Monografías de la Real Academia de Ciencias de Zaragoza **33**, 177–196, (2010).

On the computation of symmetric Szegő-type quadrature formulas

A. Bultheel

Department of Computer Science, K.U.Leuven
Celestijnenlaan 200 A, B-3001 Heverlee, Belgium.

and

R. Cruz-Barroso, P. González-Vera, F. Perdomo-Pío

Department of Mathematical Analysis, La Laguna University
38271 La Laguna, Tenerife, Canary Islands, Spain

"*Dedicated to Prof. Manuel Calvo Pinilla for his 65-th birthday and as an acknowledgement of his decisive contribution to the development of Numerical Analysis in La Laguna University*".

Abstract

By $z = e^{i\theta}$ and $x = \cos\theta$, one may relate $x \in I = (-1, 1]$, with $\theta \in (-\pi, \pi]$ and a point z on the complex unit circle $\mathbb{T}$. Hence there is a connection between the integrals of 2π-periodic functions, integrals of functions over I and over $\mathbb{T}$. The well known Gauss quadratures approximate the integrals over I and their circle counterparts are the Szegő quadratures. When none, one or both endpoints of I are added to the usual Gauss nodes, one obtains the Gauss-type (Radau and Lobatto) quadratures. The circular counterparts are called Szegő-type quadratures. If the integrand and the weight function are symmetric for upper and lower half of $\mathbb{T}$, the choice of complex conjugate Szegő nodes with equal weights seems to be natural, and in that case, the Gauss nodes in I are just the projections of the Szegő nodes. Also the weights are related, and it becomes numerically interesting to compute the Szegő quadrature from the corresponding Gauss quadrature which reduces the computational cost considerably. Especially when the weights and nodes are computed via an eigenvalue problem, which for Gauss works with a tri-diagonal Jacobi matrix, but requires an upper Hessenberg matrix in the Szegő case.

Key words: Gauss-type quadrature formulas, Jacobi matrices, Szegő-type quadrature formulas, Hessenberg matrices, symmetric weight functions, Rogers-Szegő q-polynomials.

1 Introduction

Since the publication in 1989 of the paper [35] by W.B. Jones, O. Njåstad and W.J. Thron along with the recent works by B. Simon [39]-[42] among others (see also some of the most relevant contributions of the "Spanish Mathematical Community" on Orthogonal Polynomials, e.g. [1], [3]-[4], [10]-[12], [23] or [26]), the theory of orthogonal polynomials on the unit circle introduced by Szegő in [45] has become an interesting research topic both from a theoretical and from an applied point of view. In this respect, when dealing with the approximate calculation of a weighted integral of a 2π-periodic function or more generally a weighted integral over the unit circle, the so-called Szegő quadrature formulas introduced in [35] (see also [13], [29, Chapter 4], [30]-[31] and [44]) appear and represent the analog on the unit circle of the Gaussian Formulas. As it is known, a fundamental aspect of a family of quadrature rules is the efficient computation of its nodes and weights. Thus, the computation of the Gaussian formulas leads to an eigenvalue problem involving certain tri-diagonal (Jacobi) matrices meanwhile the Szegő formula can be efficiently computed in terms of an eigenvalue problem involving certain Hessenberg matrices [30]-[31] (see also [11] and [14] for an alternative approach).

In this paper, we will be mainly concerned with the computation of the Szegő formulas when both the weight function in the integral and the nodes in the quadrature rules satisfy symmetry properties. For this purpose, the well known connection between the theory of Orthogonal Polynomials on the unit circle and the real line will be used in order to drastically reduce the computational effort of such rules.

Thus, in order to make the paper self-contained it has been organized as follows: Sections 2 and 3 are dedicated to collect some preliminary results concerning the most relevant aspects of both Gaussian and Szegő formulas. In Section 4 the above symmetry properties are exposed and the characterization of the corresponding symmetric Szegő-type quadrature formulas deduced. The computation features are given in Section 5 meanwhile some numerical illustrative experiments are finally carried out in Section 6.

2 Preliminary results: Jacobi matrices and Gauss-type formulas

Given the integral,

$$I_\sigma(f) = \int_a^b f(x)\sigma(x)dx, \tag{1}$$

σ being a weight function on [a,b], by an n-point Gaussian formula $I_n(f) = \sum_{j=1}^{n} A_j f(x_j)$ for $I_\sigma(f)$ or σ we mean a quadrature formula so that $I_\sigma(P) = I_n(P)$ for any polynomial $P \in \mathcal{P}_{2n-1}$; in the sequel, $\mathcal{P}_k$ denotes the space of polynomials of degree less than or equal to k and $\mathcal{P}$ the space of all polynomials i.e., $\mathcal{P} = \cup_{k=0}^{\infty} \mathcal{P}_k$. A characterization of these rules is given in the following result (see e.g. [36, pp. 101-103] and [45, Theorem 3.4.2]),

