Monografías de la Real Academia de Ciencias de Zaragoza **33**, 225–232, (2010).

A note on typical sections of isotropic convex bodies

David Alonso-Gutiérrez, Jesús Bastero, Julio Bernués

IUMA, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

Abstract

Let $K \subset \mathbb{R}^n$ be a centrally symmetric isotropic convex body. We prove that for random $F \in G_{n,k}$, and k slowly growing to infinity, the central section $|F^\perp \cap K|_{n-k}^{1/k}$ is almost constant. A simple approach using standard concentration of measure arguments is given.

1 Introduction and notation

Let $K \subset \mathbb{R}^n$ be a symmetric convex body. We say K is isotropic if it is of volume 1 and there exists a constant $L_K > 0$ called isotropy constant of K such that $L_K^2 = \int_K \langle x, \theta \rangle^2 \, dx, \forall \theta \in S^{n-1}$.

Since the works of [H], [B] or [MP] we know of the close relation between the isotropy constant and the size of the central sections of K. It is well known that for any $1 \leq k \leq n$ there exist $c_1(k), c_2(k) > 0$ such that for every subspace $F \in G_{n,k}$ (the Grassmann space)

$$\frac{c_1(k)}{L_K} \leq |F^\perp \cap K|_{n-k}^{1/k} \leq \frac{c_2(k)}{L_K}$$

where $|\cdot|_m$ is the Lebesgue measure in the appropiate m dimensional space.

Well known estimates (see [H], [MP] and [Kl]) imply $c_1(k) \geq c_1$ and $c_2(k) \leq c_2 k^{1/4}$, where $c_1, c_2 > 0$ are absolute numerical constants. These bounds are the best ones known to be valid for *every* subspace $F \in G_{n,k}$.

For random sections, much better estimates are possible. The following result was proved in [ABBP],

There exist absolute constants $c_1, c_2, c_3 > 0$ with the following property: If K is an isotropic convex body in $\mathbb{R}^n$ and $1 \leqslant k \leqslant \sqrt{n}$ then, the set of subspaces $F \in G_{n,k}$ such that

$$\frac{c_1}{L_K} \leqslant |K \cap F^\perp|_{n-k}^{1/k} \leqslant \frac{c_2}{L_K}$$

has Haar probability $\geq 1 - e^{-c_3 \frac{n}{k}}$

In [EK] the authors prove a version of the central limit theorem for convex bodies. Its proof uses the strong concentration behavior of the Euclidean norm on K, [Kl2], and a delicate study of the marginal distribution of some intermediate measures, namely the convolution of the uniform measure on K with an independent gaussian vector. As a consequence of it it is easy to check that

For $\varepsilon = \frac{1}{n^{c_1}}$, $k \leq n^{c_2}$ the set of subspaces $F \in G_{n,k}$ such

$$\frac{1-\varepsilon}{\sqrt{2\pi}L_K} \leq |K \cap F^{\perp}|_{n-k}^{1/k} \leq \frac{1+\varepsilon}{\sqrt{2\pi}L_K}$$

has Haar probability $\geq 1 - c_3 e^{-n^{c_4}}$.

These two results are different: the second one gives better constants ($\sim \frac{1}{\sqrt{2\pi}}$) but a worse dependence on k and on the estimate of the Haar probability.

In this note we use a simpler approach to the question. Our final result is weaker in k than the one deduced from [EK] and it provides better estimate of the Haar probability. But the main advantadge, we think, is that the arguments are simpler and the tools used are of independent interest: First we estimate Lipschitz constant of the section function $F \in G_{n,k} \to |F^{\perp} \cap K|_{n-k}$ (Proposition 2.3), for $k = 1$ this was proved in [ABP]. Then we apply the concentration of measure phenomenum on $G_{n,k}$ (equipped with the right distance (Proposition 2.2)). In this way we measure the closeness between the section function and its expectation. Finally, by expressing this expectation as a marginal, we related it to the marginal of a gaussian distribution. For that final step, we unavoidably use the concentration of the Euclidean norm on K, [Kl2] in the version stated in [BB]. Our result is

Theorem 2.8. *Let $K \subset \mathbb{R}^n$ isotropic. For all $\varepsilon > 0$, $1 \leq k \leq \frac{c\varepsilon \log n}{(\log\log n)^2}$, the set A of subspaces $F \in G_{n,k}$ such that*

$$\frac{1-\varepsilon}{\sqrt{2\pi}L_K} \leq |K \cap F^{\perp}|_{n-k}^{1/k} \leq \frac{1+\varepsilon}{\sqrt{2\pi}L_K} \tag{1.1}$$

holds, has probability $\mu(A) \geq 1 - c_1 e^{-c_2 n^{0.9}}$.

In $\mathbb{R}^n$, $|\cdot|$ denotes the Euclidean norm and B_2^n the Euclidean ball. For any k-dimensional subspace $F \subset \mathbb{R}^n$ we denote $S_F = S^{n-1} \cap F$ and by P_F the orthogonal projection onto F. $G_{n,k}$ is the grassmaniann space of all k dimensional subspaces of $\mathbb{R}^n$ and its Haar probability is denoted by μ. For any linear map T from $\mathbb{R}^n$, $\|T\|$ denotes the operator norm and $\|T\|_{HS} := \Big(\sum_{j=1}^{n} |T(e_j)|^2\Big)^{1/2}$, for (any) orthonormal basis (e_j) of $\mathbb{R}^n$, its Hilbert-Schmidt norm.

2 The result

In the first part we estimate the Lipschitz constant of the function $F \to |F^\perp \cap K|_{n-k}$ and also review concentration inequalities with respect to several natural distances on $G_{n,k}$. We start with the latter.

The following lemma constructs a suitable orthonormal basis for two subspaces E and F and will be very useful for our purposes

Lemma 2.1 ([GM], Lemma 4.1) *Let $E, F \in G_{n,k}$ such that $F^\perp \cap E = 0$. Then there exists $u_1, \dots u_k$ orthonormal basis of E such that the family $v_1, \dots v_k$ given by $v_j = \frac{P_F(u_j)}{|P_F(u_j)|}$ is an orthonormal basis of F. In particular, $\langle u_j, v_i \rangle = |P_F(u_j)|\ \delta_i^j$.*

The space $G_{n,k}$ appears in the literature equipped with a number of different distances. In the following Proposition, we estimate the equivalence constants between them. It is probably folklore but we include for the reader's convenience. The fact that one can move from one distance to another will be useful while computing the Lipschitz constant and also when considering the concentration phenomena on $G_{n,k}$.

The elements of the orthogonal group $O(n)$ will be denoted by $U = (u_1 \dots u_n)$ so the columns (u_i) form an orthonormal basis in $\mathbb{R}^n$.

Proposition 2.2 *For $E, F \in G_{n,k}$ we consider the following distances*

$d_0(E,F) = \max\{d(x, S_F) \mid x \in S_E\}$, *$d$ is the euclidean distance.*

$d_1(E,F) = \inf\{\varepsilon > 0 \mid S_E \subset S_F + \varepsilon B_2^n, S_F \subset S_E + \varepsilon B_2^n\}$

$$d_2(E,F) = \inf\{\Big(\sum_{j=1}^k |u_j - v_j|^2\Big)^{1/2}\ E = \langle u_j \rangle_1^k,\ F = \langle v_j \rangle_1^k \text{ orthon. basis}\}$$

$$d_3(E,F) = \inf\{\Big(\sum_{j=1}^n |u_j - v_j|^2\Big)^{1/2}\ E = \langle u_j \rangle_1^k,\ F = \langle v_j \rangle_1^k \text{ orthon. basis}\}$$

$d_4(E,F) = \|P_E - P_F\|_{HS}$

$d_5(E,F) = \inf\{\|U - V\|_{HS} \mid U, V \in O(n), E = \langle u_1 \dots u_k \rangle, F = \langle v_1 \dots v_k \rangle\}$

$d_6(E,F) = \|P_E - P_F\|$

Then, d_2, d_3, d_4, d_5 are equivalent with numerical equivalence constants, $d_0 = d_1$, $d_1 \leq d_2 \leq \sqrt{2k}\ d_1$ and $d_6 \leq d_4 \leq \sqrt{2k} d_6$.

$d_0 = d_1$: d_1 is the Hausdorff distance between S_E and S_F which also reads

$$d_1(E,F) = \max\big\{\max_{x \in S_E} d(x, S_F), \max_{y \in S_F} d(y, S_E)\big\}$$

so $d_0 \leq d_1 \leq \sqrt{2}$ and it is enough to check that the two inner maxima are equal.

If $E \cap F^\perp \neq 0$ then $d_0(E,F) = \sqrt{2}$. Suppose $E \cap F^\perp = 0$. For any $x \in S_E, y \in S_F$, $|x - y|^2 = 2 - 2\langle x, y \rangle = 2 - 2\langle P_F(x), y \rangle$. So, $d^2(x, S_F) = 2 - 2 \sup_{y \in S_F} \langle P_F(x), y \rangle = 2 -$

$2|P_F(x)| = \big|x - \dfrac{P_F(x)}{|P_F(x)|}\big|^2$. Let $x_0 \in S_E$ that maximizes $d(x, S_F)$ on S_E or equivalently that minimizes $|P_F(x)|$. Denote $y_0 = \frac{P_F(x_0)}{|P_F(x_0)|}$ (observe $P_F(x_0) \neq 0$). By the arguments in [GM] Lemma 4.1, $P_F(x_0)$ is orthogonal to $E \cap x_0^{\perp}$ and so $P_E P_F(x_0)$ is parallel to x_0. Write $P_E(y_0) = \lambda x_0$. Then $\lambda = \langle P_E(y_0), x_0\rangle = \langle y_0, P_E(x_0)\rangle = |P_F(x_0)|$ and $\frac{P_E P_F(x_0)}{|P_E P_F(x_0)|} = x_0$. Therefore, $d(y_0, S_E) = d(x_0, S_F)$ and so $\max\{d(y, S_E) \mid y \in S_F\} \geq \max\{d(x, S_F) \mid x \in S_E\}$. Exchange E, F and equality follows.

$d_1 \leq d_2 \leq \sqrt{2k}\ d_1$: It is proved in [GM], Lemma 4.1.

$\frac{1}{\sqrt{2}} d_2 \leq d_4 \leq \sqrt{2}\ d_2$: Let $F^{\perp} \cap E := E_0$ and write the orthogonal decomposition $E = E_0 \oplus E_1$ with $E_1 \cap F^{\perp} = 0$. By Lemma 2.1, there exists an orthonormal basis in E_1, (u_j), such that $v_j = \frac{P_F(u_j)}{|P_F(u_j)|}$ is an orthonormal system in F. Now add vectors to complete an orthonormal basis in E (by adding vectors in E_0) and in F that we also denote as u_j and v_j. Trivially,

$$\|P_E - P_F\|_{HS}^2 \geq \sum_{j=1}^{k} |(P_E - P_F)(u_j)|^2$$

If $u_j \in E_1$ then, since $\langle u_j, v_j\rangle = |P_F(u_j)|$ (Lemma 2.1),

$$|(P_E - P_F)(u_j)|^2 = 1 - |P_F(u_j)|^2 \geq 1 - |P_F(u_j)| = \frac{1}{2}|u_j - v_j|^2$$

If $u_j \in E_0$ and $v_j \in F$ then $|(P_E - P_F)(u_j)|^2 = 1$. Also, since $\langle u_j, v_j\rangle = 0$ and so $|u_j - v_j|^2 = 2$.

For the second inequality, let $(u_j), (v_j)$ be orthonormal basis of $E, F \in G_{n,k}$ we write $P_E = \sum_{j=1}^k u_j \otimes u_j$ and $P_F = \sum_{i=1}^k v_i \otimes v_i$ and by definition

$$\|P_E - P_F\|_{HS}^2 = 2k - 2\sum_{i,j=1}^{k} \langle u_j, v_i\rangle^2 \leq 2\sum_{j=1}^{k}(1 - \langle u_j, v_j\rangle^2) \leq 2\sum_{j=1}^{k} |u_j - v_j|^2$$

since $1 - \langle u_j, v_j\rangle^2 \leq 2(1 - \langle u_j, v_j\rangle) = |u_j - v_j|^2$.

$d_2 \leq d_3 \leq \sqrt{5} d_2$: By definition $d_3^2(E,F) = d_2^2(E,F) + d_2^2(E^{\perp}, F^{\perp})$. Now, $d_2^2(E^{\perp}, F^{\perp}) \leq 2d_4^2(E^{\perp}, F^{\perp}) = 2d_4^2(E,F) \leq 4d_2^2(E,F)$. With similar arguments one proves $d_2 \leq d_5 \leq 3d_2$.

$d_6 \leq d_4 \leq \sqrt{2k} d_6$: For T linear $\|T\| \leq \|T\|_{HS} \leq \sqrt{\dim(T(\mathbb{R}^n))}\|T\|$. □

Proposition 2.3 *Let $K \subset \mathbb{R}^n$ isotropic. The function given by $G_{n,k} \ni E \to |E^{\perp} \cap K|_{n-k}$ is Lipschitz and for all $E, F \in G_{n,k}$ we have the estimate*

$$\big|\ |E^{\perp} \cap K|_{n-k} - |F^{\perp} \cap K|_{n-k}\big| \leq \frac{(c\mathcal{L}_k)^{2k}}{L_K^k}\|P_E - P_F\|_{HS}$$

where $\mathcal{L}_k := \sup\{L_M \mid M \subset \mathbb{R}^k,$ convex body isotropic$\}$.

In order to prove it, one more lemma will be used. An equivalent version of it for $k = 1$ is due to Busemann.

Lemma 2.4 ([B]) *If $K \subset \mathbb{R}^n$ is a convex body and $E \in G_{n,k}$ then the function given by*

$$E^{\perp} \ni \theta \to \|\theta\| := \frac{|\theta|}{|K \cap E(\theta)|_{k+1}}$$

is a norm on $E^{\perp}$ where $E(\theta) = E \oplus \langle \theta \rangle$.

Proof of Proposition 2.3. Suppose $F^{\perp} \cap E = 0$ and let $E = \langle u_1 \dots u_k \rangle, F = \langle v_1 \dots v_k \rangle$ be the orthonormal basis in Lemma 2.1. Denote $E_0^{\perp} = E^{\perp}$, $E_j^{\perp} = v_1^{\perp} \cap \dots \cap v_j^{\perp} \cap u_{j+1}^{\perp} \cap \dots \cap u_k^{\perp}$ and $E_k^{\perp} = F^{\perp}$. Then

$$\big|\, |E^{\perp} \cap K|_{n-k} - |F^{\perp} \cap K|_{n-k} \big| \leq \sum_{j=1}^{k} \big|\, |E_j^{\perp} \cap K|_{n-k} - |E_{j-1}^{\perp} \cap K|_{n-k} \big|$$

Let us estimate (say) the first summand. Set $\bar{E} = E^{\perp} \cap v_1^{\perp} = E_1^{\perp} \cap u_1^{\perp}$. Then, by Lemma 2.1, $E^{\perp} = \bar{E} \oplus P_{E^{\perp}}(v_1)$ and $E_1^{\perp} = \bar{E} \oplus P_{E_1^{\perp}}(u_1)$ so we can apply Lemma 2.4 to $\bar{E}$

$$\big|\, |E^{\perp} \cap K|_{n-k} - |E_1^{\perp} \cap K|_{n-k} \big| = \left| \frac{|P_{E^{\perp}}(v_1)|}{\|P_{E^{\perp}}(v_1)\|} - \frac{|P_{E_1^{\perp}}(u_1)|}{\|P_{E_1^{\perp}}(u_1)\|} \right|$$

and since $|P_{E_1}(u_1)| = |\langle u_1, v_1 \rangle| = |P_E(v_1)|$ and the triangle inequality,

$$\left| \frac{|P_{E^{\perp}}(v_1)|}{\|P_{E^{\perp}}(v_1)\|} - \frac{|P_{E_1^{\perp}}(u_1)|}{\|P_{E_1^{\perp}}(u_1)\|} \right| \leq \frac{|P_{E_1^{\perp}}(u_1)|}{\|P_{E_1^{\perp}}(u_1)\| \, \|P_{E^{\perp}}(v_1)\|} \|P_{E_1^{\perp}}(u_1) - P_{E^{\perp}}(v_1)\|$$

Finally, observe that $|P_{E_1^{\perp}}(u_1) - P_{E^{\perp}}(v_1)| = (1 - \langle u_1, v_1 \rangle)|u_1 - v_1|$ and apply Hensley's estimate [H] to conclude with

$$\big|\, |E^{\perp} \cap K|_{n-k} - |E_1^{\perp} \cap K|_{n-k} \big| \leq \frac{(1 - \langle u_1, v_1 \rangle)}{(1 - \langle u_1, v_1 \rangle^2)^{1/2}} |u_1 - v_1| \frac{(c\mathcal{L}_k)^{2k}}{L_K^k}$$

Since we can also suppose $\langle u_1, v_1 \rangle \geq 0$, the first quotient above is bounded by 1. So,

$$\big|\, |E^{\perp} \cap K|_{n-k} - |F^{\perp} \cap K|_{n-k} \big| \leq \sqrt{k} \Big(\sum_{j=1}^{k} |u_j - v_j|^2 \Big)^{1/2} \frac{(c\mathcal{L}_k)^{2k}}{L_K^k}$$

By the proof of Proposition 2.2, $\Big(\sum_{j=1}^{k} |u_j - v_j|^2 \Big)^{1/2} \leq \sqrt{2} \|P_E - P_F\|_{HS}$. In the general case, if $F^{\perp} \cap E := E_0$ then we can write $E = E_0 \oplus E_1$ with $E_1 \cap F^{\perp} = 0$. Choose an orthonormal basis of E_0 and proceed as in the previous case. □

We recall the following celebrated result by M. Gromov and V. Milman, see for instance [MS].

Theorem 2.5 (Concentration of measure) *There exist absolute constants $c_1, c_2 > 0$ such that* *i) For every* $A \subset G_{n,k}$ *and every* $\delta > 0$

$$\mu(A_\delta) \geq 1 - \frac{c_1}{\mu(A)} \exp\left(-c_2\delta^2 n\right)$$

where $A_\delta = \{E \in G_{n,k};\ \exists\ F \in A, d_5(E,F) \leq \delta\}$

ii) For $f\colon G_{n,k} \to \mathbb{R}$ *a Lipschitz function with Lipschitz constant* σ*, that is* $|f(E) - f(F)| \leq \sigma d_5(E,F)$*,*

$$\mu\left\{E \in G_{n,k}; |f(E) - \mathbb{E}\,(f)| \leq a\right\} \geq 1 - c_1 \exp\left(-\frac{c_2 a^2 n}{\sigma^2}\right) \qquad \forall\ a > 0$$

Remark 2.6 *If* $d, \tilde{d}$ *are two distances on* $G_{n,k}$ *such that* $d \leq M\tilde{d}$ *for some* $M > 0$ *then a concentration inequality for* $\tilde{d}$ *with bound* $c_1 \exp\left(-c_2\delta^2 n\right)$ *implies one for* d *with bound* $c_1 \exp\left(\frac{-c_2\delta^2 n}{M^2}\right)$*. Similarly for Lipschitz functions. It is then possible to state concentration inequalities for the different distances (Proposition 2.2) on* $G_{n,k}$*.*

The last main ingredient is the concentration of $|\cdot|$ on K

Theorem 2.7 *[Kl2]. Let* $K \subset \mathbb{R}^n$ *be an isotropic convex body. Then,*

$$|\{x \in K : \big||x| - \sqrt{n}L_K\big| > t\sqrt{n}L_K\}|_n \leq c\exp(-Cn^\alpha t^\beta) \tag{2.2}$$

for all $0 \leq t \leq 1$ *and* $\alpha = 0.33, \beta = 3.33$*.*

It was proved by [So] (with sharp exponents α and β) for normalized unit balls of $\ell_p^n, 1 \leq p$ and in full generality in [Kl2].

As an application of the results we show the announced

Theorem 2.8 *Let* $K \subset \mathbb{R}^n$ *isotropic. For all* $\varepsilon > 0$*,* $1 \leq k \leq \frac{\varepsilon \log n}{(\log\log n)^2}$*, the set* A *of subspaces* $E \in G_{n,k}$ *such that*

$$\frac{1-\varepsilon}{\sqrt{2\pi}L_K} \leq |E^\perp \cap K|_{n-k}^{1/k} \leq \frac{1+\varepsilon}{\sqrt{2\pi}L_K}$$

holds, has probability $\mu(A) \geq 1 - c_1 \exp -c_2 n^{0.9}$

Consider the function $f\colon G_{n,k} \to \mathbb{R}$, $f(E) = |E^\perp \cap K|_{n-k}$. By Proposition 2.3 and Theorem 2.5 we have

$$\mu\left\{E \in G_{n,k}; |f(E) - \mathbb{E}\,(f)| \leq \varepsilon\mathbb{E}\,(f)\right\} \geq 1 - c_1 \exp\left(-\frac{c_2^k L_K^{2k} (\mathbb{E}\,(f))^2 \varepsilon^2 n}{(\mathcal{L}_k)^{2k}}\right)$$

On the other hand, denote (as in [BB]) $F_K(t,E) := \big|\{x \in K : |P_E(x)| \leq t\}\big|$, $t \geq 0$, the marginal measure on E of the euclidean ball of radius t and $\Gamma_K^k(t)$ the k-dimensional

Gaussian measure (centered with variance L_K^2) of $\{s \in \mathbb{R}^k : |s| \leq t\}$. Theorem 3.5 in [BB] and Theorem 2.7 readily imply

$$\left| \frac{\int_{G_{n,k}} F_K(t,E)\, d\mu(E)}{\Gamma_K^k(t)} - 1 \right| \leq \frac{c_1}{n^{0.09}} \qquad \forall\, t \geq 0$$

Taking limits as $t \to 0$ (see Corollary 3.6 in [BB]) yields

$$\left| \frac{\mathbb{E}\,(f)}{\frac{1}{(\sqrt{2\pi}L_K)^k}} - 1 \right| \leq \frac{c_1}{n^{0.09}} \Big(\leq \frac{\varepsilon}{3} \Big)$$

By the triangle inequality

$$\left| \frac{f(E)}{\frac{1}{(\sqrt{2\pi}L_K)^k}} - 1 \right| \leq \frac{\mathbb{E}\,(f)}{\frac{1}{(\sqrt{2\pi}L_K)^k}} \left| \frac{f(E)}{\mathbb{E}(f)} - 1 \right| + \left| \frac{\mathbb{E}\,(f)}{\frac{1}{(\sqrt{2\pi}L_K)^k}} - 1 \right|$$

So, if $\left| \frac{f(E)}{\mathbb{E}(f)} - 1 \right| \leq \frac{\varepsilon}{3}$, then $\left| \frac{f(E)}{\frac{1}{(\sqrt{2\pi}L_K)^k}} - 1 \right| \leq (1 + \frac{\varepsilon}{3})\frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon$ and conclude, using also $\mathcal{L}_k \leq ck^{1/4}$

$$\begin{aligned} &\mu\big\{E \in G_{n,k}; \big|f(E) - \frac{1}{(\sqrt{2\pi}L_K)^k}\big| \leq \frac{\varepsilon}{(\sqrt{2\pi}L_K)^k}\big\} \geq \\ &\geq \mu\left\{E \in G_{n,k}; |f(E) - \mathbb{E}\,(f)| \leq \frac{\varepsilon}{3}\mathbb{E}\,(f)\right\} \geq 1 - c_1 \exp\left(-\frac{c_2^k \varepsilon^2 n}{k^{k/2}}\right) \end{aligned}$$

The hypothesis on k implies $\varepsilon \geq \frac{(\log\log n)^2}{\log n}$ and $k^{k/2} \ll n^{0,1}$, so

$$\mu\big\{E \in G_{n,k}; \big|f(E) - \frac{1}{(\sqrt{2\pi}L_K)^k}\big| \leq \frac{\varepsilon}{(\sqrt{2\pi}L_K)^k}\big\} \geq 1 - c_1 \exp(-c_2 n^{0.9})$$

□

Acknowledgements

Partially supported by MTM2007-61446 and DGA E-64.

References

[ABBP] D. Alonso, J. Bastero, J. Bernués and G. Paouris, *High dimensional sections of isotropic convex bodies*, J. Math. Anal. Appl. **361** (2010), pp. 431-439.

[ABP] M. Anttila, K. Ball, I. Perissinaki, *The central limit theorem for convex bodies*, Trans. Amer. Math. Soc. **355** (2003), pp. 4723-4735.

[B] K. Ball, *Logarithmic concave functions and sections of convex sets in $\mathbb{R}^n$*, Studia Math. **88** (1988), pp. 69-84.

[BB] J. Bastero and J. Bernués, *Asymptotic behaviour of averages of k-dimensional marginals of measures on $\mathbb{R}^n$*, Studia Math. **190** (2009), pp. 1-31.

[EK] R. Eldan and B. Klartag, *Pointwise Estimates for Marginals of Convex Bodies*, Journal Functional Analysis **254** (2008), pp. 2275-2293.

[GM] A. Giannopoulos and V. Milman, *Mean width and diameter of proportional sections of a symmetric convex body*, J. Reine Angew. Math. **497** (1998), pp. 113-139.

[H] D. Hensley, *Slicing convex bodies, bounds of slice area in terms of the body's covariance*, Proc. Amer. Math. Soc. **79** (1980), pp. 619-625.

[Kl] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. and Funct. Anal. (GAFA) **16** (2006) 1274-1290.

[Kl2] B. Klartag, *Power-law estimates for the central limit theorem for convex sets*, Journal Functional Analysis **245** (2007), pp. 284-310.

[MP] V. Milman and A. Pajor, *Isotropic positions and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, GAFA Seminar 87-89, Springer Lecture Notes in Math. **1376** (1989), pp. 64-104.

[MS] V. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Math. **1200**, Springer, (1986).

[So] S. Sodin, *Tail-sensitive Gaussian asymptotics for marginals of concentrated measures in high dimension*,GAFA Seminar,2004-05 Springer Lecture Notes in Math. **1910** (2007), pp. 271-295.