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Groups with families of generalized normal subgroups

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To Manolo with affection in his 65th birthday

Introduction

In his treatment of the solvability of polynomial equations, *Évariste Galois* coined the term *group* and established a connection, now known as *Galois theory*, between the nascent Theory of Groups (formerly Theory of *Finite* Groups) and Field Theory, giving rise to one of the main historical sources of the Theory of Groups (being the others Number Theory and Geometry). In the well know Galois' correspondence mentioned above, Galois emphasized the fundamental role of some subgroups of the Galois group that are invariant under certain automorphisms, namely its normal subgroups. If G is a group (in its abstract form), we recall that a subgroup N of G is said to be a *normal* subgroup if it is invariant under *inner* automorphisms (G–invariant), that is $N^x := x^{-1}Nx = N$ for each $x \in N$. For example *every subgroup of an abelian group is normal.* In studying number fields (finite Galois extensions of the field $\mathbb{Q}$ of rational numbers), R. Dedekind [5] was able to determine the form of a non-abelian (finite) group with normal subgroups only (*Hamiltonian* groups), a result extended by R. Baer [1] to the general case.

Theorem 1 *Let G be a non-abelian group in which all subgroups are normal. Then $G = Q \times B \times D$, where Q is a copy of the quaternion of order 8, B is an elementary abelian 2–group and D is a periodic abelian group with all elements of odd order.*

Here, a group is said to be *periodic* if their elements have finite order, and *bounded* (or that has *finite exponent*) if these orders are bounded. Opposite to this, *a torsion-free group* is a group with no non-trivial elements of finite order.

We recall that a group G is called a *Dedekind group* if every subgroup of G is normal. By Theorem 1, the class of Dedekind groups is the union of the class of Hamiltonian groups and that of abelian groups.

It is well known that *being a normal subgroup* is not a transitive property, and to make that concept transitive, it is introduced the concept of subnormality. A subgroup H of a group G is said to be *subnormal* if there is a finite chain of intermediate subgroups

$$H = H_0 \leq H_1 \leq \cdots H_i \leq H_{i+1} \leq \cdots H_n = G$$

such that H_i is normal in H_{i+1} for every $0 \leq i \leq n-1$. This is a fairly generalization of the concept of normal subgroup. A natural extension of these concepts is that of an ascendant subgroup. H is said to be an *ascendant subgroup of* G if there exists an *ascending series* from H to G, that is, a chain of normal subgroups well-ordered by inclusion and indexed by the corresponding ordinal numbers

$$H = H_0 \leq H_1 \leq \ \cdots \ \leq H_\alpha \leq H_{\alpha+1} \leq \ \cdots \ \leq H_\gamma = G$$

with the additional stipulation that for each limit ordinal λ, H_λ is the union of all H_β, $\beta < \lambda$. The most easy way of realizing an ascending series is constructing a Prüfer group. If p is a prime, *the Prüfer p–group*

$$C_{p^\infty} = \langle x_1, \cdots, x_n \cdots \mid x_1^p = 1,\ x_n^p = x_{n-1}(n > 1)\rangle$$

is an infinite abelian group whose proper subgroups are all finite. In fact, the subgroups of C_{p^∞} are the terms of the ascending series

$$\langle 1\rangle \leq C_1 \leq \cdots \leq C_n \leq \cdots \bigcup_{n\geq 1} C_n = C_{p^\infty},$$

where $C_n = \langle x_n\rangle$ for every $n \geq 1$. By the way, this an obvious example of a *locally finite group*, a group whose finitely generated subgroups are finite.

If $x, y \in G$, then $xy = yx(x^{-1}y^{-1}xy)$, and then the commutativity of x and y is measured by the so called *commutator* of x and y, namely $[x, y] := x^{-1}y^{-1}xy$, because we immediately have that $xy = yx \Leftrightarrow [x, y] = 1$. If $H, K \leq G$ and $S \subseteq G$, these considerations lead to the construction of the subgroups of G,

$$[H, K] = \langle [x, y] \mid x \in H, y \in K\rangle \text{ and } C_G(S) = \{x \in G \mid [x, y] = 1 \text{ for all } y \in S\}.$$

The most important cases are $[H, G]$ and $\zeta(G) = C_G(G)$, which are called *the commutator subgroup* of H by G and *the center* of G, respectively. Clearly G is abelian if and only if $[G, G] = \langle 1\rangle$ if and only if $\zeta(G) = G$. Roughly speaking, we could say that we may construct generalizations of an abelian group making trivial commutators of higher weight or stabilizing the natural pre-images of the subsequent centers. By definition, *the upper central series of* G is the ascending chain of subgroups

$$\langle 1\rangle = \zeta_0(G) \leq \zeta_1(G) \leq \cdots \zeta_\alpha(G) \leq \zeta_{\alpha+1}(G) \leq \cdots$$

given by $\zeta_{i+1}(G)/\zeta_i(G) = \zeta(G/\zeta_i(G)),\ i \geq 0$. Note that $\zeta_1(G) = Z(G)$. On the other hand *the lower central series of* G is the descending chain of subgroups

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_\alpha(G) \geq \gamma_{\alpha+1}(G) \geq \cdots$$

given by $\gamma_{i+1}(G) = [\gamma_i(G), G],\ i \geq 0$. Note that $\gamma_2(G) = [G, G]$. A group G is said to be *nilpotent* if there is some $c \geq 0$ satisfying one of the following equivalent conditions: (i) $\zeta_c(G) = G$; and (ii) $\gamma_{c+1}(G) = \langle 1\rangle$. More generally G is said to be *hypercentral* if there exists an ordinal α such that $\zeta_\alpha(G) = G$. Finally *the derived series of* G is the descending chain of subgroups

$$G = G^{(0} \geq\ \cdots G^{(n} \geq \cdots$$

given by $G^{(i+1} = [G^{(i}, G^{(i}],\ i \geq 0$. Here also $G^{(1} = [G, G]$. The group G is said to be *soluble* if there is some $d \geq 0$ such that $G^{(d} = \langle 1\rangle$. It is very easy to see that a nilpotent group is soluble although the converse is not true. Finite soluble groups were fundamental in the Galois' characterization of the solvability of polynomial equations by radicals.

The next result is standard inside the Theory of Groups and is similar to Theorem 1. It characterizes *nilpotent* groups in the finite case.

Theorem 2(W. Burnside) *For a finite group G the following conditions are equivalent:*

(1) G is nilpotent;

(2) Every subgroup of G is subnormal; and

(3) If H is a proper subgroup of G then H is properly contained in its normalizer (the largest subgroup of G in which H is normal) $N_G(H) = \{x \in G \mid H^x = H\}$.

It is worth mentioning that the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) of Theorem 2 hold for arbitrary groups though the equivalence is false in general and gives rise to several classes of *generalized nilpotent groups*. The falsity holds for infinite groups as the following example shows. If G is a hypercentral group, it is very easy to show that every subgroup of G is ascendant. However if P is a Prüfer 2–group, we construct *the infinite dihedral group*,

$$D = \langle P, y \mid y^2 = 1, x^y = x^{-1} \text{ para todo } x \in P\rangle.$$

The group D is hypercentral but the subgroup $\langle y\rangle$ is not subnormal.

The aim of this survey paper is to review some families of subgroups that generalize normal subgroups as well as the classes of groups involved.

1 Subnormal subgroups

We begin stating a result that locates the groups under consideration.

Theorem 1.1 (A. I. Maltsev [16]). *A hypercental group is locally nilpotent, that is their finitely generated subgroups are nilpotent.*

A group G is said to satisfy *the normalizer condition* (or *G is an N–group*) if $H \neq N_G(H)$ for each proper subgroup H (see Theorem 2). Since every proper ascendant subgroup is properly contained in its normalizer, G is an N–group if and only if every subgroup of G is ascendant.

Theorem 1.2 (B. I. Plotkin [20]). *A group whose subgroups are ascendant is locally nilpotent.*

That is, an N–group is locally nilpotent. As we mentioned above, hypercentral groups are N–groups, but the converse is far from being true. In this setting one of the most celebrated examples in the Theory of Groups is given in the following result.

Theorem 1.3 (H. Heineken, I. J. Mohamed [11]). *There exists a p–group G, p a prime, satisfying the following properties:*

(1) G contains an elementary abelian normal p–subgroup A such that G/A is a Prüfer p–group;

(2) every proper subgroup of G is subnormal and nilpotent; and

(3) $\zeta(G) = \langle 1 \rangle$.

In relation with subnormal and ascendant subgroups of a group, some distinguished subgroups of the group can be constructed. That construction arises from the following results

Theorem 1.4 (R. Baer [2], K. W. Gruenberg [9]). *Let H and K be two finitely generated nilpotent subgroups of the group G. If H and K are subnormal (respectively ascendant), then so is $\langle H, K \rangle$.*

Let G be a group. Then the subgroup $B(G)$ generated by all subnormal cyclic subgroups of G is called *the Baer radical of G*, and the subgroup $Gr(G)$ generated by all ascendant cyclic subgroups of G is called *the Gruenberg radical of G*. Clearly, both subgroups are locally nilpotent normal subgroups of G. A group G is called *a Baer group* if $G = B(G)$ holds, and *a Gruenberg group* if $G = Gr(G)$.

We mention that every countable locally nilpotent group can be expressed as the union of an ascending chain of finitely generated nilpotent subgroups and therefore it is a Gruenberg group. But for uncountable groups it is not true, as the following result shows.

Theorem 1.5 (M. I. Kargapolov [13]). *There is a locally finite p–group that is not a Gruenberg group.*

Groups whose subgroups are subnormal were studied by many authors. In this area many interesting were obtained. We mention here only certain satisfactory structural results.

Theorem 1.6 (W. Möhres [18]). *A group whose subgroups are all subnormal is soluble.*

Theorem 1.7 (W. Möhres [17]). *A bounded group whose subgroups are all subnormal is nilpotent.*

Theorem 1.8 (W. Möhres [19]). *A hypercentral group whose subgroups are all subnormal is nilpotent.*

Theorem 1.9 (H. Smith [22]). *A torsion-free group in which all subgroups are subnormal is nilpotent.*

The last results are somewhat specific but we quote for their interest.

Theorem 1.10 (C. Casolo [3], H. Smith [22]). *Let G be a periodic group in which all subgroups are subnormal. If $\bigcap_\alpha \gamma_\alpha(G) = \langle 1 \rangle$, then G is nilpotent.*

Theorem 1.11 (C. Casolo [3]). *Let G be a periodic group in which all subgroups are subnormal. Then G contains a nilpotent normal subgroup H such that G/H is a divisible abelian group of finite special rank.*

2 Groups with many ascendant subgroups

The condition to be an ascendant subgroup is very wide than to be a subnormal subgroup. It is the main reason why the groups whose subgroups are all ascendant were not studied too well. There are many partial results about these groups, but in general its study is very difficult. There are quite a few general results on the structure of these groups. Some authors started consider groups in which the family of non-ascendant subgroups is not empty but it is very small. Some examples of this are

- S. N. Chernikov [4] who studied groups whose subgroups are either ascendant or finite.
- H. Heineken and L. A. Kurdachenko [10], who studied groups whose subgroups are either subnormal or finitely generated.
- H. Smith [24, 25], who studied groups whose subgroups are either subnormal or nilpotent.

as well as many others.

In studying the structure of groups whose subgroups belong to two types, there is an interesting approach that gives rise to obtain more information. Many important types of subgroups have their *antipodes*, i.e. subgroups that have diametrically opposite properties with respect to the original. For example, if H is a subgroup of G, then $H \leq N_G(H) \leq G$. If H is normal in G then $N_G(H) = G$. Therefore subgroups with the property $H = N_G(H)$ are in the antipodes to normal subgroups. These subgroups are called *self-normalizing*. As we mentioned above, subnormal subgroups and ascendant subgroups cannot be self-normalizing, so that we may conclude that self-normalizing subgroups are in the antipodes of subnormal and ascendant subgroups. Moreover we also apply Theorem 2 to deduce that *a nilpotent group has no proper self-normalizing subgroups.*

We also note that if H is a subgroup properly contained in its normalizer, that is $H \neq N_G(H)$, then $H^g = H$ for each $g \in N_G(H)$. If moreover $g \notin H$, then it is trivial that $g \notin \langle H, H^g \rangle$. A subgroup H of a group G is called *abnormal* if $g \in \langle H, H^g \rangle$ for every element $g \in G$. Therefore, we conclude that *a nilpotent group has no proper abnormal subgroups*, and we see that abnormal subgroups also are in the antipodes of normal, subnormal and ascendant subgroups. Thus in a certain sense we could say that the subgroups of a group that have some defining properties and those that have the antipodes with respect to these properties are located at the opposite ends of the group, while the other subgroups have some kind of mixed intermediate positions between these two ends. If a group G has few subgroups of mixed intermediate positions, it appears that the structure of G is more transparent. Therefore the following question can be naturally raised: *characterize groups whose subgroups have only a certain property and its antipode.* The first example appear considering nilpotent groups. Actually a nilpotent group only has subnormal subgroups and has neither abnormal subgroups nor self-normalizing subgroups. One of the first investigations carrying out this approach was the paper by A. Fattahi [8], where finite groups with normal and abnormal subgroups only were described. Later on, G. Ebert and S. Bauman [6] studied finite groups every subgroup of which is either subnormal or abnormal. Infinite groups with these properties and their generalizations were described by M. de Falco, L. A. Kurdachenko and I. Ya. Subbotin [7], and later L. A. Kurdachenko and H. Smith [15] studied groups whose subgroups are either subnormal or self-normalizing. We quote here the main results of these papers as well as latest results from which the previous are now a consequence.

Theorem 2.1 (L. A. Kurdachenko, J. Otal, A. Russo, G. Vincenzi [14]). *Let G be a locally finite group and suppose that G is not locally nilpotent. If every finitely generated non-ascendant subgroup of G is self-normalizing then there exist a prime p and a nilpotent normal subgroup A of G with no elements of order p such that the following conditions* *hold*

(1) $G = AP$ *and* $A \cap P = \langle 1 \rangle$, *where* $P = \langle x \rangle$ *is a cyclic* p*-subgroup and* $C_P(A) = \langle g^p \rangle$;

(2) *the commutator subgroup* $[G, G] = A$; *and*

(3) P *is self-centralized, that is* $C_G(P) = P$.

Conversely, if the group G *satisfies the conditions (1)-(3), then every subgroup of* G *is either ascendant or self-normalizing.*

This result can be put in the usual form of these results.

Corollary 2.2. *Let* G *be a locally finite group and suppose that* G *is not locally nilpotent. Then every non-ascendant subgroup of* G *is self-normalizing if and only if there exist a prime* p *and a nilpotent normal subgroup* A *of* G *with no elements of order* p *such that the following conditions hold*

(1) $G = AP$ *and* $A \cap P = \langle 1 \rangle$, *where* $P = \langle x \rangle$ *is a cyclic* p*-subgroup and* $C_P(A) = \langle g^p \rangle$;

(2) *the commutator subgroup* $[G, G] = A$; *and*

(3) P *is self-centralized, that is* $C_G(P) = P$.

As we mentioned above we are able to obtain previous results.

Corollary 2.3 (L. A. Kurdachenko, H. Smith [15])**.** *Let* G *be a locally finite group and suppose that* G *is not locally nilpotent. Then every non-subnormal subgroup of* G *is self-normalizing if and only if there exist a prime* p *and a nilpotent normal subgroup* A *of* G *with no elements of order* p *such that the following conditions hold*

(1) $G = AP$ *and* $A \cap P = \langle 1 \rangle$, *where* $P = \langle x \rangle$ *is a cyclic* p*-subgroup and* $C_P(A) = \langle g^p \rangle$;

(2) *the commutator subgroup* $[G, G] = A$; *and*

(3) P *is self-centralized, that is* $C_G(P) = P$.

Corollary 2.4 (L. A. Kurdachenko, H. Smith [15])**.** *Let* G *be a locally finite group and suppose that* G *is not a Dedekind group. Then every non-normal subgroup of* G *is self-normalizing if and only if there exist a prime* p *and an abelian normal subgroup* A *of* G *with no elements of order* p *such that the following conditions hold*

(1) $G = AP$ *and* $A \cap P = \langle 1 \rangle$, *where* $P = \langle x \rangle$ *is a cyclic* p*-subgroup and* $C_P(A) = \langle g^p \rangle$;

(2) *the commutator subgroup* $[G, G] = A$;

(3) P *is self-centralized, that is* $C_G(P) = P$; *and*

(4) every subgroup of A is G–invariant.

Corollary 2.5. *Let G be a locally finite group and suppose that G is not locally nilpotent. Then every non-ascendant subgroup of G is abnormal if and only if there exist a prime p and a nilpotent normal subgroup A of G with no elements of order p such that the following conditions hold*

(1) $G = AP$ and $A \cap P = \langle 1 \rangle$, where $P = \langle x \rangle$ is a cyclic p-subgroup and $C_P(A) = \langle g^p \rangle$;

(2) the commutator subgroup $[G, G] = A$; and

(3) P is self-centralized, that is $C_G(P) = P$.

Corollary 2.6 (M. de Falco, L. A. Kurdachenko, I. Ya. Subbotin [7])**.** *Let G be a locally finite group and suppose that G is not locally nilpotent. Then every non-subnormal subgroup of G is abnormal if and only if there exist a prime p and a nilpotent normal subgroup A of G with no elements of order p such that the following conditions hold*

(1) $G = AP$ and $A \cap P = \langle 1 \rangle$, where $P = \langle x \rangle$ is a cyclic p-subgroup and $C_P(A) = \langle g^p \rangle$;

(2) the commutator subgroup $[G, G] = A$; and

(3) P is self-centralized, that is $C_G(P) = P$.

Corollary 2.7. *Let G be a locally finite group and suppose that G is not a Dedekind group. Then every non-normal subgroup of G is abnormal if and only if there exist a prime p and an abelian normal subgroup A of G with no elements of order p such that the following conditions hold*

(1) $G = AP$ and $A \cap P = \langle 1 \rangle$, where $P = \langle x \rangle$ is a cyclic p-subgroup and $C_P(A) = \langle g^p \rangle$;

(2) the commutator subgroup $[G, G] = A$;

(3) P is self-centralized, that is $C_G(P) = P$; and

(4) every subgroup of A is G–invariant.

For non-periodic groups, we have

Theorem 2.8 (L. A. Kurdachenko, J. Otal, A. Russo, G. Vincenzi [14])**.** *Let G be a group and suppose that every finitely generated subgroup is either ascendant or self-normalizing. If G is not periodic, then G is a Gruenberg group.*

Corollary 2.9. *Let G be a group whose subgroups are either ascendant or self-normalizing. If G is not periodic then G is a Gruenberg group.*

We apply our study to hyperabelian groups, a class of generalized soluble groups. We recall that a group G is said *hyperabelian* if there exists an ascending series $\{H_\alpha\}_{\alpha<\gamma}$ from the trivial subgroup $\langle 1\rangle = H_0$ to the whole group $G = H_\gamma$ such that $H_{\alpha+1}/H_\alpha$ is abelian for every ordinal α.

Theorem 2.10 (L. A. Kurdachenko, J. Otal, A. Russo, G. Vincenzi [14]). *Let G be a hyperabelian group whose subgroups are either ascendant or self-normalizing. If G is locally nilpotent, then every subgroup of G is ascendant.*

Corollary 2.11. *Let G be a hyperabelian group whose subgroups are either ascendant or self-normalizing. If G is not periodic, then G is locally nilpotent. In particular, every subgroup of G is ascendant.*

With some extra work we find out a little more.

Corollary 2.12 (L. A. Kurdachenko, H. Smith [15]). *Let G be a group whose subgroups are either subnormal or self-normalizing. If G is locally nilpotent, then every subgroup of G is subnormal.*

Proof. If G is finitely generated, then G is nilpotent, and the proof is over. Suppose that G has no a finite set of generators. Let $F \leq G$ be a finitely generated subgroup of G. Pick $x \notin F$. Then $\langle x, F\rangle$ is nilpotent and so $F \neq N_{\langle x,F\rangle}(F)$. Thus F is subnormal. Let

$$F = F_0 \trianglelefteq F_1 \trianglelefteq \cdots \trianglelefteq F_n = G$$

be a subnormal series of F in G, that is $F_n = F^G$, $F_{n-1} = F^{F_n}$, . . . , $F_1 = F^{F_2}$, where $X^Y = \langle x^y = y^{-1}xy \mid x \in X, y \in Y\rangle$. Then F_1 is the product of the nilpotent normal subgroups F^x, $x \in F_2$, and it is known that F_1 is hyperabelian. By Theorem 2.10, F_1 has no self-normalizing subgroups and thus every subgroup of F_1 is subnormal. By Theorem 1.6, F_1 is soluble. Now F_2 is the product of the soluble normal subgroups F_1^x, $x \in F_3$, and it is known that F_2 is hyperabelian. As above, we see that F_3 is hyperabelian. Proceeding in this way, after finitely many steps we see that G is hyperabelian. By Theorem 2.10, G has no self-normalizing subgroups, and hence every subgroup of G is subnormal. $\square$

Corollary 2.13 (L. A. Kurdachenko, H. Smith [15]). *Let G be a group whose subgroups are either subnormal or self-normalizing. If G is not periodic, then every subgroup of G is subnormal. In particular, if G is torsion-free, then G is nilpotent.*

Corollary 2.14. *Let G be a group whose subgroups are either normal or self-normalizing. If G is not periodic, then G is abelian.*

Corollary 2.15. *Let G be a group whose subgroups are either subnormal or abnormal. If G is locally nilpotent, then every subgroup of G is subnormal.*

Corollary 2.16. *Let G be a group whose subgroups are either subnormal or abnormal. If G is not periodic, then every subgroup of G is subnormal.*

We mention that in [7] the latter was proved with the additional condition $G \neq [G, G]$.

Corollary 2.17. *A non-periodic group whose subgroups are either normal or abnormal is abelian.*

To finish this Section we mention the related result obtained in the paper [15].

Theorem 2.18 (L. A. Kurdachenko, H. Smith [15])**.** *Let G be a group whose subgroups are all subnormal. Suppose that there is $n \geq 1$ such that G is generated by elements of order at most n. Then G is nilpotent.*

3 Permutable subgroups

A subgroup H of a group G is said to be *permutable in G* (or quasi-normal in G), if $HK = KH$ for every subgroup K of G. This concept arises as a generalization of that of normal subgroup since it is immediate that a normal subgroup is permutable. The study of the properties of the permutable subgroups started a rather long time ago (see, for example [21]), where groups whose subgroups are all permutable were described. Before than giving that description we recall the following result that establish a certain connection among the concepts involved in this paper.

Theorem 3.1 (S. E. Stonehewer [26])**.** *A permutable subgroup of a group G is ascendant in G.*

In this case, by Theorems 3.1 and 1.2, G is locally nilpotent. Application of the results of a paper by K. Iwasawa [12] give us the following description.

Theorem 3.2. *Let G be a group whose subgroups are all permutable.*

(1) If G is periodic, then G can be expressed as a direct product

$$G = Dr_p G_p,$$

where G_p is the Sylow p–subgroup of G, and the following conditions holds:

(1A) if $p \neq 2$, then either G_p is abelian or $G_p = B_p\langle a_p\rangle$, where B_p is an abelian subgroup of exponent p^k, and there is a positive integer t such that $t = 1 + p^m$, for some $m \leq k \leq m + d$, where $p^d = |G_p/B_p|$, and $a_p^{-1} b a_p = b^t$ for all $b \in B_p$; and

(1B) if $p = 2$, then either G_p is a Dedekind group or $G_p = B_p\langle a_p\rangle$, where B_p is an abelian normal subgroup of exponent p^k, and there is a positive integer t such that $t = 1 + p^m$, where $p^d = |G_p/B_p|$, and $a_p^{-1}ba_p = b^t$ for all $b \in B_p$.

In both cases G_p is nilpotent, and bounded in the non-abelian case; and

(2) If G is not periodic, then

(2A) the set T consisting of all elements of G having finite order is a subgroup of G;

(2B) T and G/T are abelian;

(2C) every subgroup of T is G–invariant; and

(2D) if the abelian factor-group G/T has positive torsion-free rank, then G is abelian.

If G further is torsion-free, then G is abelian.

As a consequence of Theorem 2.1, we can now obtain the following result.

Theorem 3.3. *Let G be a locally finite group and suppose that G is not locally nilpotent. Then every non-permutable subgroup of G is self-normalizing if and only if there exist a prime p and an abelian normal subgroup A of G with no elements of order p such that the following conditions hold*

(1) $G = AP$ and $A \cap P = \langle 1\rangle$, where $P = \langle x\rangle$ is a cyclic p-subgroup and $C_P(A) = \langle g^p\rangle$;

(2) the commutator subgroup $[G, G] = A$;

(3) P is self-centralized, that is $C_G(P) = P$; and

(4) every subgroup of A is G–invariant.

Applying Theorem 2.10 and with some extra work, we are able to obtain.

Proposition 3.4. *Let G be a group whose subgroups are either permutable or self-normalizing. If G is locally nilpotent, then every subgroup of G is permutable.*

Proof. If G is finitely generated, then G is nilpotent, and the proof is over. Suppose that G has no a finite set of generators. Let $F \leq G$ be a finitely generated subgroup of G. Pick $x \notin F$. Then $\langle x, F\rangle$ is nilpotent and so $F \neq N_{\langle x,F\rangle}(F)$. Thus F is ascendant. Let

$$F = F_0 \trianglelefteq F_1 \trianglelefteq \cdots \trianglelefteq F_\alpha \leq F_{\alpha+1} \trianglelefteq \cdots F_\gamma = G$$

be an ascending series between F and G, and define $L_1 = \langle F^x \mid x \in F_2\rangle$. Any F^x is normal in F_1 if $x \in F_2$, and it readily follows that L_1 is hyperabelian. By Theorem 2.10, L_1 has no self-normalizing subgroups, and hence every subgroup of L_1 is permutable. By Theorem

3.2, L_1 is metabelian, that is an abelian extension of an abelian group. Proceeding in the same way we see that F_3 is metabelian, and applying transfinite induction we obtain that F^G is also metabelian. Hence G contains an abelian normal subgroup. Using transfinite induction again, we deduce that G itself is hyperabelian. By Theorem 2.10, G has no self-normalizing subgroups, and hence every subgroup of G is permutable, as required. □

Corollary 3.5. *Let G be a group whose subgroups are either permutable or self-normalizing. If G is not periodic, then every subgroup of G is permutable. If G further is torsion-free, then G is abelian.*

A subgroup H of a group G is said to be *contranormal* if $H^G = H$; it is clair that this concept defines subgroups that are some kind of antipodes of subnormal and normal subgroups. In the paper M. de Falco, L.A. Kurdachenko and I.Ya. Subbotin [7] the following description of groups whose subgroups are either subnormal or contranormal was obtained.

Theorem 3.6 (M. de Falco, L. A. Kurdachenko, I. Ya. Subbotin [7])**.** *Let G be a group such that $G \neq [G,G]$. Every non-subnormal subgroup of G is contranormal if and only one of the following holds.*

(1) Every subgroup of G is subnormal;

(2) G is a Baer group and has a normal subgroup H whose subgroups are subnormal such that G/H is a Prüfer p–group for some prime p; or

(3) $G = [G,G]P$, where $P = \langle g \rangle$ is a cyclic contranormal subgroup and there is a prime q such that every subgroup of $[G,G]\langle g^q \rangle$ is subnormal.

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