Theorem 2.1 *Let $\{Q_k\}_{k=0}^{\infty}$ be the sequence of orthonormal polynomials for σ. Then, $I_n^\sigma(f) = \sum_{j=1}^{n} A_j f(x_j)$ is the n-point Gaussian formula for $I_\sigma(f)$, if and only if,*

1. *$\{x_j\}_{j=1}^{n}$ are the zeros of any orthogonal polynomial of degree n with respect to σ.*
2. *$A_j = \left(\sum_{k=0}^{n-1} |Q_k(x_j)|^2 \right)^{-1} > 0$, for all $j = 1, \ldots, n$ (Christoffel numbers).*

$I_n^\sigma(f)$ as given in Theorem 2.1 is *optimal* in the sense there exists $P \in \mathcal{P}_{2n}$ such that $I_n^\sigma(P) \neq I_\sigma(P)$.

On the other hand, efficient computation of the weights and nodes for $I_n^\sigma(f)$ has been carried out by means of the so-called Jacobi matrices associated with the three-term recurrence relation satisfied by the sequence $\{Q_k\}_{k=0}^{\infty}$. Indeed, it is known that it holds,

$$xQ_n(x) = a_{n+1}Q_{n+1}(x) + b_n Q_n(x) + a_n Q_{n-1}(x), \quad n \geq 0, \quad Q_{-1} \equiv 0,$$

so that by setting,

$$\mathcal{J} = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & \cdots \\ 0 & 0 & a_3 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2}$$

then, the eigenvalues of the n-th truncation of the matrix $\mathcal{J}$ give us the set of nodes $\{x_j\}_{j=1}^{n}$ in $I_n^\sigma(f)$ and the square of the first component of the eigenvector of unit length corresponding to the eigenvalue x_j yields the weight A_j, for $j = 1, \ldots, n$. As seen, in a Gaussian formula no freedom is left to fix some nodes in advance so that the remaining nodes and weights can be chosen to produce quadrature formulas with similar features to the Gaussian ones, that is, positive weights and exactly integrating polynomials with as high degree as possible. This kind of quadratures which are of a great interest in the construction of methods to numerically solve differential and integral equations, have been studied in the last decades producing satisfactory results only in a few particular cases . Thus, in the simplest situation when $[a, b]$ is finite, say $[a, b] = [-1, 1]$, quadrature formulas with prescribed nodes at ± 1 can be constructed and exhibiting similar characteristics to the Gaussian ones. These are the so-called Gauss-Radau and Gauss-Lobatto formulas as summarized in the following (for a more general situation see the recent papers [7] and [15]),

Theorem 2.2 *Given the integral $I_\sigma(f)$ in (1) and $r, s \in \{0,1\}$, consider the n-point quadrature rule:*

$$I_n^{r,s}(f) = rA_n^+ f(1) + sA_n^- f(-1) + \sum_{j=1}^{n-r-s} A_j^{r,s} f(x_j^{r,s}).$$

Then $I_n^{r,s}(P) = I_\sigma(P)$, for all $P \in \mathcal{P}_{2n-1-r-s}$, if and only if,

1. *$I_n^{r,s}(P) = I_\sigma(P)$, for all $P \in \mathcal{P}_{n-1}$ (that is, it is of interpolatory type).*

2. *The nodes $\{x_j^{r,s}\}_{j=1}^{n-r-s}$ are the zeros of any orthogonal polynomial of degree $n-r-s$ with respect to the weight function $\sigma_{r,s}(x) = (1-x)^r(1+x)^s\sigma(x)$, $x \in [-1,1]$. Furthermore, the weights A_n^+, A_n^- and $A_j^{r,s}$ for all $j = 1, \ldots, n-r-s$ are positive and it holds that,*

$$A_j^{r,s} = \frac{\tilde{A}_j^{r,s}}{(1-x_j^{r,s})^r(1+x_j^{r,s})^s}, \quad j = 1, \ldots, n-r-s,$$

$\{\tilde{A}_j^{r,s}\}_{j=1}^{n-r-s}$ being the Christoffel numbers for $\sigma_{r,s}$.

Thus,

1. As $r+s=0$, $I_n^{0,0}$ is the n-point Gauss-formula.

2. As $r+s=1$, $I_n^{1,0}$ and $I_n^{0,1}$ are the n-point Gauss-Radau formulas.

3. As $r+s=2$, $I_n^{1,1}$ is the n-point Gauss-Lobatto formula.

Sometimes, we will refer to these quadratures as *Gauss-type formulas* so that they can be efficiently computed in terms of an eigenvalue problem involving Jacobi matrices. Indeed, let $\mathcal{J}$ be the Jacobi matrix associated with the weight function σ, set the Darboux transform $\tilde{\sigma}(x) = (x-\beta)\sigma(x)$ with $\beta \in \mathbb{R}$ such that $Q_n(\beta) \neq 0$ for all $n = 1, \ldots$, $\{Q_k\}_{k=0}^{\infty}$ being the sequence of orthonormal polynomials for σ and denote by $\tilde{\mathcal{J}}$ the Jacobi matrix associated with $\tilde{\sigma}$. Then, it holds that (see e.g. [4])

$$\tilde{\mathcal{J}} = UL + \beta I, \tag{3}$$

where I denotes the unit matrix and $\mathcal{J} - \beta I = LU$. That is, once we have obtained the LU decomposition of the known matrix $\mathcal{J} - \beta I$, (3) gives the Jacobi matrix $\tilde{\mathcal{J}}$ associated with $\tilde{\sigma}$.

When this is restricted to a finite section of the Jacobi matrix (2), it is equivalent with the eigenvalue techniques proposed by Gautschi and Golub (see [21]-[22], [28] and also [7]). Indeed, if $\beta \in \{-1, 1\}$ and we want β to be a node of the quadrature, then we modify the last b_{n-1} and require that the corresponding $\hat{Q}_n = Q_n - \hat{b}_{n-1}Q_{n-1}$ has a zero in β, which leads to $\hat{b}_{n-1} = Q_n(\beta)/Q_{n-1}(\beta)$. By changing b_{n-1} into $b_{n-1} + \hat{b}_{n-1}$ we get a modified truncated Jacobi matrix $\hat{\mathcal{J}}_n$ which will deliver the nodes and weights of the Gauss-Radau formula like in the Gauss case. Similarly, one may consider $\hat{Q}_n = Q_n - \hat{b}_{n-1}Q_{n-1} - \hat{a}_{n-1}Q_{n-2}$ and solve for $\hat{a}_{n-1}$ and $\hat{b}_{n-1}$ by requiring that $\hat{Q}_{n-1}(\pm 1) = 0$, which leads to the system

$$\begin{pmatrix} Q_{n-1}(1) & Q_{n-2}(1) \\ Q_{n-1}(-1) & Q_{n-2}(-1) \end{pmatrix} \begin{pmatrix} \hat{b}_{n-1} \\ \hat{a}_{n-1} \end{pmatrix} = \begin{pmatrix} Q_n(1) \\ Q_n(-1) \end{pmatrix}.$$

Modifying the truncated Jacobi matrix by replacing (b_{n-1}, a_{n-1}) with $(b_{n-1} + \hat{b}_{n-1}, a_{n-1} + \hat{a}_{n-1})$ gives a matrix $\hat{\mathcal{J}}_n$ that provides the nodes and weights of the Gauss-Lobatto formula through its eigenvalue decomposition as in the classical Gauss case.

3 Integration of periodic functions

Suppose now we are concerned with the approximate calculation of the integral,

$$I_\omega(g) = \int_{-\pi}^{\pi} g(\theta)\omega(\theta)d\theta,$$

g and ω being 2π-periodic functions and ω a weight function on $[-\pi, \pi]$. Without loss of generality we will assume the normalization $\int_{-\pi}^{\pi} \omega(\theta)d\theta = 1$. For this purpose, we will use an n-point quadrature rule like,

$$I_n^\omega(g) = \sum_{j=1}^{n} \lambda_j g(\theta_j), \quad \{\theta_j\}_{j=1}^n \subset (-\pi, \pi], \quad \theta_j \neq \theta_k \text{ if } j \neq k,$$

but now imposing that $I_n^\omega(T) = I_\omega(T)$, for any trigonometric polynomial $T(\theta) = \sum_{k=0}^{N}(a_k \cos k\theta + b_k \sin k\theta)$ with as high degree N as possible. In this respect, it is known that $N \leq n - 1$ (see [36, pp. 73-74]) and that the case $N = n - 1$ gives rise to the quadrature formulas with the maximum trigonometric degree of precision which come characterized in terms of the so-called bi-orthogonal systems of trigonometric polynomials associated with ω (see [18] or [44] for further details). Alternatively, taking into account that any 2π-periodic function on $\mathbb{R}$ can be viewed as a function defined on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ we could write,

$$I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta,$$

to be approximated by,

$$I_n^\omega(g) = \sum_{j=1}^{n} \lambda_j f(z_j), \quad \{z_j\}_{j=1}^n \subset \mathbb{T}, \quad z_j \neq z_k \text{ if } j \neq k, \tag{4}$$

such that

$$I_n^\omega(L) = I_\omega(L), \text{ for all } L \in \Lambda_{-(n-1),n-1}, \tag{5}$$

where for p and q are integers with $p \leq q$, $\Lambda_{p,q} = \text{span}\{z^k : p \leq k \leq q\}$ and $\Lambda = \text{span}\{z^k : k \in \mathbb{Z}\}$. Here the functions in Λ are called Laurent polynomials so that if $T(\theta)$ is a trigonometric polynomial of degree m then one can write $T(\theta) = L(e^{i\theta})$ with $L \in \Lambda_{-(m-1),m-1}$. