)10$; $n = 11(1)14$ and $n = 15(1)19$ and $n = 20(1)24$, while Figure 5 shows them for $n=6(1)24$. The envelope curve L of the normalized error curves of Tau polynomials is clearly visible in these figures. The patterns of intersections became visible, as white lines, only in the last Figure.

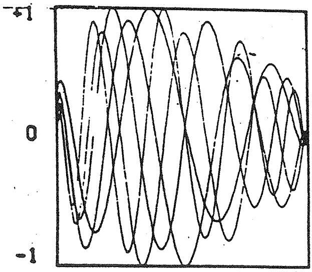

Figure 1: n = 6(1)10

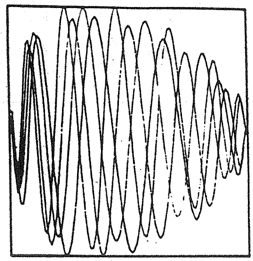

Figure 2: n = 11(1)14)

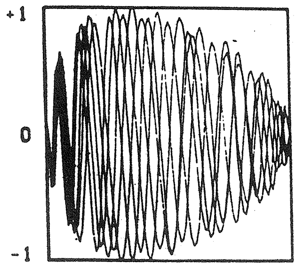

Figure 3: n = 15(1)19

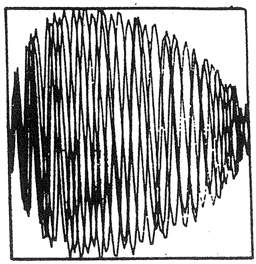

Figure 4: n = 20(1)24)

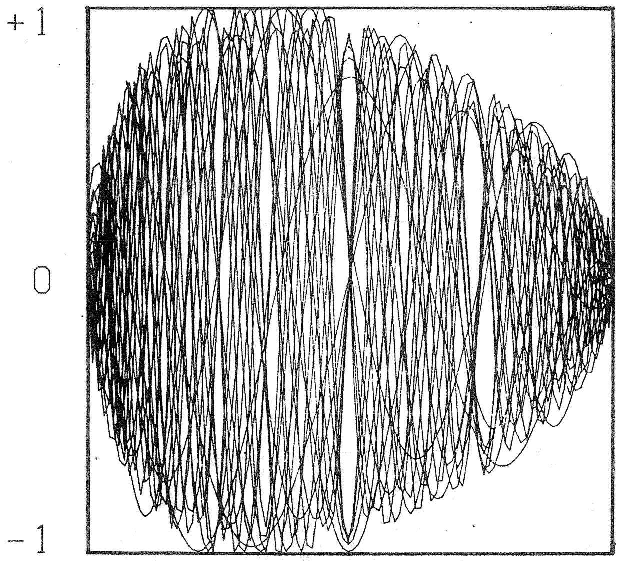

Figure 5:n = 6(1)24

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KERNELS AND POLYNOMIC MODIFICATIONS IN LEMNISCATES

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1.- INTRODUCTION

Let $\Gamma: |A(z)| = R$ be a lemniscate, where $A(z) = \sum_{j=0}^{h} a_j z^{h-j}$ is a monic complex polynomial, whose roots $\{\alpha_i\}_{i=1}^{h}$ are simples. We consider the orthonormal polynomials sequence (O.N.P.S.) $\{\hat{P}_n(z)\}_{n \in \mathbb{N}}$ with respect to a m-distribution function $\mu(z)$ defined over the curve.

As we know the following inner product associated with $\mu(z)$ is defined as:

$$\langle z^k, z^j \rangle_\mu = \int_\Gamma z^k \bar{z}^j \, d\mu(z) = c_{kj}$$

The descomposition

$\mathbb{P}_n = A\,\mathbb{P}_{n-h} \oplus [A\,\mathbb{P}_{n-h}]^{\perp n}$, $(n \geq h)$, where $\mathbb{P}_n$ is the linear space of complex polynomials whose degree is less than $n+1$, is central in the study of the recurrence properties of such a sequence. In a particular case, a basis of the orthogonal complement subspace $([A\,\mathbb{P}_{n-h}]^{\perp n})$ is given in [1], and satisfies some interpolation properties.

In this paper, we study a new basis related to a problem of polynomic modifications of $\mu(z)$.

2.- POLYNOMIC MODIFICATIONS OF A M-DISTRIBUTION FUNCTION

Let $\mu(z)$ be a m-distribution function defined over Γ.

The m-distribution functions $\mu^{(i)}(z)$ are defined from $\mu(z)$ in the following way:

$$d\mu^{(i)}(z) = |z - \alpha_i|^2 \, d\mu(z) \qquad i = 1,\ldots,h$$

These transformations are similar to those realized in [4].

We call $\{\hat{P}_n^{(i)}(z)\}_{n \in \mathbb{N}}$ the O.N.P.S. associated with $\mu^{(i)}(z)$ (see [2] and [6]).

Then, we have:

1) The sequence $\{(z-\alpha_i)\,\hat{P}_j^{(i)}(z)\}_{j=0}^{n-1}$ is an orthonormatized basis of $(z-\alpha_i)\mathbb{P}_{n-1}$, related to $\mu(z)$.

2) $K_n(z, \alpha_i)$ is a basis of $[(z-\alpha_i)\mathbb{P}_{n-1}]^{\perp n}$ related to $\mu(z)$, where $K_n(z,y)$ is

We then define the polynomic modifications $\tau_i(z)$ of $\mu(z)$ by:

$$d\,\tau_i(z) = \left| \frac{\hat{A}(z)}{z-\alpha_i} \right|^2 d\,\mu(z) \qquad i=1,\dots,h,$$

where $\hat{A}(z) = \frac{A(z)}{R}$

Obviusly, these m-distributions satisfy: $|z-\alpha_i|^2\, d\,\tau_i(z) = d\,\mu(z)$, because the operator $\hat{A}(z)$ is isometric.

We call $\{\hat{Q}_n^{(i)}(z)\}_{n\in\mathbb{N}}$ and $L_n^{(i)}(z, y)$, the O.N.P.S. and n-Kernel - associated with $\tau_i(z)$.

It is very easy to prove the following:

1) If $\{\hat{P}_n(z)\}_{n\in\mathbb{N}}$ is an O.N.P.S. related to $\mu(z)$, then $\{(z-\alpha_i)\hat{P}_n(z)\}_{n\in\mathbb{N}}$ is an O.N.P.S. related to $\tau_i(z)$.

2) $L_n^{(i)}(z, \alpha_i)$ is orthogonal to $\{(z-\alpha_i)\,\hat{P}_j(z)\}_{j=0}^{n-1}$ related to $\tau_i(z)$.

Further, $L_n^{(i)}(z, \alpha_i)$ is a basis of $[(z-\alpha_i)\,\mathbb{P}_{n-1}]^{\perp n}$

Proposition 1

The sequence $\left\{ \frac{\hat{A}(z)}{z-\alpha_i} L_{n-h+1}^{(i)}(z, \alpha_i)\right\}_{i=1}^{h}$ is orthogonal to the sequence $\{\hat{A}(z)\hat{P}_j(z)\}_{j=0}^{n-h}$ related to $\mu(z)$, and it is a basis of $[A\,\mathbb{P}_{n-h}]^{\perp n}$.

Proof:

The polynomial $L_n^{(i)}(z,y)$ verifies the reproductive property of n-Kernels:

$$< P(z),\, L_n^{(i)}(z,y) >_{\tau_i} = P(y), \qquad \forall\, P(y) \in \mathbb{P}_n$$

Then, for all $j=0,1,\dots n-h$, we have:

$$<\hat{A}(z)\hat{P}_j(z),\, \frac{\hat{A}(z)}{z-\alpha_i} L_{n-h+1}^{(i)}(z,\alpha_i) >_\mu =$$

$$= <(z-\alpha_i)\frac{\hat{A}(z)}{z-\alpha_i}\hat{P}_j(z),\, \frac{\hat{A}(z)}{z-\alpha_i} L_{n-h+1}^{(i)}(z,\alpha_i)>_\mu =$$

$$= <(z-\alpha_i)\hat{P}_j(z),\, L_{n-h+1}^{(i)}(z,\alpha_i)>_{\tau_i} = 0$$

Thus: $\frac{\hat{A}(z)}{z-\alpha_i} L_{n-h+1}^{(i)}(z,\alpha_i) \in [A\,\mathbb{P}_{n-h}]^{\perp n}$ $(i=1, \dots, h)$

On the other hand, if

$$\sum_{i=1}^{h} \lambda_i \frac{\hat{A}(z)}{z-\alpha_i} L_{n-h+1}^{(i)}(z,\alpha_i) = 0,$$

then $\lambda_i = 0$ $(i=1,2,\dots h)$.

Since $[A\,\mathbb{P}_{n-h}]^{\perp n}$ is a h-dimensional vector space, proposition 1 is proved.

Corollary

The basis $\left\{ \dfrac{\hat{A}(z)\, L^{(i)}_{n-h+1}(z, \alpha_i)}{(z-\alpha_i)\hat{A}'(\alpha_i) L^{(i)}_{n-h+1}(\alpha_i, \alpha_i)} \right\}_{i=1}^{h}$

of $[A\, \mathbb{P}_{n-h}]^{\perp n}$ is the same as the basis $\{\phi^{(i)}_n(z)\}_{i=1}^{h}$ defined in [5].

3.- AN ORTHOGONAL BASIS OF $[A\, \mathbb{P}_{n-h}]^{\perp n}$

From $\mu(z)$, we define the following m-distribution functions:

$$d\mu_1(z) = |z-\alpha_1|^2\, d\mu(z), \qquad d\mu_2(z) = |z-\alpha_2|^2\, d\mu_1(z), \dots\; d\mu_k(z) = |z-\alpha_k|^2\, d\mu_{k-1}(z) \qquad (k \le h)$$

We call $K^{(j)}_n(z,y)$ the n-Kernel associated with $\mu_j(z)$, $K^{(o)}_n(z,y) = K_n(z,y)$ and $\mu_o(z) = \mu(z)$.

Proposition 2

The sequence $\left\{ \prod_{i=1}^{k} (z-\alpha_i)\, K^{(k)}_{n-k}(z, \alpha_{k+1}) \right\}_{k=0}^{h-1}$ is an orthogonal basis of $[A\, \mathbb{P}_{n-h}]^{\perp n}$ related to $\mu(z)$, where $\prod_{i=1}^{0}(z-\alpha_i) K^{(0)}_n(z, \alpha_i) = K_n(z, \alpha_i)$ and $\prod_{i=1}^{k}(z, \alpha_i) = (z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_k)$

Proof:

If $j=0,1,\dots n-h$, then

$$\left\langle \hat{A}(z)\hat{P}_j(z),\ \prod_{i=1}^{k}(z-\alpha_i)\, K^{(k)}_{n-k}(z, \alpha_{k+1}) \right\rangle_\mu =$$

$$= \left\langle \frac{1}{R} \cdot \prod_{l=k+1}^{h}(z-\alpha_l) \prod_{i=1}^{k}(z-\alpha_i)\hat{P}_j(z),\ \prod_{i=1}^{k}(z-\alpha_i) K^{(k)}_{n-k}(z, \alpha_{k+1}) \right\rangle_\mu =$$

$$= \left\langle \frac{1}{R} \cdot \prod_{l=k+1}^{h}(z-\alpha_l)\, \hat{P}_j(z),\ K^{(k)}_{n-k}(z, \alpha_{k+1}) \right\rangle_{\mu_K} = 0$$

Thus:

$$\prod_{i=1}^{k}(z-\alpha_i)\, K^{(k)}_{n-k}(z, \alpha_{k+1}) \in [A\, \mathbb{P}_{n-h}]^{\perp n}$$

Finally, the orthogonality of the polynomials is easily demonstrated.

Thus, proposition 2 is proved.

It is also possible to solve an orthogonal basis of $[A\, \mathbb{P}_{n-h}]^{\perp n}$ when $A(z)$ has multiple zeros.

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RECURSIVE INVERSION OF HANKEL MATRICES

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Abstract

In this paper we obtain a recursive method for inversing Hankel matrices. This procedure generalizes the algorithm by Trench, for the inversion of positive definite Hankel matrices, to the case of arbitrary Hankel matrices. The algorithm that we describe is related with the formal orthogonal polynomials introduced by Brezinski and Draux, and with the reproducing kernels defined by the authours in a preceding paper. We conclude with some applications to the Padé approximants and to the ϵ-algorithm.

1 Introduction

Let $\{ c_i \}_{i\in N}$ be a sequence of real numbers. By a Hankel matrix we mean a matrix $A_k = (a_{ij})_{i,j=0}^{k-1}$ such that $a_{ij} = c_{i+j}$ for $i,j=0...k-1$

The main goal in this paper is to propose a recursive algorithm for the inversion of a Hankel matrix, assuming that it is invertible. The algorithm by W.F. Trench described in [6] is the base that inspires this work. In practice, Trench's algorithm is based in the application of a bordering method and the special structure of the Hankel matrix. For the inversion of A_{k+1}, this algorithm requires that the matrices A_k and A_{k-1} are invertible too. In this paper we get going over this condition and we obtain a similar algorithm relating the inverses of two consecutive regular Hankel matrices: A_k and $A_{s(k)}$ (where $s(k)$ denotes the smallest interger j, such that $j>k$ and A_j is regular).

In section 4 we relate this algorithm with concepts as the formal orthogonal polynomials with respect to the functional associated with the moments c_i's, which were introduced by C. Brezinski [2] and A. Draux [3], the recurrence relation and the Christoffel-Darboux relation for orthogonal polynomials given by Draux and the reproducing Kernels for non-definite linear functionals , which were introduced by the authors in [4] . This relation have been noted in Trench's paper for the case of positive definite Hankel matrices and Lanczos' orthogonal polynomials associated with. We want to remark that one of the steps of the recursion described in the algorithm provides a new method for the computation of the coefficients of the recurrence formula for the formal orthogonal polynomials. Finally in section 5, we show some applications of this algorithm to the formal orthogonal polynomials, Padé approximants associated with a formal power series, the ϵ-algorithm, etc.

2 Obtaining the algorithm

Let $\{ c_i \}_{i\in N}$ be a sequence of real numbers and let us consider the secuence of Hankel matrices $\{ A_n \}_{n\in N}$.

Let us suppose that A_k is invertible and denote the inverse matrix of A_k by $B_k = (b_{rs}^{(k)})_{r,s=0,\dots,k-1}$. Obviously, B_k is symmetric.

We will denote by $p(k)$, the greatest integer j, such that $j<k$ and A_j is invertible. $s(k)$ will denote the smallest integer j , such that $j>k$ and A_j is invertible and h will be equal to $h=s(k)-k$.

We have by definition,

$$\sum_{j=0}^{k-1} c_{r+j}\, b_{js}^{(k)} = \delta_{rs} \qquad 0\leqslant r,s\leqslant k-1 \tag{1}$$

Definition. For $0\leqslant s\leqslant k-1$ and $k\leqslant i\leqslant s(k)-1$ we define the coefficients

$$u_{si}^{(k)} = -\sum_{j=0}^{k-1} c_{i+j}\, b_{js}^{(k)} \tag{2}$$

It is evident that $u_{si}^{(k)}$ are the only solution of the systems

$$\sum_{j=0}^{k-1} c_{r+j}\, u_{ji}^{(k)} = - c_{r+i} \qquad 0\leqslant r\leqslant k-1 \qquad k\leqslant i\leqslant s(k)-1 \tag{3}$$

The elements defined in (2) are not independent, but all of them are related. To show this, we need the following lemma, which proof is an easy exercise.

Lemma 1. The elements $u_{s,k}^{(k)}$ verify the following equations:

$$\sum_{j=0}^{k-1} c_{r+j}\, u_{j,k}^{(k)} = - c_{r+k} \qquad 0\leqslant r\leqslant s(k)-2 \tag{4}$$

From now on, we will write $u_{s,k}^{(k)} = u_s^{(k)}$ for $s=0,\dots,k-1$ and

$$\begin{aligned} u_{j,j}^{(k)} &= 1 \quad \text{for } k\leqslant j\leqslant s(k)-1, \quad u_{r,j}^{(k)} = 0 \quad \text{for } r\neq j \ k\leqslant r\leqslant s(k)-1 \\ b_{rs}^{(k)} &= 0 \quad \text{if either } r>k-1 \text{ or } s>k-1 \text{ or } r=-1 \text{ or } s=-1 \end{aligned} \tag{5}$$

Now, let us see that the elements defined in (2) are related with the $u_{s,k}^{(k)}$.

Lemma 2. If $U_h^{(k)} = (u_{ij})_{i,j=0}^{h-1}$ denote the lower triangular matrix with elements $u_{ij} = u_{k+j-i}^{(k)}$ then, for $0\leq s\leq s(k)$, it is satisfied that

$$\left[u_{s,k}^{(k)} \quad ,\ldots, \quad u_{s,s(k)-1}^{(k)} \right]^T = U_h^{(k)^{-1}} \left[u_s^{(k)} \quad \ldots \quad u_{s-h+1}^{(k)} \right]^T \tag{6}$$

Proof. Writing matricially the equations (4) and multiplying to the left this equation by the matrix $U_h^{(k)}$, we obtain a matricial equation equivalent to (3). And we can obtain the above mentioned relation only by identifying the columns of the matrices involved in these equations.

Let us return now to the deduction of the algorithm. We define

$$\lambda_j^{(k)} = \sum_{r=0}^{k} c_{j+r}\, u_r^{(k)} \qquad \text{for } s(k)-1\leq j\leq s(k)+h-2 \tag{7}$$

Theorem 1. For $0\leq r,s\leq s(k)-1$ it is satisfied that

$$b_{rs}^{(s(k))} = b_{rs}^{(k)^{-1}} + \lambda_{s(k)-1}^{(k)} (u_r^{(k)} \;\ldots\; u_{r-h+1}^{(k)}) \begin{bmatrix} B_1^{(k)} & \ldots & B_h^{(k)} \\ \vdots & \cdot & \vdots \\ B_h^{(k)} & \ldots & 0 \end{bmatrix} \begin{bmatrix} u_s^{(k)} \\ \vdots \\ u_{s-h+1}^{(k)} \end{bmatrix}$$

where if we write $D_j^{(k)} = \sum_{r=s(k)-1}^{j} \lambda_r^{(k)} u_{s(k)-j+r}^{(k)}$ for $k-1\leq j\leq k+h-2$ then the $B_i^{(k)}$ are the solution of the following triangular system

$$\begin{bmatrix} 0 & \ldots & D_{s(k)-1}^{(k)} \\ \vdots & \cdot & \vdots \\ D_{s(k)-1}^{(k)} & \ldots & D_{s(k)+h-2}^{(k)} \end{bmatrix} \begin{bmatrix} B_1^{(k)} \\ \vdots \\ B_h^{(k)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Proof. By substracting (1) and (2) and by using (5) we have that

$$\sum_{j=0}^{s(k)-1} c_{r+j}\, b_{js}^{(k)} = \delta_{rs} - \delta_{rk} u_{sk}^{(k)} - \ldots - \delta_{rs(k)-1} u_{s,s(k)-1}^{(k)} \tag{8}$$

for $0\leq r\leq s(k)-1$ and $0\leq s\leq k-1$. For a fixed s, (8) can be considered as a system of s(k) equations with the unknows $b_{js}^{(k)}$ $0\leq j\leq k-1$, which solution

$$b_{rs}^{(k)} = b_{rs}^{(s(k))} - b_{rk}^{(s(k))} u_{sk}^{(k)} - \ldots - b_{rs(k)-1}^{(s(k))} u_{s,s(k)-1}^{(k)} \tag{9}$$

for $0\leq r\leq s(k)-1$ and $0\leq s\leq k-1$.

For $k\leq r\leq s(k)-1$ we have that $b_{rs}^{(k)} = 0$ and therefore

$$b_{rs}^{(s(k))} = b_{rk}^{(s(k))} u_{sk}^{(k)} + \ldots + b_{rs(k)-1}^{(s(k))} u_{s,s(k)-1}^{(k)} \tag{10}$$

for $0\leq s\leq k-1$ and by (7) we have that

$$b_{rs}^{(s(k))} = (b_{rk}^{(s(k))} ,\ldots, b_{rs(k)-1}^{(s(k))}) (u_{sk}^{(k)} ,\ldots, u_{ss(k)-1}^{(k)})^T =$$

$$= (b_{rk}^{(s(k))} ,\ldots, b_{rs(k)-1}^{(s(k))}) U_h^{(k)^{-1}} (u_s^{(k)} ,\ldots, u_{s-h+1}^{(k)})^T \tag{11a}$$

for $0\leq s\leq k-1$

$$b_{rs}^{(s(k))} = (b_{rk}^{(s(k))} ,\dots, b_{rs(k)-1}^{(s(k))}) (0 ,\dots, 1 ,\dots, 0)^T =$$

$$= (b_{rk}^{(s(k))} ,\dots, b_{rs(k)-1}^{(s(k))}) {U_h^{(k)}}^{-1} (0,\dots,1,u_{k-1}^{(k)},\dots,u_s^{(k)})^T \quad (11b)$$

for $k \leq s \leq s(k)-1$.

By multiplying (11) by c_{r+j} and summing for $j=0,\dots,s(k)-1$ and for $r=k,\dots,s(k)-1$, we can obtain that :

$$(b_{rk}^{(s(k))} ,\dots, b_{rs(k)-1}^{(s(k))}) {U_h^{(k)}}^{-1} (0,\dots,0,\lambda_{s(k)-1}^{(k)},\dots,\lambda_{s+h-1}^{(k)})^T = \delta_{rs} \quad (12)$$

for $k \leq r,s \leq s(k)-1$ and where δ_{rs} is the Kronecker symbol.

And in this way, (11) can be written as follows

$$\begin{bmatrix} b_{k,s}^{(s(k))} \\ \vdots \\ b_{s(k)-1,s}^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \lambda_{s(k)-1}^{(k)} \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ \lambda_{s(k)-1}^{(k)} & \cdots & \lambda_{s(k)+h-2}^{(k)} \end{bmatrix}^{-1} \begin{bmatrix} u_s^{(k)} \\ \vdots \\ u_{s-h+1}^{(k)} \end{bmatrix}$$

By sustituting in (9), we have that

(13)

$$b_{rs}^{(s(k))} = b_{rs}^{(k)} + {\lambda_{s(k)-1}^{(k)}}^{-1} (u_r^{(k)} \; \cdots \; u_{r-h+1}^{(k)}) \begin{bmatrix} B_1^{(k)} & \cdots & B_h^{(k)} \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ B_h^{(k)} & \cdots & 0 \end{bmatrix} \begin{bmatrix} u_s^{(k)} \\ \vdots \\ u_{s-h+1}^{(k)} \end{bmatrix}$$

The quantities $B_i^{(k)}$ of theorem 1 can be computed with a different way as the following theorem shows.

Theorem 2. The coefficients $B_1^{(k)},\dots,B_h^{(k)}$ satisfy the equations

$${\lambda_{s(k)-1}^{(k)}}^{-1} \begin{bmatrix} 0 & \cdots & \lambda_{s(k)-1}^{(k)} \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ \lambda_{s(k)-1}^{(k)} & \cdots & \lambda_{s(k)+h-2}^{(k)} \end{bmatrix} \begin{bmatrix} B_1^{(k)} \\ \vdots \\ B_h^{(k)} \end{bmatrix} = \begin{bmatrix} \lambda_{k-1}^{(p(k))} \\ \vdots \\ \lambda_{k+h-2}^{(p(k))} \end{bmatrix} {\lambda_{k-1}^{(p(k))}}^{-1} \quad (14)$$

Proof. From lemma 2, we have that

$$\sum_{j=0}^{k} c_{r+j}\, u_j^{(k)} = 0 \quad (0 \leq r \leq s(k)-2) \qquad \sum_{j=0}^{p(k)} c_{r+j}\, u_j^{(p(k))} = 0 \quad (0 \leq r \leq k-2)$$

and by summing we can see that

$$\sum_{j=k-i-1}^{k} u_j^{(k)} \lambda_{j+i}^{(p(k))} = 0 \qquad \text{for } 0 \leq i \leq h-1 \quad (15)$$

If we add the equation $u_k^{(k)} \lambda_{k-1}^{(p(k))} = \lambda_{k-1}^{(p(k))}$ the system of equations (15) can be written in matricial form as follows

$$U_h^{(k)} (\lambda_{k-1}^{(p(k))} ,\dots, \lambda_{k+h-2}^{(p(k))})^T = (\lambda_{k-1}^{(p(k))}, 0 ,\dots, 0)^T$$

and therefore $U_h^{(k)}$ is the inverse matrix of the lower triangular matrix $L_h^{(p(k))} = (l_{ij})_{i,j=0}^{h-1}$ with elements $l_{ij} = \lambda_{k+j-i-1}^{(p(k))} {\lambda_{k-1}^{(p(k))}}^{-1}$

So, by using the equality (13), we have that the coefficients

$B_1^{(k)}, \ldots, B_h^{(k)}$ satisfy the equation (14).

Theorems 1 and 2 provide a method to compute $B_{s(k)}$ from B_k. But we can reduce the problem of computing B_k to a simpler form, because, as the next theorem states, the elements of each anti-diagonal in the matrix B_k are related by an easy formula

Theorem 3. By using the notation (5) for $0 \leq r, s \leq k$ it is satisfied that

$$b_{rs}^{(k)} = b_{r-1,s+1}^{(k)} + \lambda_{k-1}^{(p(k))^{-1}} (u_r^{(p(k))} u_{s+1}^{(k)} - u_{s+1}^{(p(k))} u_r^{(k)})$$

Proof. From (1), and by shifting the indices and operating conveniently we can obtain that

$$\sum_{j=0}^{k-1} c_{r+j} b_{j-1,s+1}^{(k)} = \delta_{rs} - \delta_{rk-1} u_{s+1}^{(k)} - c_{k+r} b_{k-1,s+1}^{(k)} \qquad (16)$$

for $0 \leq r \leq k-1$ $0 \leq s \leq k-2$

For a fixed s, the solution of the system (16) can be written as follows

$$b_{r-1,s+1}^{(k)} = b_{rs}^{(k)} - b_{r,k-1}^{(k)} u_{s+1}^{(k)} - b_{k-1,s+1}^{(k)} \sum_{j=0}^{k-1} b_{rj}^{(k)} c_{k+j}$$

$$= b_{rs}^{(k)} - b_{r,k-1}^{(k)} u_{s+1}^{(k)} + b_{k-1,s+1}^{(k)} u_r^{(k)} \qquad (17)$$

By using theorem 1 for k instead of s(k), we can obtain that $b_{r,k-1}^{(k)} = \lambda_{k-1}^{(p(k))^{-1}} u_r^{(p(k))}$ and in this way, (17) can be written as

$$b_{rs}^{(k)} = b_{r-1,s+1}^{(k)} + \lambda_{k-1}^{(p(k))^{-1}} (u_r^{(p(k))} u_{s+1}^{(k)} - u_{s+1}^{(p(k))} u_r^{(k)}) \qquad (18)$$

Therefore, the problem of inversing A_k can be reduced to compute $u_r^{(p(k))}$ and $u_r^{(k)}$, but, as we will see next, these quantities can be computed with a recurrence formula.

Theorem 4. The coefficients $u_i^{(k)}$ satisfy the recurrence relation

$$u_{s+h}^{(s(k))} = \sum_{j=0}^{h} B_j^{(k)} u_{s+h-j}^{(k)} - \lambda_{k-1}^{(p(k))^{-1}} \lambda_{s(k)-1}^{(k)} u_{s+h}^{(p(k))}$$

For $s=-h,\ldots,k$, where the coefficients $B_j^{(k)}$ for $j=1,\ldots,h$ are the quantities of theorem 1 and $B_0^{(k)}$ is given by:

$$B_0^{(k)} = \lambda_{k-1}^{(p(k))^{-1}} \lambda_{k+h-1}^{(p(k))} - \sum_{j=1}^{h} B_j^{(k)} \lambda_{s(k)+j-1}^{(k)}$$

Proof. The equation (18) can also be written as follows

$$b_{rs}^{(k)} = b_{r-h,s+h}^{(k)} + \lambda_{k-1}^{(p(k))^{-1}} \sum_{j=1}^{h} (u_{r-j+1}^{(p(k))} u_{s+j}^{(k)} - u_{s+j}^{(p(k))} u_{r-j+1}^{(k)}) \qquad (19)$$

By sustituting (17) in the equation (2) we can obtain

$$u_s^{(k)} = - \sum_{r=0}^{s(k)-1} c_{k+r} b_{rs}^{(k)} = - \sum_{r=0}^{s(k)-1} c_{s(k)+r} b_{r,s+h}^{(k)} -$$

$$- \lambda_{k-1}^{(p(k))^{-1}} \sum_{j=1}^{h} u_{s+j}^{(k)} \lambda_{k+j-1}^{(p(k))} + \lambda_{k-1}^{(p(k))^{-1}} u_{s+h}^{(p(k))} \lambda_{s(k)-1}^{(k)} \qquad (20)$$

But by the theorem 1, sustituting in (20), we have that

$$u_s^{(k)} = u_{s+h}^{(s(k))} + \lambda_{s(k)-1}^{(k)^{-1}} (\lambda_{s(k)}^{(k)} \ \dots \ \lambda_{s(k)+h-1}^{(k)}) \begin{bmatrix} B_1^{(k)} & \dots & B_h^{(k)} \\ \vdots & & \vdots \\ B_h^{(k)} & \dots & 0 \end{bmatrix} \begin{bmatrix} u_{s+h}^{(k)} \\ \vdots \\ u_{s+1}^{(k)} \end{bmatrix}$$

$$- \lambda_{k-1}^{(p(k))^{-1}} (\lambda_k^{(p(k))} \dots \lambda_{k+h-1}^{(p(k))}) \begin{bmatrix} u_{s+h}^{(k)} \\ \vdots \\ u_{s+1}^{(k)} \end{bmatrix} + \lambda_{k-1}^{(p(k))^{-1}} \lambda_{s(k)-1}^{(k)} u_{s+h}^{(p(k))} \qquad (21)$$

and if we write $B_0^{(k)} = \lambda_{k-1}^{(p(k))^{-1}} \lambda_{k+h-1}^{(p(k))} - \sum_{j=1}^{h} B_j^{(k)} \lambda_{s(k)-1}^{(k)^{-1}} \lambda_{s(k)+h-1}^{(k)}$

we have that

$$\lambda_{s(k)-1}^{(k)^{-1}} \begin{bmatrix} 0 & \dots & \lambda_{s(k)-1}^{(k)} \\ \vdots & & \vdots \\ \lambda_{s(k)-1}^{(k)} & \dots & \lambda_{s(k)+h-1}^{(k)} \end{bmatrix} \begin{bmatrix} B_0^{(k)} \\ \vdots \\ B_h^{(k)} \end{bmatrix} = \begin{bmatrix} \lambda_{k-1}^{(p(k))} \\ \vdots \\ \lambda_{k+h-1}^{(p(k))} \end{bmatrix} \lambda_{k-1}^{(p(k))^{-1}} \qquad (22)$$

Therefore, the equation (21) can be rewritten as follows:

$$u_{s+h}^{(s(k))} = \sum_{j=0}^{h} B_j^{(k)} u_{s+h-j}^{(k)} - \lambda_{k-1}^{(p(k))^{-1}} \lambda_{s(k)-1}^{(k)} u_{s+h}^{(p(k))} \qquad (23)$$

which is valid for $s = 0,\dots,k$, and a similar proof shows that this equation is also valid for $s = -h,\dots,-1$.

The equations (13) , (16) , (18) and (23) are the basis of our generalization of Trench's algorithm. In the next section these equations are placed in an order which is convenient for the computation.

3 Stating the algorithm

Let us suppose that we need to inverse A_n, then we will compute the elements $b_{rs}^{(n)}$ as follows:

1. Initialization of the method.

   $u_j^{(-1)} = 0 \quad j \in N \ ; \quad u_0^{(0)} = 1$ and $u_j^{(0)} = 0 \quad j \neq 0$

   $\lambda_j^{(-1)} = \delta_{-1,j}$ and determine $s(0)-1 = \min \{ i \in N;\ c_i \neq 0 \}$

   compute $\lambda_j^{(0)} = c_j$ for $j = s(0)-1,\dots,2s(0)-1$

2. Recursion: we know the coefficients $\{ u_r^{(p(k))} \}$ and $\{ u_r^{(k)} \}$, with the convention $u_j^{(i)} = 0$ for either $j<0$ or $j>i$. Then we compute

$$\lambda_j^{(k)} = \sum_{r=0}^{k} c_{j+r} u_r^{(k)} \qquad \text{for } j \geq k$$

and we determine $s(k)$ from the condition $s(k)-1 = \min \{ j \geq k \; ; \; \lambda_j^{(k)} \neq 0 \}$

If $s(k) = n$ go to 3. Otherwise compute $\lambda_i^{(k)}$ and $\lambda_j^{(p(k))}$ for $i=s(k)-1,\ldots,s(k)+h-1$ and $j=k-1,\ldots,k+h-1$ and compute the $B_j^{(k)}$ for $j=0,\ldots,h$ solving the system (22)

Next, compute the coefficients $\{u_j^{(s(k))}\}$ by using the recurrence relation

$$u_{s+h}^{(s(k))} = \sum_{j=0}^{h} B_j^{(k)} u_{s+h-j}^{(k)} - \lambda_{k-1}^{(p(k))^{-1}} \lambda_{s(k)-1}^{(k)} u_{s+h}^{(p(k))}$$

for $s=-h,\ldots,k$. Sustitute k by $s(k)$ and come back to 2.

3. We have that $s(k)=n$. First, compute the $b_{rs}^{(k)}$ by using

$$b_{rs}^{(k)} = b_{r-1,s+1}^{(k)} + \lambda_{k-1}^{(p(k))^{-1}} (u_r^{(p(k))} u_{s+1}^{(k)} - u_{s+1}^{(p(k))} u_r^{(k)})$$

for $0 \leq r,s \leq k-1$. Next, we compute $\lambda_i^{(k)}$ and $\lambda_j^{(p(k))}$ for $i=s(k)-1,\ldots,s(k)+h-2$ and $j=k-1,\ldots,k+h-2$, and solve the system (14) to obtain the $B_j^{(k)}$ for $j=1,\ldots,h$. Then we obtain the $b_{rs}^{(n)}$ from the $b_{rs}^{(k)}$ by using the equation (13).

4 Conection with the formal orthogonal polynomials

Let $\{ c_i \}_{i \in N}$ be a sequence of real numbers. Let us consider a linear functional c which is defined on the space Π of the real polynomials by

$$c(x^i) = c_i \qquad i= 0,1,2,\ldots \qquad (24)$$

Moreover, let us consider orthogonal polynomials $Q_k(x)$ in the sense that

$$c(x^i Q_k(x)) = 0 \qquad \text{for} \quad i=0,1,\ldots, k-1 \qquad (25)$$

These polynomials exist and are uniquely determined (except a multiplicative factor) if the Hankel matrices A_k are inversible.

Let I be the index set given by $I = \{ i \in N \; ; \; H_i(c_0) \neq 0 \}$

Let us denote by $\{ P_i(x) \}$ a basis of Π such that: $P_0(x) = 1$,
$P_i(x)$ is the unitary orthogonal polynomial if $i \in I$ and
$P_i(x) = x^{i-p(i)} P_{p(i)}(x)$ if $i \notin I$

It is well-know that these formal orthogonal polynomials satisfy a recurrence relation with three terms (see Draux [3]). In fact, for $k \in I$ we have that

$$P_{s(k)}(x) = (x \, w_{s(k)-k-1}(x) + B_k) P_k(x) + C_k P_{pr(k)}(x) \qquad (26)$$

where $w_{s(k)-k-1}(x)$ is an unitary polynomial of degree $s(k)-k-1$, and B_k and C_k are constants, determined from the orthogonality relations.

Now, suppose that $k \in I$ and let us define the matrix H from the relation $H_k = L_k A_k L_k^T$, where the matrix L_k is an square matrix whose elements in the j-th row are the coefficients $p_i^{(j)}$ in the expansion:

$$P_j(x) = \sum_{i=0}^{j} p_i^{(j)} x^i \quad , \quad p_j^{(j)} = 1$$

Then, from the orthogonality conditions, it is readily obtained that $H_{k-1} = (h_{ij})$ is a block-diagonal matrix with elements $h_{ij} =$

$= c(P_i P_j)$ for $0 \leq i, j \leq k-1$ and such that on the diagonal there is an alternance of diagonal blocks, with elements $c(P_i^2)$, where $i \in I$, and Hankel blocks which elements above the antidiagonal are equal to zero.

Obviously, for $k \in I$, L_k and H_k are invertible and we have that

$$A_k = L_k^{-1} \; H_k \; (L_k^T)^{-1} \quad \text{and} \quad A_k^{-1} = L_k^T \; H_k^{-1} \; L_k \tag{27}$$

In [4], we have defined for $k \in I$ the reproducing Kernel of order $k-1$ associated with the linear functional c by the expression

$$K_{k-1}(x,t) = (P_0(x), \ldots, P_{k-1}(x)) \; H_k^{-1} (P_0(t), \ldots, P_{k-1}(t))^T \tag{28}$$

and therefore, we have that

$$K_{k-1}(x,t) = (1, \ldots, x^{k-1}) \; L_k^T \; H_k^{-1} \; L_k \; (1, \ldots, t^{k-1})^T$$

From (27) we have that $K_{k-1}(x,t)$ is the generating function of the inverse matrix of A_k, that is: $\quad K_{k-1}(x,t) = \sum_{r,s=0}^{k-1} b_{rs}^{(k)} \; x^r \; t^s$

These reproducing kernels satisfy the usual properties, for example the Christoffel-Darboux relation

$$(x-t) \; K_{k-1}(x,t) = t_k^{-1} \; (P_k(x) P_{p(k)}(t) - P_k(t) P_{p(k)}(x)) \tag{29}$$

where $t_k = c(P_{k-1} P_{p(k)})$

In this context, the equations of the preceding algorithm can be interpreted in the following way:

- The equations (2) for $i=k$ are the orthogonality conditions (25) and then, the $u_j^{(k)}$ for $0 \leq j \leq k$ are the coefficients of the unitary orthogonal polynomial of degreee k.
- The equation (13) is the expression (28) of the Kernel in terms of the elements of the basis of Π.
- The equation (23) is the recurrence relation with three terms for three consecutive orthogonal polynomials. We note that the equations (22) form a new method for the computation of the coefficients of the recurrence relation.
- Last, the equation (18) is the Christoffel-Darboux relation for the reproducing kernels.

5 Some applications.

The proposed algorithm can be used to compute the formal orthogonal polynomials associated with a linear functional c, which is defined from its moments. In this way, the algorithm can be used in the study of problems related with formal orthogonal polynomials: Padé approximants and the ϵ-algorithm.

Padé approximants.

Let f be a power series $\quad f(x) = \sum_{i=0}^{\infty} c_i \; x^i \quad$ and m, n two natural numbers. The [m/n]-Padé approximant (if exists) is a rational function

$$[m/n]_f(x) = \sum_{i=0}^{m} a_i x^i \Big/ \sum_{i=0}^{n} b_i x^i = w_m(x) / v_n(x)$$

such that $f(x) - [m/n]_f(x) = O(x^{m+n+1})$. If $v_n(x) \neq 0$ then it has to be

$$v_n(x)\, f(x) - w_m(x) = O(x^{m+n+1}) \qquad (30)$$

As it is well know, the Padé approximant exist if the Hankel determinant $H_n(c_{m-n+1})$ is different from zero, $H_n(c_{m-n+1}) \neq 0$

In this case, if we define $P_n(x) = x^n v_n(x^{-1}) = \sum_{i=0}^{n} b_{n-i} x^i$, we have that $P_n(x)$ is the formal orthogonal polynomial of degree n associated with the linear functional $c^{(m-n+1)}$ defined by

$$c^{(m-n+1)}(x^i) = c_{m-n+i+1}$$

Therefore, our algorithm can be interpreted as a method for computing the denominators of the regular Padé approximants in a diagonal of the Padé table. Later, we can compute the numerators by using the relation (30). This algorithm can readily be obtained if we sustitute $u_j^{(k)}$ by $b_{k-j}^{(k)}$ in the recursion of section 3.

This method is a generalization of the first bordering method of Brezinski.

The ϵ-algorithm

Let $\{ S_n \}$ be a sequence of numbers. The Shank's transformation provides a set of sequences by using the following relations:

$$e_k(S_n) = H_{k+1}^{(n)}(S_n) / H^{(n)}(\Delta^2 S_n)$$

The ϵ-algorithm by Wynn is a method to avoid the computation of the the Hankel determinants that appear in Shank's transformation. If we define $\epsilon_{-1}^{(n)} = 0$ and $\epsilon_0^{(n)} = S_n$ for $n=0,1,\ldots$, then the table for the ϵ-algorithm can be computed by using the relations

$$\epsilon_{k+1}^{(n)} = \epsilon_{k-1}^{(n+1)} + (\epsilon_k^{(n+1)} - \epsilon_k^{(n)})^{-1}$$

and it is well know that $\epsilon_{2k}^{(n)} = e_k(S_n)$.

An algebraic interpretation of the computation of the quantity $\epsilon_{2k}^{(n)}$, which was found by Brezinski, provides the basis for the application of our algorithm. If we solve the system,

$$\sum_{j=0}^{k} S_{n+i+j}\, a_j = 1 \qquad \text{for } i=0,\ldots,k$$

then $\epsilon_{2k}^{(n)} = 1 / \sum_{i=0}^{k} a_i$. As it can easily be checked, a_i is the sum of the elements of the i-th row in the inverse of the Hankel matrix $A_{k+1}^{(n)}(S_n)$. Therefore, our method for computing recursively the inverse of a Hankel matrix, provides a method for computing the elements in the table of the ϵ-algorithm, avoiding the non-defined elements.

Initialization of the method

$u_j^{(-1)} = 0 \quad j\in N \; ; \quad u_0^{(0)} = 1 \quad \text{and} \quad u_j^{(0)} = 0 \quad j\neq 0$

$d^{(-1)} = 0 \quad d^{(0)} = 1 \quad \lambda_j^{(-1)} = \delta_{-1,j} \quad j\geq -1$

Determine $s(0)-1 = \min \{ i\in N;\ S_{n+i} \neq 0 \}$ and compute

$\lambda_j^{(0)} = S_{n+j}$ for $j=s(0)-1,\dots,2s(0)-1$, $\epsilon_{2s(0)-2}^{(n)} = S_{n+s(0)-1}$

Computation of $\epsilon_{2s(k)-2}^{(n)}$ for a fixed n.

$\lambda_j^{(k)} = \sum_{r=0}^{k} S_{n+j+r}\, u_r^{(k)}$ for $j\geq k$

Determine s(k) by the condition $s(k)-1 = \min \{ j\geq k \; ; \; \lambda_j^{(k)} \neq 0 \}$

Compute $\lambda_i^{(k)}$ and $\lambda_{j-1}^{(p(k))}$ for $i=s(k)-1,\dots,s(k)+h-1$ and $j=k-1,\dots,k+h-1$ and solve the system (22)

After this, compute the coefficients $\{ u_j^{(s(k))} \}$ by using the recurrence relation (23) and compute:

$$d^{(s(k))} = \left(\sum_{j=0}^{h} B_j^{(k)} \right) d^{(k)} - {\lambda_{k-1}^{(p(k))}}^{-1} \lambda_{s(k)-1}^{(k)}\, d^{(p(k))}$$

Then obtain $\left(\epsilon_{2s(k)-2}^{(n)} \right)^{-1} = \left(\epsilon_{2k-2}^{(n)}\right)^{-1} + {\lambda_{s(k)-1}^{(k)}}^{-1}\, {d^{(k)}}^{2} \sum_{j=1}^{h} j\, B_j^{(k)}$

Acknowledgments

The authours want to thank Professor C. Brezinski for his kind suggestions on the applications of section 5.

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SEMI CLASSICAL WEIGHTS $(-\infty,+\infty)$
SEMI HERMITE ORTHOGONAL POLYNOMIALS

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ABSTRACT

All semi classical weights continuous and positive in $(-\infty,+\infty)$ are derived first using elementary integration techniques, and the finite moment relationship is derived.

The Semi Hermite polynomials, orthogonal with respect to the semi classical weight, depend on two fixed polynomials. Some elementary situations generalizing the Hermite case are examined.

I. INTRODUCTION

In two recent papers, Maroni [1,2] characterized algebraically the complete class of semiclassical orthogonal polynomials. This class is defined by a regular linear form $\mathcal{L}$, involving two fixed polynomials acting on the space of real polynomials.

A Semi Classical Orthogonal Polynomial Sequence (SCOPS) [3] is an, Orthogonal Polynomial Sequence (orthogonal with respect to $\mathcal{L}$) such that the derivative (or difference) polynomials are quasi orthogonal. The quasi orthogonality concept can be defined in several ways [1, 3, 4, 5] (not necessarly equivalent) but for the uses of this paper the following restricted definition will be sufficient [4] :

$P_n(x)$ and $P_m(x)$ are said quasi orthogonal of order k if:

$$(P_n, P_m) = 0 \quad \text{for} \quad |n-m| > k \tag{1}$$

k being the smallest integer such that the appropriate scalar product is zero.

On the other side, Bonan, Lubinsky and Nevai [5] gave recently the explicit measures of the Semiclassical class generalizing the approach of Hendriksen and Van Rossum [3] .

In contrast to these two different characterizations of the Semiclassical class, giving in full generality the linear form or the measures, we want to give here in explicit form the weigths in $(-\infty,+\infty)$ responsible for the quasi orthogonality of the sequence of derivative (or difference) polynomial.

The corresponding polynomials begin to be useful in Field Theory and in Statistical Mechanics. For instance, Amundsen and Damgaard [6] introduce Polynomials orthogonal in $(-\infty,+\infty)$ with respect to the weight:

$$(x) = e^{-\alpha x^2 - \beta x^4} \qquad (\beta > 0) \tag{2}$$

in order to compute the vacuum to vacuum amplitude in an unitarily invariant field theory in zero dimension. Bonan and Nevai [7] independently studied also these polynomials giving the second order differential equation satisfied by these polynomials. (See also Maroni [1]). Explicit tabulation are given in ref [6] involving modified Bessel functions of non integer order, and approximation in the Padé Spirit are studied by Chalbaud and Martin [8] . These polynomials belong to the semiclassical class in $(-\infty,+\infty)$ and merit therefore a special attention.

II. CONTINUOUS SEMICLASSICAL WEIGHTS IN $(-\infty,+\infty)$ -SEMI HERMITE ORTHOGONAL POLYNOMIALS

1. The starting point is the following elementary problem in differential equation: [3] .

Find all positive solutions $\rho(x)$ in $(-\infty,+\infty)$ of the first order linear differential equation with polynomial coefficients:

$$A(x)\rho'(x) + B(x)\rho(x) = 0 , \tag{3}$$

such that all moment C_n are finite:

$$C_n = \int_{-\infty}^{+\infty} x^n \rho(x)\,dx < \infty \tag{4}$$

$A(x)$ and $B(x)$ are real polynomials of degree respectively α and β (with no commun factors) and $A(x)$ is supposed to be monic (leading coefficient equal to +1).

It is clear that the existence of C_o implies that

$$\lim_{x\to\pm\infty} \rho(x) = 0 \tag{5}$$

The solution of eq. 3:

$$\rho(x) = e^{-\int_{-\infty}^{x} \frac{B(s)}{A(s)}\,ds} , \tag{6}$$

and the assumption $\rho(x) = 0$, excludes all real roots of $A(x)$ as

seen immediately from the partial fraction decomposition of $\frac{B(x)}{A(x)}$.
$A(x)$ can therefore be written in the form:

$$A(x) = \prod_{j=1}^{I} [(x-a_i)^2+b_i^2]^{r_i} \qquad b_i \neq 0 \qquad \forall_i \tag{7}$$

and the degree of $A(x)$ ($\alpha = 2 \sum_{j=1}^{I} r_i$) is therefore even .

On the other side, the asymptotic behaviour of $\rho(x)$ (from eq. 6) when x goes to plus infinity, gives

$$\rho(x) \sim e^{-\lambda' x^{\beta-\alpha+1}} \tag{8}$$

$\lambda' = \frac{\lambda}{\beta-\alpha+1}$, λ being the leading coefficient of $B(x)$.
Now $\lim_{x\to+\infty} x^n \rho(x) = 0$ implies $\beta > \alpha$ and $\lambda > 0$, and $\lim_{x\to-\infty} \rho(x) = 0$ implies now that β is odd .

The constraints imposed by equation (3) and (4) and positivity of $\rho(x)$ in $(-\infty,+\infty)$ give therefore

$$\alpha \text{ even }, \quad \beta \text{ odd} \qquad \alpha < \beta \quad \text{and} \quad \lambda > 0 \tag{9}$$

which is a little bit more precise that the conditions given by Hendriksen and van Rossum [3] .

2. Let us call Semi-Hermite the orthogonal polynomials $H_n^{(A,B)}(x)$ of degree n with respect to the weight $\rho(x)$ solution of eq(3) with restrictions (9) .

Explicitely:

$$\int_{-\infty}^{+\infty} H_n^{(A,B)}(x)\, H_m^{(A,B)}(x)\, \rho(x)\, dx = \delta_{mn}, \tag{10}$$

and let us now construct all possible weights $\rho(x)$.

The condition $\alpha < \beta$ given an integral part in the partial fraction expansion of $\frac{B}{A}$ that we denote $-Q(x)$ of even degree , pure fractional terms and after integration logarithmic and arctg terms. Collecting all possible terms, the SEMIHERMITE weight can be written in full generality as:

$$\rho(x) = e^{-Q(x)} \left[\left(\prod_{i=1}^{I} [(x-a_i)^2 + b_i^2]^{\lambda_i} \; e^{\mu_i \operatorname{arctg}(v_i x+\varepsilon_i)} \right] \cdot \left[\prod_{i=1}^{I} e^{\sum_{k=1}^{r_i-1} \frac{m_{ki}x + n_{ki}}{[(x-a_i)^2+b_i^2]^k}} \right] \tag{11}$$

where $\lambda_i, \mu_i, \nu_i, \varepsilon_i, m_{ki}$ and n_{ki} are real constants and $Q(x)$ polynomial of even degree with leading positive coefficient .

3. In the theory of semi classical orthogonal polynomials, the finite linear recurrence relation between moment C_n is essentially responsible for the quasi orthogonality of the derivative polynomials.

The recurrence isimmediately derived from eq. 3 after multiplication by x^{n+1} and integration by part between $(-\infty,+\infty)$.

The result gives

$$\sum_{i=o}^{\alpha} a_i (n+1+\alpha-i) C_{n+\alpha-i} = \sum_{k=o}^{\beta} b_k C_{n+\beta+1-k}$$

with

$$A(x) = \sum_{i=o}^{\alpha} a_i x^{\alpha-i} \qquad a_o = 1$$

$$B(x) = \sum_{k=o}^{\beta} b_k x^{\beta-k} \qquad b_o = \lambda \qquad (12)$$

This fundamental relation permits to construct all moments C_n , and therefore all polynomials $H_n^{(A,B)}(x)$, starting from the fundamental set $[C_o C_1 \dots C_\beta]$.

The r^{th} derivative $[H_n^{(A,B)}(x)]^{(r)}$ being quasi orthogonal of order $r(\beta-1)$ (of class $rs = r(\beta-1)$ [1,3,4]) with respect to the weight $[A(x)]^r \rho(x)$, it is also easy to derive a moment relationship between the moments $C_n^{(r)}$ of the weights $\rho_r = A^r \rho(x)$.

The differential equation satisfied by $\rho_r(x)$ is from eq. 3

$$A(x)\rho_r'(x) + \rho_r(x)[B-rA'] = 0 \qquad (13)$$

Multiplication by x^{n+1} and integration as before gives:

$$\sum_{j=o}^{\alpha} a_j [n+1+(r+1)(\alpha-j)] C_{n+\alpha-j}^{(r)} = \sum_{k=o}^{\beta} b_k \; C_{n+\beta+1-k}^{(r)} \qquad (14)$$

4. Let us describe briefly the simplest semi orthogonal weights in $(-\infty,+\infty)$.

a) The classical Hermite case occurs of course when all constants $\lambda_i, \mu_i, \nu_i, \varepsilon_i, m_{ki}$ and n_{ki} are zero and $A(x) = 1$. The standard choice $Q(x) = x^2$ gives the differential equation

$$\rho_r'(x) + 2x\rho_r(x) = 0 \qquad \rho_r(x) = e^{-x^2} \qquad (15)$$

and the moment relationship

$$(n+1)\,C_n^{(r)} = 2C_{n+2}^{(r)} \qquad \forall n,\ \forall r \tag{16}$$

b) The case $\lambda_i, \mu_i, \nu_i, \varepsilon_i, m_{ki}, n_{ki}$ equal to zero and $A(x) = 1$, $Q(x) = \frac{C}{4}(x-b)^4 + \frac{K}{2}(x-b)^2$ $(C > 0)$, was considered by Amundsen and Damgaard [6] and by Bonan and Nevai [7].

The weight differential equation is, for all r:

$$\rho_r'(x) + [C(x-b)^3 + K(x-b)]\rho_r(x) = 0 \tag{17}$$

and the generalized $(\forall r)$ moment recurrence (with $b = 0$ to simplify):

$$(n+1)\,C_n^{(r)} = C\,C_{n+3}^{(r)} + K\,C_{n+2}^{(r)} \tag{18}$$

generalizes the Hermite recurrence relation.

c) Searching for simplicity, the most representative case reducing also to Hermite corresponds to the choice:

$$A(x) = x^2+a^2, \quad \rho(x) = e^{-x^2} \cdot e^{\mu \,\mathrm{arctg}\, x/a} \quad \text{and}$$

$$B(x) = -A\frac{\rho'}{\rho} = 2x^3 + 2xa^2 - \mu a \tag{19}$$

The recurrence relation between the moment becomes $(r = 1)$:

$$2C_{n+4} = (n+1)a^2C_n + \mu a C_{n+1} + C_{n+2}[n+3-2a^2]$$

which reduces again to the Hermite one for large a, and to first order in $1/a$ to:

$$(n+1)\,C_n + \frac{1}{a}\mu C_{n+1} = 2C_{n+2} \tag{20}$$

d) The last example gives a new type of possible weights:

$$A(x) = (x^2+a^2)^2, \quad \rho(x) = e^{-x^2} \cdot e^{\nu/x^2+a^2},$$

$$B(x) = 2x(x^2+a^2)^2 - 2\nu x, \tag{21}$$

and a recurrence relation of type generalizing again Hermite $(a \to \infty)$:

$$(n+5)\,C_{n+4} + 2a^2(n+3)\,C_{n+2} + (n+1)\,a^4C_n =$$

$$= 2C_{n+6}+4a^2C_{n+4}+2(a^4-\nu)C_{n+2} \tag{22}$$

III. DISCRETE SEMI CLASSICAL WEIGHT IN $(-\infty,+\infty)$

We start now, in parallelism with the continuous case, with a difference equation:

$$A(x)\rho(x+1)-B(x)\rho(x) = 0 , \tag{23}$$

which can also be written in the important alternate ways:

$$A(x)\Delta\rho(x) + [A(x)-B(x)]\rho(x) = 0 \tag{24}$$

or $\Delta[\sigma(x)\rho(x)] = \tau(x)\rho(x)$

where $\Delta f(x) = f(x+1)-f(x)$

$$\sigma(x) = A(x-1) \quad \text{and} \quad \tau(x) = -A(x-1)+B(x) \tag{25}$$

$A(x)$ and $B(x)$ are real polynomials as before of degree respectively α and β ($A(x)$ is again monic). The constraints on solution of Eq. 23 are now:

$$\rho(k) > 0 \qquad k \in N , \qquad C_n = \sum_{k=-\infty}^{+\infty} \rho(k)k^n < \infty \qquad \forall n ,$$

which again implies $\lim\limits_{k\to\pm\infty} \rho(k) = 0$.

Let $\lambda = \lim\limits_{x\to+\infty} \frac{B(x)}{A(x)}$ and consider the solutions of the difference equation:

$$\rho(k+1) -\lambda\rho(k) = 0 , \tag{27}$$

which are: $\rho(k) = S(k)\lambda^k$ (28)

with $S(k)$ any periodic function of period 1 : $[S(k+1) = S(k)]$.

The degree of $B(x)$ cannot be equal to the degreee of $A(x)$ because λ finite implies that $\rho(k)$ diverges at $k = +\infty (\lambda > 1)$ or diverges at $k = -\infty$ $(\lambda < 1)$.

Each linear factor $(x-a)$ in $A(x)$ with a real gives a contribution of type $\Gamma(x-a)$ in the denominator of $\rho(x)$ which must be rejected being in contradiction with the positivity of $\rho(x)$ ($\rho(x)$ becomes 0 if $k-a$ is a negative integer or zero).

The remaining possibility for $A(x)$ is therefore:

$$A(x) = \prod_{t=1}^{T} [(x-a_t)^2+b_t^2]^{r_t} \qquad b_t \neq 0 \tag{29}$$

of even degree $\alpha = 2 \sum_{t=1}^{T} r_t$.

For analogous reasons, $B(x)$ cannot have real roots and therefore β , the degree of $B(x)$ must also be even.

The solution of the difference equation (23) should contain therefore power and ratio of Euler gamma function of the type $|\Gamma(x-a) + ib||^2$ $(b \neq 0)$.

The behaviour of the Γ function along a line parallel to the real axis is easely deduced from the Gauss relation: $(z = x+iy)$.

$$\frac{\pi^2}{|\Gamma(z)\Gamma(1-z)|^2} = (\sin \pi x \mathrm{Ch} \pi y)^2 + (\cos \pi x \ \mathrm{Sh} \pi y)^2 \qquad (30)$$

For fixed $y = b$ the right hand side is bounded when x goes to + or - infinity and therefore $|\Gamma(z)|$ goes to zero when x goes to - infinity because $|\Gamma(z)|$ goes to + infinity when x goes to + infinity.

The remaining case $\alpha > \beta$ or $\alpha < \beta$ must therefore be also eliminated from the exclusive behavior of $|\Gamma(z)|$ at + on $-\infty$ infinity appearing in the numerator of denominator of the solution of (23).

We must therefore conclude to the nonexistence of discrete semi Hermite orthogonal polynomials.

IV. REMARKS AND CONCLUSIONS

Semi Hermite Weight verify interesting properties generalizing Hermite Weight.

1. $H_n^{(A,rB)}(x)$ are Semi Hermite orthogonal polynomials with respect to the weight $[\rho^{(A,B)}(x)]^r$ $r > 0$.

$(A,B_1,\pm B_2)$

2. $H_n^{(A,B_1 \pm B_2)}(x)$ are Semi Hermite polynomials orthogonal respectively to the weight $\rho_1 \cdot \rho_2 (+)$ and $\rho_1/\rho_2 (-)$, where $\rho_i(x)$ $(i=1,2)$ are the Semi classical weight solutions of the equations:

$$A(x)\rho_i'(x) + B_i(x)\rho_i(x) = 0 \qquad (31)$$

The class $\beta - 1$ is conserved or may be lower in the ratio case; condition 9 must also be fulfilled by the difference polynomial $B_1(x) - B_2(x)$.

3. Expansion of the polynomial $A(x)[H_n^{(A,B)}(x)]'$ in the base of the $H_m^{(A,B)}(x)$, shows immediately, using quasiorthogonality of the discrete polynomial, the relation:

$$A(x)[H_n^{(A,B)}(x)]' = \sum_{m=n-\beta}^{n+\alpha-1} C_{nm} H_m^{(A,B)}(x) \tag{32}$$

This relation, generalizes the Appel property for the Classical Hermite polynomial $(A(x) = 1 , \alpha = 0 , \beta = 1)$.

4. Discrete Hermite orthogonal polynomial does not exist in the classical and semiclassical case as a consequence of the asymptotic behavior of the Γ function at $\pm\infty$. If we restrict the domain to $(0\ \infty)$ semi classical polynomial exist [9] , [10] , called generalized Meixner, with $\alpha = \beta$. In the same way $\alpha > \beta$ would give other discrete semi Laguerre orthogonal polynomials that would generalize the Charlier polynomials.

ACKNOWLEDGEMENTS

It is a pleasure to thank Dr. P. Maroni and his orthogonal polynomial team in Paris VI, and Dr. A.P. Magnus from Louvain-la-Neuve for constructive comments on this paper.

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HAHN POLYNOMIALS AS EIGENVECTORS OF POSITIVE OPERATORS

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Abstract:

It is proved that Hahn polynomials are eigenvectors of positive operators of Bernstein type. The eigenvalues are also computed exactly. This extend previous results of Derriennic and the author on Legendre and Jacobi polynomials.

1. INTRODUCTION

In his thesis, Durrmeyer [6] introduced the modified Bernstein operators in $L^2(0,1)$:

$$\tilde{B}_n f = (n+1) \sum_{i=0}^{n} \left(\int_0^1 b_i^n(t) f(t) dt \right) b_i^n$$

where $b_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}$, $0 \leq i \leq n$. They were studied and generalized later by Deriennic in [4] and [5] , who proved that the eigenvectors of $\tilde{B}_n$ are the shifted Legendre polynomials $\{p_i, \ 0 \leq i \leq n\}$ orthogonal on $(0,1)$ w.r.t. the classical scalar product:

$$(f,g) = \int_0^1 f(t) g(t) dt$$

More precisely, she proved that $\tilde{B}_n p_i = \lambda_{n,i} p_i$ where

$$\lambda_{n,i} = \frac{n! \quad (n+1)!}{(n-i)! \ (n+i+1)!} \ .$$

This result was extended by the author in an unpublished report [8] to shifted Jacobi polynomials $\{p_i^w \ , \ 0 \leq i \leq n\}$ orthogonal on $(0,1)$ w.r.t. the scalar product:

$$(f,g)_w = \int_0^1 w(t) f(t) g(t) dt$$

where $w(t) = t^\alpha (1-t)^\beta$ $(\alpha, \beta > -1)$

Defining the Bernstein-Jacobi operators:

$$J_n^w f = \sum_{i=0}^{n} (f, \tilde{b}_i^n)_w \ b_i^n$$

where $\tilde{b}_i^n = b_i^n/(e_o,b_i^n)_w$ and $e_o(x) = 1$, we get :

$$J_n^w\, p_i^w = \lambda_{n,i}^w\, p_i^w \quad \text{where :}$$

$$\lambda_{n,i}^w = \frac{n!\ \Gamma(\alpha+\beta+n+2)}{(n-i)!\ \Gamma(\alpha+\beta+n+i+2)} \quad , \quad 0 \leq i \leq n .$$

In this paper, we prove a similar result for Hahn polynomials $\{Q_{N,i}^n\ ,\ 0 \leq i \leq n \leq N\}$ orthogonal on $[0,N]$ w.r.t. the scalar product:

$$<f,g>_w = \sum_{x=o}^{n} w(x)\,f(x)\,g(x)$$

where $w(x) = \binom{x+\alpha}{\alpha}\binom{N-x+\beta}{\beta}$ $\qquad (\alpha,\beta > -1)$

(see for example [1] , chapter 5 of [2] or chapter 2 of [7]) . The corresponding positive operators are:

$$H_{N,n}^w f = \sum_{i=o}^{n} <f,\tilde{b}_{N,i}^n>_w\, b_{N,i}^n \qquad (1)$$

where $b_{N,i}^n(x) = \binom{x}{i}\binom{N-x}{n-i}\Big/\binom{N}{n}$ $\qquad (2)$

and $\tilde{b}_{N,i}^n(x) = b_{N,i}^n(x)\Big/<e_o,b_{N,i}^n>_w$

As for the operators J_n^w in $L_w^2(0,1)$, these operators are positive and self-adjoint on $l_w^2[0,N]$. Moreover we have :
$H_{N,n}^w\, Q_{N,i}^w = \lambda_{n,i}^w(N)\, Q_{N,i}^w$ where:

$$\lambda_{N,i}^w(N) = \frac{n(n-1)\ldots.(n-i+1)\ (N+\alpha+\beta+2)\ \ldots..(N+\alpha+\beta+i+1)}{N(N-1)\ldots.(N-i+1)\ (n+\alpha+\beta+2)\ \ldots..(n+\alpha+\beta+i+1)}$$

for $1 \leq i \leq n \leq N$, and $\lambda_{n,o}^w(N) = 1$.

Throughout the paper, we use the notation $e_r(x) = x^r$ for all $r \geq 0$.

2. THE DISCRETE BERNSTEIN BASIS (DBB)

The polynomials $\{b_{N,i}^n(x)\ ,\ 0 \leq i \leq n\}$ defined by (2) form a basis of $\mathbb{P}_n[0,N]$ and verify the following properties:

(i) $b_{N,i}^n(x) \geq 0$ for all x integer in $[0,N]$.

(ii) $\sum_{i=o}^{n} b_{N,i}^n(x) = \binom{N}{n}^{-1} \sum_{i=o}^{n} \binom{x}{i}\binom{N-x}{n-i} = 1$

by Vandermonde's convolution identity [3] .

(iii) $\sum_{i=0}^{n} \frac{iN}{n} b_{N,i}^{n}(x) = e_1(x) = x$.

The two last results are particular cases of :

Lemma 1. For all $r \geq 1$, we have:

$$x(x-1)\ldots(x-r+1) = \sum_{i=r}^{n} i(i-1)\ldots(i-r+1)\, b_{N,i}^{n}(x) .$$

This is proved by using generating functions, as for the following:

Lemma 2. For all $0 \leq i \leq n$:

$$\langle e_0, b_{N,i}^{n}\rangle_w = \sum_{x=0}^{N} w(x) b_{N,i}^{n}(x) =$$

$$= \binom{N}{n}^{-1} \binom{\alpha+i}{\alpha} \binom{\beta+n-i}{\beta} \binom{N+\alpha+\beta+1}{n+\alpha+\beta+1}$$

the basis $\{b_{N,i}^{n}(x)\ ,\ 0 \leq i \leq n\}$ is called the discrete Bernstein basis (DBB) of $\mathbb{P}_n[0,N]$.

3. THE BERNSTEIN HAHN OPERATORS

By definition (1) and Lemma 2, we have

$$H_{N,n}^{w} f = \binom{N}{n} \binom{N+\alpha+\beta+1}{n+\alpha+\beta+1}^{-1} \sum_{i=0}^{n} \binom{\alpha+i}{\alpha}^{-1} \binom{\beta+n-i}{\beta}^{-1} \langle f, b_{N,i}^{n}\rangle_w b_{N,i}^{n} \qquad (3)$$

Therefore, property (ii) is equivalent to $H_{N,n}^{w} e_n = e_0$. We have the more general result:

Theorem 1. $H_{H,n}^{w}(\mathbb{P}_r) = \mathbb{P}_r$ for $0 \leq r \leq n \leq N$.

Proof: From (3) and (iii), it is easy to deduce:

$$H_{N,n}^{w} e_1 = \frac{n}{N} \frac{N+\alpha+\beta+2}{n+\alpha+\beta+2} e_1 + \frac{(\alpha+1)(N-n)}{n+\alpha+\beta+2} e_0$$

therefore $H_n^{w}(\mathbb{P}_1) = \mathbb{P}_1$.

Now, it suffices to prove that $p_r = H_{N,n}^{w} e_r$ is exactly of degree r , for all $0 \leq r \leq n$. As a consequence of lemma 1, this is equivalent to proving that the coefficients of p_r w.r.t. the DBB are polynomials in i of degree at most r. After some simple computations, one gets :

$$p_r = \binom{N+\alpha+\beta+1}{n+\alpha+\beta+1}^{-1} \sum_{i=o}^{n} \delta_{i,r}\, b^n_{N,i} \quad , \text{ where } : \tag{4}$$

$$\delta_{i,r} = \sum_{y=o}^{N-n} (y+i)^r \binom{y+\alpha+i}{\alpha+i} \binom{N-i+\beta-y}{n-i+\beta}$$

$$= \sum_{y=o}^{M} (y+i)^r \binom{y+a}{a}\binom{P-a-y}{p-a} \tag{5}$$

with the notations $M = N-n$, $a = \alpha+i$, $b = \beta+n-i$, $p = n+\alpha+\beta = a+b$ and $P = N+\alpha+\beta = p+M$. Now let :

$$f_o(u) = u^i (1-u)^{-(a+1)} = \sum_{k\geq o} \binom{k+a}{a} u^{k+i}$$

$$f_1(u) = u\, f'_o(u) = f_o(u) \left\{ i + (a+1) \frac{u}{1-u} \right\} = \sum_{k\geq o} (k+i) \binom{k+a}{a} u^{k+i} .$$

Using the notations $w = u/(1-u)$, $p_{10}(i) = i$ and $p_{11}(i) = a+1 = \alpha+i+1 \in \mathbb{P}_1[i]$, we get :

$$f_1(u) = f_o(u) \cdot \sum_{j=o}^{1} p_{1j}(i) v^j .$$

By induction on r , we then prove that $f_r(u) = uf'_{r-1}(u)$ $(r \geq 1)$ is equal to :

$$f_o(u) \sum_{j=o}^{r} p_{rj}(i) v^j$$

where $p_{rj}(i) \in \mathbb{P}_r[i]$ verifies the recurrence relation:

$$p_{rj}(i) = (i+j) p_{r-1,j}(i) + (i+\alpha+j)\, p_{r-1,j-1}(i) \tag{6}$$

On the other hand, we have:

$$f_r(u) = \sum_{k\geq o} (k+i)^r \binom{k+a}{a} u^{k+i}$$

Now let $g(u) = (1-u)^{-(b+1)} = \sum_{\ell\geq o} \binom{\ell+b}{b} u^\ell$, then :

$$u^{-i} f_r(u) g(u) = \sum_{m\geq o} \left\{ \sum_{k=o}^{m} (k+i)^r \binom{k+a}{a} \binom{m-k+b}{b} \right\} u^m$$

in which the coefficient of u^M is exactly $\delta_{i,r}$ (5) . This function is also equal to :

$$u^{-i} f_o(u)\ g(u) \sum_{j=o}^{r} p_{rj}(i) v^j$$

$$= \sum_{j=o}^{r} p_{rj}(i) u^j (1-u)^{-(a+b+j+2)}$$

$$= \sum_{k\geq o} \left\{ \sum_{j=o}^{r} p_{rj}(i) \binom{k+a+b+j+1}{a+b+j+1} u^j \right\} u^k$$

$$= \sum_{m\geq o} \sum_{j=o}^{\inf(r,m)} p_{rj}(i) \binom{m+a+b+1}{a+b+j+1} u^m$$

Since the coefficient of u^M is $\delta_{i,r}$, we get :

$$\delta_{i,r} = \sum_{j=o}^{r} p_{rj}(i) \binom{N+\alpha+\beta+1}{n+\alpha+\beta+j+1} \qquad (7)$$

which is in $\mathbb{P}_r[i]$, q.e.d.

4. EIGENVALUES AND EIGENVECTORS OF $H^w_{N,n}$

Let us recall a result given in Derriennic [4] :

Theorem 2. Let H be a Hilbert space and $\{H_m,\ m \geq 0\}$ be an increasing sequence of subspaces verifying $\dim H_m = m$. Let $L \in \mathcal{L}(H)$ be self-adjoint and verify $L(H_m) \subset H_m$ for all $m \geq 0$, then the orthogonal sequence $\{v_m,\ m \geq 0\}$ defined by $v_m \in H_m$, $v_m \perp H_{m-1}$ $(m \geq 1)$, is a sequence of eigenvectors of L .

Lemma 3. $H^w_{N,n}$ is self-adjoint in the space $l^2_w[0,N] = \mathbb{R}^{N+1}_w$ w.r.t. the scalar product $<f,g>_w$.

The proof is obvious since:

$$<H^w_{N,n}\ f,g> = \sum_{i=o}^{n} <f,\tilde{b}^n_{N,i}> \cdot <g,b^n_{N,i}> =$$

$$= \sum_{i=o}^{n} <g,\tilde{b}^n_{N,i}> \cdot <f,b^n_{N,i}> = <f,H^w_{N,n}\ g>$$

Theorem 3. The eigenvectors of $H^w_{N,n}$ are the Hahn polynomials

$\{Q^w_{N,i}\ ,\ 0 \le i \le n \le N\}$ and the corresponding eigenvalues are:

$$\lambda^w_{n,i}(N) = \frac{n(n-1)\ldots(n-i+1)}{N(N-1)\ldots(N-i+1)}\ \frac{(N+\alpha+\beta+2)}{(n+\alpha+\beta+2)}\ \ldots\ \frac{(N+\alpha+\beta+i+1)}{(n+\alpha+\beta+i+1)}$$

for $1 \le i \le n$, with $\lambda^w_{n,0}(N) = 1$.

Proof: From theorem 2 and Lemma 3, we get:

$$H^w_{N,n}\ Q^w_{N,r} = \lambda^w_{n,r}(N)\ Q^w_{N,r} \qquad \text{for} \qquad 0 \le r \le n$$

the eigenvalue being yet unkown . But we have also $p_r = H^w_{N,n}\ e_r = \lambda^w_{n,r}(N) e_r + q_{r-1}$ where $q_{r-1} \in \mathbb{P}_{r-1}$; thus it suffices to compute the coefficient of x^r in p_r . From the proof of theorem 1, this amounts to compute the coefficient of i^r in $\delta_{i,r}(7)$. Using the recurrence (6) and an induction of r , this coefficient is proved to be :

$$\sum_{j=o}^{r} \binom{r}{j} \binom{N+\alpha+\beta+1}{N-n-j} = \binom{N+\alpha+\beta+r+1}{n+\alpha+\beta+r+1}$$

The coefficient of i^r in p_r is then , by (4) :

$$A_r = \binom{N+\alpha+\beta+1}{n+\alpha+\beta+1}^{-1} \binom{N+\alpha+\beta+r+1}{n+\alpha+\beta+r+1}$$

By lemma 1, the coefficient of i^r in e_r is :

$$B_r = \binom{N}{n} \binom{N-r}{n-r}^{-1}$$

Therefore, the r^{-th} eigenvalue of $H^w_{N,n}$ is :

$$\lambda^w_{n,r}(N) = B_r^{-1}\ A_r \qquad , \quad \text{q.e.d.}$$

6. MISCELLANEOUS RESULTS

Lemma 4. (A possibly new identity for Hahn polynomials). Let $Q^w_{N,i}(x)$ be the i-th orthonormal Hahn polynomial. There holds, for all $0 \le n \le N$ and $(x,y) \in [0,N]^2$:

$$\sum_{i=o}^{n} \lambda^w_{n,i}(N) Q^w_{N,i}(x)\ Q^w_{N,i}(y) = \sum_{i=o}^{n} \tilde{b}^n_{N,i}(x) b^n_{N,i}(y) \ .$$

This is a simple corollary of theorem 3 : to prove it, take the scalar product by any $f \in \mathbb{P}_n$, expressed in the orthonormal basis of Hahn polynomials, of the two Kernels above.

Lemma 5. (Coefficients of Hahn polynomials in the DBB)

$$Q^{w}_{N,k}(x) = \binom{k+\beta}{\beta} \sum_{j=0}^{n} (-1)^j \binom{k}{j} \binom{\alpha+j}{\alpha}^{-1} \binom{\beta+k-j}{\beta}^{-1} b^{n}_{N,j}(x)$$

if this Hahn polynomial is defined by the Olinde-Rodrigues formula (cf. [1], p. 40) :

$$\binom{x+\alpha}{\alpha}\binom{N-x+\beta}{\beta}\binom{N}{k} Q^{w}_{N,k}(x) = \binom{k+\beta}{\beta} \Delta^k \left[\binom{x+\alpha}{\alpha+k}\binom{N-x+\beta+k}{\beta+k}\right]$$

This is a simple corollary of Leibniz formula

Lemma 6. (Some further properties of $H^{w}_{N,n}$). In $\mathbb{R}^{N+1} \sim \mathbb{R}^{[0,N]}$, take the norm :

$$||f||_p = \left(\sum_{x=0}^{N} w(x) \, |f(x)|^p\right)^{1/p}$$

Then $H^{w}_{N,n}$ is an operator of norm 1 in $\mathcal{L}(\mathbb{R}^{N+1})$. Moreover, there holds:

$$\sum_{x=0}^{N} w(x) \, H^{w}_{N,n} f(x) = \sum_{x=0}^{N} w(x) \, f(x)$$

Proof: The last property can be written as:

$$\langle e_o, H^{w}_{N,n} f \rangle_w = \langle H^{w}_{N,n} e_o, f \rangle_w = \langle e_o, f \rangle_w$$

which is true for $H^{w}_{N,n}$ is self-adjoint and $H^{w}_{N,n} e_o = e_o$. The first one can be proved by using the Hölder inequalities in $l^p_w[0,N]$ and $\mathbb{R}^{n+1}$, together with property (ii) of §2 and $\langle e_o, \tilde{b}_{N,i} \rangle_w = 1$.

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On Zeros of General Orthogonal Polynomials on the Unit circle

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ABSTRACT: The location of zeros of orthogonal polynomials formally associated to a moment functional on the unit circle is considered. By using known theorems of stability theory about discrete Lyapunov matrix equation, conclusions that generalize the classical results are obtained.

1.– Introduction

It is well known that all the zeros of every polynomial of a sequence of orthogonal polynomials associated to a positive measure on the unit circle are inside the unit circle. In this communication we generalize this result to general orthogonal polynomials. As in Geronimus [1], we understand for a sequence of general orthogonal polynomials associated to a moment functional, the formally orthogonal polynomials associated to a moment functional, such that all the principal minors of the moment matrix do not vanish, though they are not necessarily positive. The generalization obtained is that every polynomial has no zeros over the unit circle and, as in the classical case, the number of zeros inside and outside the unit circle is given by some inequalities with parameters depending on the principal minors of the moment matrix.

The approximation made in Geronimus to the location of zeros of orthogonal polynomials is essentially the application of the well known Schur algorithm, see Marden [2]. It is possible to use the same method to analize the location of zeros of general orthogonal polynomials. Here we give an alternate approach through the Lyapunov matrix equation, a cornerstone in the theory of stability. We prove that the $(n-1)$-th section of the moment matrix is a solution of certain Lyapunov matrix equation associated to the n-th monic orthogonal polynomial. By using general results related to stability theory, we get conclusions about location of the zeros of these polynomials.

2.- The discrete Lyapunov matrix equation

Let A and Q be complex matrices of n–order. The matrix equation:

$$X - A^*XA = Q \tag{1}$$

where A^* is the transposed conjugate matrix of A, is usually known in the literature on stability theory as the discrete Lyapunov matrix equation. Here we shall give an overview of some relevant properties which we will use later.

Our interest is in properties of Hermitian solutions of the equation in case that Q is a semidefinite Hermitian matrix. Let us suppose $P = P^*$ is such a solution. Let λ be an eigenvalue of A and $v \neq 0$ be a corresponding eigenvector. Then it is easily seen that:

$$(1 - |\lambda|^2)v^*Pv = v^*Qv. \tag{2}$$

Hence if Q is a positive or negative definite Hermitian matrix we can conclude that all the eigenvalues of A have module different from 1. In some special cases, when Q is semidefinite the assertion holds. In this case, if Q is positive semidefinite of rank m, it can be descomposed as:

$$Q = C^*C. \tag{3}$$

with C an $m \times n$ matrix.

Then if the couple (A, C) is observable, that is, if condition:

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \tag{4}$$

is satisfied, we have likewise that all the eigenvalues of A also perform $|\lambda| \neq 1$. Briefly, the reason is that for any eigenvector v of A we can write:

$$Cv \neq 0 \tag{5}$$

otherwise the observability condition will not be fulfilled. Thus we also have $v^*Qv > 0$ and it does not matter if Q is merely semidefinite positive, and the asserted eigenvalues property of A follows. A resembling argument can be used in case Q is only negative semidefinite.

Next we shall see that there exists an interesting relation between the number of eigenvalues of A inside and outside the unit circle, and the number of positive and negative eigenvalues of the Hermitian solutions of the Lyapunov equation.

First let us suppose that all the eigenvalues of A are inside the unit circle. In this case we have:

$$\lim_{k\to\infty} A^k = 0 \tag{6}$$

and it is an easy task to check that equation (1) has one and only one solution given by:

$$X = \sum_{k=0}^{\infty}(A^*)^k Q A^k. \tag{7}$$

If Q is positive (negative) semidefinite and observability condition (4) is satisfied, this matrix is Hermitian positive (negative) definite, so all its eigenvalues are real and positive (negative).

When all the eigenvalues of A are outside the unit circle, then all the eigenvalues of the inverse matrix A^{-1} are inside the unit circle, and writing equation (1) in the equivalent form:

$$X - (A^{-1})^* X (A^{-1}) = -(A^{-1})^* Q A^{-1} \tag{8}$$

it is seen that the equation has one and only one solution given by:

$$X = -\sum_{k=1}^{\infty}(A^*)^{-k} Q A^{-k} \tag{9}$$

which, if the observability condition is assumed, turns to be Hermitian negative (positive) definite, as long as Q is a positive (negative) semidefinite matrix. In this case, all the eigenvalues of the solution are negative (positive).

In the general case, when the matrix A has eigenvalues inside and outside the unit circle, it can be proved that every Hermitian solution of the equation is nonsingular, and the number of positive and negative eigenvalues of every solution are, if Q is a positive semidefinite Hermitian matrix, respectively equal to the number of eigenvalues of A inside and outside the unit circle, and an analogous result is valid when Q is a negative semidefinite Hermitian matrix. For the proof of this result, due to Wimmer [3], we refer to Lancaster and Tismenetsky [4].

3.- Toeplitz matrices and Lyapunov equation

Let $(c_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers, such that $c_{-n} = \bar{c}_n$, and $T = [t_{ij}]$ with $t_{ij} = c_{i-j}$, be the associated infinite Hermitian Toeplitz matrix. As it is well known, we can look at T as defining a bilinear Hermitian form on the complex space $\mathbb{C}[z]$ of polynomials with complex coefficients, by extending the definition:

$$< z^i, z^j > \;=\; c_{i-j} \tag{10}$$

to every couple of polynomials by using linearity with regard to the second polynomial of the couple and Hermitian symmetry.

A crucial property of this bilinear form is:

$$< zf(z), zg(z) > \quad = \quad < f(z), g(z) > \tag{11}$$

for every pair of polynomials $f(z)$ and $g(z)$.

Now suppose that each k–section of the matrix T, that is every submatrix:

$$T_k = \begin{bmatrix} c_0 & \overline{c}_1 & \dots & \overline{c}_k \\ c_1 & c_0 & \dots & \overline{c}_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_k & c_{k-1} & \dots & c_0 \end{bmatrix} \tag{12}$$

is nonsingular. Then, as it is well known, there exist for each positive integer n a n–degree polynomial $p(z)$ such that:

$$< p(z), q(z) > \quad = \quad 0 \tag{13}$$

for every polynomial $q(z)$ such that degree $q(z) < n$. Without lost of generality, we can suppose $p(z)$ a monic polynomial as follows:

$$p(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n \tag{14}$$

Let A be the $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \dots & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & \dots & 1 & -a_1 \end{bmatrix} \tag{15}$$

and C the $1 \times n$ matrix

$$C = [0, 0, \ldots, 0, 1] \tag{16}$$

Notice that $p(z)$ is the characterictic polynomial of A.

With the previous definitions we can state our principal result:

Theorem : The matrix T_{n-1} is the solution of the discrete Lyapunov matrix equation:

$$X - A^* X A = \delta C^* C \tag{17}$$

with $\delta = \; < p(z), p(z) >$

Proof: Let us introduce the $1 \times n$ polynomial matrix:

$$U(z) = [1, z, z^2, \ldots, z^{n-1}] \tag{18}$$

which allows us to write $T_{n-1} = \ < U(z)^T, U(z) >$. Then from the identity:

$$zU(z) = U(z)A + p(z)C \tag{19}$$

and properties (10) and (11) it follows :

$$T_{n-1} = A^* T_{n-1} A + \delta C^* C \tag{20}$$

as desired.

4.- Zeros of general orthogonal polynomials

In this section, we make use of the results of the foregoing sections, in order to get conclusions about zeros of orthogonal polynomials formally associated to a moment functional.

First at all, we notice that (A, C), where A and C are the matrices respectively given in (15) and (16), is an observable pair, as an easy calculation shows. Then from equation (17) and the results of section **2**, if $\delta > 0$ the number of eigenvalues of A inside (outside) the unit circle is equal to the number of positive (negative) eigenvalues of T_{n-1}. When $\delta < 0$ we get a similar result merely by interchanging the words inside and outside. As the n-th orthogonal polynomial is the characteristic polynomial of A, we have the same proposition for the zeros of this polynomial.

On the other hand it is known, see Gantmacher [5], that for an n-order Hermitian matrix whose principal minors h_k do not vanish, the number of positive (negative) eigenvalue is the number of positive (negative) ratios in

$$\frac{h_1}{1}, \ \frac{h_2}{h_1}, \ \ldots, \frac{h_n}{h_{n-1}} \tag{21}$$

Let us introduce Δ_k as the determinant of the k-section of the matrix , and fc convenience, $\Delta_{-1} = 1$. Joining together the above result with the fact that $\delta = \Delta_n / \Delta_{n-}$ we have from the preceding consideration that the number of zeros of the n-th orthogon polynomial inside the unit circle is given by the number of positive terms in :

$$\frac{\Delta_n \Delta_{-1}}{\Delta_{n-1}\Delta_0}, \ \frac{\Delta_n \Delta_0}{\Delta_{n-1}\Delta_1}, \ \ldots, \ \frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2} \qquad (2$$

By introducing the auxiliary parameters α_k given by:

$$\alpha_k = 1 - \frac{\Delta_{k+1} \quad \Delta_{k-1}}{\Delta_k^2} \tag{23}$$

we easily obtain that:

$$\frac{\Delta_n \Delta_{k-1}}{\Delta_{n-1} \Delta_k} = (1 - \alpha_{n-1}) \dots (1 - \alpha_k) \tag{24}$$

and we have the following rule for the location, with respect to the unit circle, of zeros of orthogonal polynomials:

Let λ_n and μ_n be the number of zeros of the n-th polynomial respectively inside and outside the unit circle. Then for $(n+1)$-th polynomial we have:

a) if $\alpha_n > 1$ then $\lambda_{n+1} = \lambda_n + 1$ and $\mu_{n+1} = \mu_n$

b) if $\alpha_n < 1$ then $\lambda_{n+1} = \mu_n$ and $\mu_{n+1} = \lambda_n + 1$

In particular, if all the α_n are less than 1, all the polynomials have all their zeros inside the unit circle. This generalize Geronimus's result. The introduction of the parameters α_k is justified in order that they are the squares of the modulus of the a_k parameters used by this author.

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ORTHOGONAL POLYNOMIALS AND SUBNORMALITY OF RELATED SHIFT OPERATORS

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1. Bounded subnormal operators have been extensively studied for many recent years, mostly in connection with Function Theory (in a Complex Variable). The study of unbounded subnormals has been originated in [1]. Though in unbounded case a number of severe difficulties appear, the advantage is that one can take into account also differential operator.

In [2] Dr. Jan Stochel and the present author exhibited the fact that a simple differential operator (known sometimes as the creation operator) is subnormal. In a subsequent paper [3] we proposed another way of proving this fact, based on the observation that the operator in question may be regarded as a (weighted) shift operator related to the system of the Hermite polynomials. Here our goal is to show that, using the approach of [3], one can establish subnormality of some other operators associated with classical orthogonal polynomials, namely the Laguerre polynomials and the Charlier polynomials.

2. Recall that an operator S acting in a Hilbert space H, with domain $D(S)$, is said to be <u>subnormal</u> if there is another Hilbert space K, $K \supset H$ and a normal operator N in K such that

$$Sf = Nf, \quad f \in D(S) \subset D(N).$$

Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis in H. An operator S, with domain $D(S) = \text{span}\,\{e_n\}_{n=0}^{\infty}$ is said to be a <u>weighted shift</u> if $Se_n \in (\mathbb{C} \setminus \{0\})e_{n+1}$, $n=0,1,\ldots$.

In [3] we have proved the following

<u>Theorem</u>. Suppose S is a weighted shift operator. Then S is subnormal if and only if $\{\|S^n\|\}_{n=0}^{\infty}$ is a Stieltjes moment sequence.

This theorem suggests to use orthogonal polynomials in order to prove subnormality of some shift operators related to them. We wish to present three examples of such operators.

a) The Hermite polynomials and the related shift operator [2,3]. The Hermite polynomials

$$H_n(x) = e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad , \quad n = 1,2,\ldots, \quad H_o(x) \equiv 1 \ .$$

satisfy the following relation

$$H_{n+1}(x) = -2x\, H_n(x) + H_n'(x) \quad .$$

Thus the Hermite functions

$$h_n(x) = (-1)^n 2^{-n/2} (n!)^{-1}\, \pi^{-1/4}\, e^{-x^2/2}\, H_n(x)$$

which form an orthonormal basis in $L^2(-\infty, +\infty)$, define the weighted shift operator

$$S = 2^{-1/2}(x - \frac{d}{dx})$$

such that

$$||S^n h_o||^2 = n!$$

This means that S is subnormal.

b) The Laguerre polynomials and the related shift operator. The Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\ x^{-\alpha} e^{x} \frac{d^n}{dx^n} [x^{n+\alpha}\ e^{-x}], \quad n = 1,2,\ldots, \ L_o^{(\alpha)}=1 \ , \ \alpha>-1$$

satisfy the following relation

$$(n+1) L_{n+1}^{(\alpha)}(x) = [-x \frac{d^2}{dx^2} - (\alpha+1-2x) \frac{d}{dx}] L_n^{(\alpha)}(x)$$

So "the Laguerre functions"

$$l_n^{(\alpha)}(x) = (\frac{n!}{\Gamma(n+\alpha+1)})^{1/2} x^{\alpha/2}\ e^{-x/2}\ L_n^{(\alpha)}(x)$$

which form an orthonormal basis in $L^2(0,+\infty)$, give a rise to the family $(\alpha>-1)$ of weighted shift operators

$$S = -x \frac{d^2}{dx^2} + (x-1) \frac{d}{dx} + \frac{-x^2+2x+\alpha^2}{4x}$$

such that

$$||S^n l_o^{(\alpha)}||^2 = n!\ \Gamma(n+\alpha) \ .$$

Consequently S is subnormal.

c) The Charlier polynomials and the related shift operator.

The Charlier polynomials $C_n^{(\alpha)}$ are determined by the generating function

$$e^{-\alpha w}(1+w)^x = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x)\, w^n\, (n!)^{-1} \quad , \quad \alpha \neq 0.$$

They satisfy the following relation

$$C_{n+1}^{(\alpha)}(x+1) = (x+1)C_n^{(\alpha)}(x) - \alpha C_n^{(\alpha)}(x+1) \ .$$

Defining the "Charlier sequences"

$$c_n^{(\alpha)}(x) = (\alpha^n n!)^{-1/2} e^{-\alpha/2} \alpha^{x/2} (x!)^{-1/2} C_n^{(\alpha)}(x) \quad , \quad x=0,1,\ldots$$

we get

$$\lambda_n c_{n+1}^{(\alpha)}(x+1) = (x+1)^{1/2} c_n^{(\alpha)}(x) - \alpha^{1/2} c_n^{(\alpha)}(x+1) \quad ,$$

where $\lambda_n = (n+1)^{1/2}$. This allows us to define the finite difference operator (depending on $\alpha \neq 0$)

$$(Sf)(x+1) = (x+1)^{1/2} f(x) - \alpha^{1/2} f(x+1) \quad , \quad x=0,1,\ldots ,$$

for $f \in D(S) = \text{span}\{ c_n^{(\alpha)}(.) \}_{n=0}^{\infty}$ (the definition of $(Sf)(0)$ requires some comment: since $\{ c_n^{(\alpha)}(.) \}_{n=0}^{\infty}$ is a basis in l_+^2, $Sf(0)$ is uniquely determined and the operator itself is well defined).

Since $\| S^n c_o^{(\alpha)} \| = n!$, S is subnormal in l_+^2 .

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A GENERALIZED INVERSION FORMULA FOR THE CONTINUOUS JACOBI TRANSFORM WHEN $\alpha+\beta+1$ IS AN INTEGER

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1. Introduction

The continuous Legendre transform and its inverse were studied by Butzer, Stens and Wehrens [1]. This was extended to Jacobi transforms for $\alpha + \beta = 0$ by Deeba and Koh [3]. Unfortunately this does not include many special cases of interest.

In this work we consider a larger number of other cases in which $\alpha + \beta + 1$ is a positive integer. For the standard normalization an inversion is given in terms of a kernel defined by a series. We also introduce a renormalized transform whose range is a set of entire functions. Its inverse is then given in closed form.

2. Preliminaries:

In this section we recall some of the basic background material necessary for our investigation.

2.1 Jacobi Functions and Transforms

For any real numbers a, b and c with $c \neq 0, -1, -2, \ldots$ the hypergeometric function $F(a,b;c;z) = {}_2F_1(a,b;c;z)$ is given by

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k , \qquad |z| < 1 \tag{2.1}$$

where the series converges at $z = -1$ and $z = 1$ provided that $c - a - b + 1 > 0$ and $c - a - b > 0$ respectively.

The Jacobi function $P_\lambda^{(\alpha,\beta)}(x)$ of the first kind is defined by

$$P_\lambda^{(\alpha,\beta)}(x) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\lambda+1)} F(-\lambda, \lambda+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}) , x \in (-1,1] \tag{2.2}$$

where $\alpha, \beta > -1$, $\lambda \in \mathbb{R}$ and $\lambda + \alpha + 1 \neq 0, -1, -2, \ldots$. Since

$$P_{-\lambda}^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha-\lambda+1)\Gamma(\lambda-\alpha-\beta)}{\Gamma(1-\lambda)\Gamma(\lambda-\beta)} P_{\lambda-\alpha-\beta-1}^{(\alpha,\beta)}(x)$$

we may restrict ourselves to the case $\lambda \geqq -(\alpha+\beta+1)/2$. The function $P_\lambda^{(\alpha,\beta)}(x)$ satisfies the differential equation

$$(1-x^2)y'' + \{\beta-\alpha-(\alpha+\beta+2)x\}y' + \lambda(\lambda+\alpha+\beta+1)y = 0. \tag{2.3}$$

Let

$$L_x^{(\alpha,\beta)} = (1-x^2)\frac{d^2}{dx^2} + \{\beta-\alpha-(\alpha+\beta+2)x\}\frac{d}{dx} \tag{2.4}$$

be the differential operator associated with this equation. Then (2.3) becomes

$$L_x^{(\alpha,\beta)} P_x^{(\alpha,\beta)}(x) = -\lambda(\lambda+\alpha+\beta+1) P_\lambda^{(\alpha,\beta)}(x) \tag{2.5}$$

For integer values of λ , $P_\lambda^{(\alpha,\beta)}(x)$ reduces to the usual Jacobi polynomial as defined in [9] .

It can be shown [3] that for $\lambda,\nu \geqq -(\frac{\alpha+\beta+1}{2})$, $\lambda \neq \nu$, $\lambda \neq -(\nu+\alpha+\beta+1)$ and $-1 < \alpha,\beta$

$$\frac{1}{2^{\alpha+\beta+1}}\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_\lambda^{(\alpha,\beta)}(x) P_\nu^{(\beta,\alpha)}(-x)\,dx =$$

$$= \frac{\Gamma(\lambda+\alpha+1)\Gamma(\nu+\beta+1)}{\pi(\lambda-\nu)(\lambda+\nu+\alpha+\beta+1)}\left\{\frac{\sin\pi\lambda}{\Gamma(\nu+1)\Gamma(\lambda+\alpha+\beta+1)} - \frac{\sin\pi\nu}{\Gamma(\lambda+1)\Gamma(\nu+\alpha+\beta+1)}\right\}. \tag{2.6}$$

If we denote the Jacobi polynomial of degree n by $P_n^{(\alpha,\beta)}(x)$ then

$$\frac{1}{2^{\alpha+\beta+1}}\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)\,dx = \delta_{nm}h_n^{(\alpha,\beta)} \tag{2.7}$$

where

$$h_n^{(\alpha,\beta)} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)} . \tag{2.8}$$

Since $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ it follows that

$$\hat{P}_\lambda^{(\alpha,\beta)}(n) = \frac{1}{2^{\alpha+\beta+1}}\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_\lambda^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)\,dx =$$

$$= \frac{(-1)^n\Gamma(\lambda+\alpha+1)\Gamma(n+\beta+1)\sin\pi\lambda}{\pi(\lambda-n)(\lambda+n+\alpha+\beta+1)n!\Gamma(\lambda+\alpha+\beta+1)}, \quad \lambda \neq n \tag{2.9}$$

and hence

$$P_\lambda^{(\alpha,\beta)}(x) = \sum_{n=0}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{P}_\lambda^{(\alpha,\beta)}(n) P_n^{(\alpha,\beta)}(x), \quad x\in(-1,1]. \quad (2.10)$$

where the series converges

i) absolutely and uniformly on any compact subset of $(-1,1)$ if $-1 < \beta < 1/2$.

ii) in $L^2((1-x)^\alpha(1+x)^\beta)$ if $-1 < \beta < 1$.

iii) in the sense of generalized functions for any $-1 < \beta$.

Here we have used the fact that

$$P_\lambda^{(\alpha,\beta)}(x) = O\left(\frac{1}{\sqrt{\lambda}}\right) \quad \text{as } \lambda \to \infty \text{ uniformly in } x\in[a,b]\subset(-1,1) \quad (2.11)$$

We also note that

$$P_\lambda^{(\alpha,\beta)}(x) = O(\lambda^{\max(\alpha,\beta)}) \quad \text{as } \lambda \to \infty \quad (2.12)$$

Let $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ and $f(x) \in L^p\{w^{(\alpha,\beta)}(x)\}$, $p \geq 1$. Then the discrete Jacobi transform $\hat{f}^{(\alpha,\beta)}(n)$ of $f(x)$ is defined by

$$\hat{f}^{(\alpha,\beta)}(n) = \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} (1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) f(x)\, dx \quad (2.13)$$

and the series expansion of $f(x)$ in terms of the Jacobi polynomials is given by

$$f(x) \sim \sum_{n=0}^{\infty} \frac{1}{h_n} \hat{f}^{(\alpha,\beta)}(n) P_n^{(\alpha,\beta)}(x), \quad x \in (-1,1). \quad (2.14)$$

Analogously, if $f(x) \in L^1\{w^{(\alpha,\beta)}(x)\}$, then the continuous Jacobi transform $\hat{f}^{(\alpha,\beta)}(\lambda)$ of $f(x)$ will be defined by

$$\hat{f}^{(\alpha,\beta)}(\lambda) = \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} (1-x)^\alpha(1+x)^\beta P_\lambda^{(\alpha,\beta)}(x) f(x)\, dx; \lambda > -\frac{\alpha+\beta+1}{2} \quad (2.15)$$

The following proposition gives a series representation for the continuous Jacobi transform $\hat{f}^{(\alpha,\beta)}(\lambda)$:

Proposition 2.1. Let $f(x)$ be p times differentiable with support in $(-1,1)$, $p > 2$; then

$$\hat{f}^{(\alpha,\beta)}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n)\ \hat{P}_\lambda^{(\alpha,\beta)}(n) \qquad (2.16)$$

where the series converges uniformly on any compact subset of $[0,\infty)$ for $-1 < \beta < p - \frac{1}{2}$

Proof: By substituting the uniformly convergent series (2.14) in (2.15) , we obtain

$$\hat{f}^{(\alpha,\beta)}(\lambda) = \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} \sum_{n=0}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n) P_n^{(\alpha,\beta)}(x) P_\lambda^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)\,dx$$

$$= \frac{1}{2^{\alpha+\beta+1}} \sum_{n=0}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n) \cdot \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_\lambda^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)\ dx =$$

$$= \sum_{n=0}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n)\ \hat{P}_\lambda^{(\alpha,\beta)}(n)$$

by (2.9) . Interchanging the summation and the integration series in (2.16) converges uniformly on any compact subset of $[0,\infty)$ since

$$1/h_n^{(\alpha,\beta)} = O(n), \quad \hat{f}^{(\alpha,\beta)}(n) = O(n^{-p+\frac{1}{2}}) \text{ and } \hat{P}_\lambda^{(\alpha,\beta)}(n) = O(n^{\beta-1})$$

uniformly for $\lambda \geqq 0$ as $n \to \infty$.

Let $q = (\alpha+\beta+1)/2$ and define

$$d\sigma(\lambda) = \begin{cases} \dfrac{\Gamma^2(\lambda+q)\lambda \sin \pi\lambda\ d\lambda}{\Gamma(\lambda+\alpha-q+1)\Gamma(\lambda+\beta-q+1)} & \text{if } q \text{ is a half integer} \\[2ex] \dfrac{\Gamma^2(\lambda+q) \cos \pi\lambda\ d\lambda}{\Gamma(\lambda+\alpha-q+1)\Gamma(\lambda+\beta-q+1)} & \text{if } q \text{ is an integer} \end{cases} \qquad (2.17)$$

We shall show in the next section that $\{\hat{P}_\lambda^{(\alpha,\beta)}(n)\}$ and $\{\hat{P}_\lambda^{(\beta,\alpha)}(n)\}$ form a bi-orthogonal series with respect to $d\sigma$.

2.2 An associated orthonormal system

We begin with the complete orthonormal system on $[-\pi,\pi]$ given by

$\{\frac{1}{\sqrt{\pi}}\cos(n+\frac{1}{2})w,\ \frac{1}{\sqrt{\pi}}\sin(n+\frac{1}{2})w\}_{n=0}^{\infty}$. Their Fourier transforms will be denoted by c_n^0 and r_n^0 respectively; i.e.,

$$c_n^0(\lambda) = \frac{1}{\sqrt{2}\,\pi}\int_{-\pi}^{\pi} e^{iw\lambda}\cos(n+\tfrac{1}{2})w\,dw \qquad n = 0,1,2,...$$

$$r_n^0(\lambda) = \frac{1}{\sqrt{2}\,\pi}\int_{-\pi}^{\pi} e^{iw\lambda}\sin(n+\tfrac{1}{2})w\,dw \qquad (2.18)$$

By Plancherel's identity $\{c_n^0, r_n^0\}$ are a complete orthonormal system in $L^2(\mathbb{R})$. This system will be used with $q = (\alpha+\beta+1)/2$ a half odd integer. For q an integer we shall use the system

$$\left.\begin{matrix} c_n^e(\lambda) \\ r_n^e(\lambda)\end{matrix}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{iw\lambda}\left\{\begin{matrix}\cos nw \\ \sin nw\end{matrix}\right\} dw\ , \qquad n = 1,2,... \qquad (2.19)$$

$$c_0^e(\lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iw\lambda}\,dw$$

which again is clearly an orthonormal system. We shall show that these systems are related to $\hat{P}_\lambda^{(\alpha,\beta)}(n)$ given in (2.9). It may be expres sed as

$$\hat{P}_{\lambda-q}^{(\alpha,\beta)}(n) = \frac{\sin\pi(\lambda-n-q)(2n+2q)}{\pi(\lambda^2-(n+q)^2)\ 2(n+q)}\ \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)}\ \frac{\Gamma(\lambda+\alpha+1-q)}{\Gamma(\lambda+1-q)} \qquad (2.20)$$

By a straightforward calculation we see that

$$s_n^q(\lambda-q) = \frac{\sin\pi(\lambda-n-q)}{\pi(\lambda^2-(n+q)^2)}\ (2n+2q) = \sqrt{2}\left\{\begin{matrix} c_{n+[q]}^0(\lambda)\ ,\ q \text{ half-integer} \\ \\ r_{n+q}^e(\lambda)\ ,\ q \text{ integer}\end{matrix}\right. \qquad (2.21)$$

<u>Proposition 2.2.</u> The functions given by

$$\frac{\Gamma(\lambda+q)}{\Gamma(\lambda+\alpha+1-q)}\ 2(n+q)\ \frac{\Gamma(n+1)}{\Gamma(n+\beta+1)}\ \hat{P}_{\lambda-q}^{(\alpha,\beta)}(n) = s_n^q(\lambda-q) \qquad (2.22)$$

are orthonormal on $(0,\infty)$ with respect to Euclidean measure.

This is a consequence of the orthonormality of $s_n^q(\lambda-q)/\sqrt{2}$ on $(-\infty,\infty)$ together with its evenness when q is a half integer and oddness when q is an integer.

<u>Proposition 2.3.</u> The functions given by (2.22) satisfy

$$\int_0^{\infty} s_n^q(\lambda-q)\ s_m^q(\lambda-q)\ \frac{\Gamma(\lambda+\alpha+1-q)}{\Gamma^2(\lambda+q)}\ \Gamma(\lambda+\beta+1-q)\,d\sigma(\lambda) = \frac{(-1)^{m+[q]}}{2}\ \gamma_{q,m}\delta_{mn} \qquad (2.23)$$

where $\gamma_{q,m} = \begin{cases} 1 & , q \text{ an integer} \\ (m+q), & q \text{ a half-integer} \end{cases}$, and $d\sigma$ is the measure given by (2.17).

3. An inverse transform given by a series.

In this section we use the biorthogonality of $\{ \hat{P}^{(\alpha,\beta)}_{\lambda-q}(n)\}$ with $\{ \hat{P}^{(\beta,\alpha)}_{\lambda-q}(n)\}$ to find an inverse transform to the continuous Jacobi transform when $\alpha+\beta$ is an integer ≥ 0. The expression in Proposition 2.3 may be written as

$$\int_0^{\infty} \hat{P}^{(\alpha,\beta)}_{\lambda-q}(n)\ \hat{P}^{(\beta,\alpha)}_{\lambda-q}(m)\ d\sigma(\lambda) = \frac{(-1)^{n+[q]}(n+1)_{2q-1}}{2(n+q)} h_n^{(\alpha,\beta)} \delta_{nm}\ \gamma_{q,n} \tag{3.1}$$

where $\gamma_{q,n} = 1$ if q is an integer and $\gamma_{q,n} = (n+q)$ if q is a half integer. Let

$$R(x,\lambda) = (-1)^{[q]} \sum_{n=0}^{\infty} \frac{2(n+q)}{\gamma_{q,n} h_n^{(\alpha,\beta)} (n+1)_{2q-1}} \hat{P}^{(\beta,\alpha)}_{\lambda-q}(n)\ \hat{P}^{(\beta,\alpha)}_{n}(-x) \tag{3.2}$$

The series defining $R(x,\lambda)$ converges absolutely and uniformly on any compact subset of $(-1,1) \times [0,\infty)$ provided that

i) $\beta > -\frac{1}{2}$, for $q = \frac{1}{2}, \frac{3}{2}, \ldots$

ii) $\beta > \frac{1}{2}$, for $q = 1,2,3,\ldots$

Moreover, it is dominated by $(\lambda+1)^{-\alpha}$. We shall always assume that this is indeed the case. In the next theorem we derive an inversion formula for the continuous Jacobi transform given by (2.15).

<u>Theorem 3.1.</u> Let $f(x)$ be such that its continuous Jacobi transform has the representation (2.16) and is dominated by $O(\lambda^{-\alpha-2})$. Then

$$f(x) = \int_0^{\infty} \hat{f}^{(\alpha,\beta)}(\lambda-q)\ R(x,\lambda)\ d\sigma(\lambda) \tag{3.3}$$

where $R(x,\lambda)$ and $d\sigma(\lambda)$ are chosen from (3.2) and (2.18) according to whether q is half a positive integer or a positive integer.

4. A renormalized Jacobi transform

The standard normalization (2.2) of the Jacobi function does not give us an entire function in λ unless α is an integer. This may be rectified by adopting a normalization similar to that of the Gegenbauer polynomials when $\alpha = \beta$. Accordingly we denote φ_λ by

$$\varphi_\lambda^{(\alpha,\beta)}(x) = \binom{\lambda+\alpha+\beta}{\beta} P_\lambda^{(\alpha,\beta)}(x) = \frac{\Gamma(\lambda+2q)}{\Gamma(\beta+1)} \frac{1}{\Gamma(\lambda+1)(\alpha+1)} F(-\lambda,\lambda+2q;\alpha+1;\frac{1-x}{2}) . \tag{4.1}$$

where $2q = \alpha + \beta + 1$ is an integer as before. Then $\varphi_\lambda^{(\alpha,\beta)}$ as the product of a polynomial and an entire function is itself entire. Moreover, $\varphi_{\lambda-q}^{(\alpha,\beta)}(x)$ is either an even or odd function of λ about 0 according to where $2q$ is respectively odd or even.

$$F(\lambda) = F^{(\alpha,\beta)}(\lambda) = \frac{1}{2^{2q}} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta \varphi_\lambda^{(\alpha,\beta)}(x) f(x)\, dx =$$

$$= \binom{\lambda+2q-1}{\beta} \hat{f}^{(\alpha,\beta)}(\lambda) \tag{4.2}$$

will be the modified Jacobi transform.

Proposition 4.1. Let F be the modified Jacobi transform of a C^∞ function f with support in $(-1,1)$, $\alpha+\beta+1 = 2q$ an integer, $\beta > -1$. Then

(i) F is an entire function in λ

(ii) $F(-\lambda-q) = F(\lambda-q)(-1)^{2q+1}$

(iii) $$F(\lambda) = \sum_{n=0}^{\infty} F(n)\, s_n^q(\lambda) \tag{4.3}$$

where $s_n^q(\lambda)$ is the orthonormal sequence given by (2.21) and the convergence is uniform on compact subsets of $(-\infty,\infty)$.

(iv) $F(\lambda) = O(\lambda^{-p})$ as $\lambda \to \infty$ for all $p \geq 0$.

In the next section we shall devise another inverse given by a kernel with a closed form expression.

5. An inverse operator with a closed form kernel.

In order to derive the closed form we shall proceed inductively beginning with the two cases $q = 1/2$ and $q = 1$. In both cases the inverse operator is known or easy to derive.

We shall state the main result of this paper in the following theorem and provide no proof for it here. The details of the proof will be published somewhere else since they are too long to be given

Theorem 5.1. Let $f \in \mathcal{D}(-1,1)$ and let $F(\lambda)$ be its (α,β) Jacobi transform, given by (4.2) where $\alpha+\beta+1 = 2q$, a positive integer. Then the inverse transform is given by

$$f(x) = (-1)^q 4\Gamma(\alpha+1)\Gamma(\beta+1)\int_0^\infty F(\lambda-q)\varphi_{\lambda-q}^{(\beta,\alpha)}(-x)\frac{\cos\pi\lambda}{(\lambda^2-1)\ldots(\lambda^2-(q-1)^2)}\,d\lambda$$

when $2q$ is even and by

$$f(x) = (-1)^{q-\frac{1}{2}} 4\Gamma(\alpha+1)\Gamma(\beta+1)\int_0^\infty F(\lambda-q)\varphi_{\lambda-q}^{(\beta,\alpha)}(-x)\frac{\lambda\sin\pi\lambda}{(\lambda^2-(\frac{1}{2})^2)\ldots(\lambda^2-(q-1)^2)}\,d\lambda$$

when $2q$ is odd

where the integrals converge uniformly for x in interior intervals of $(-1,1)$.

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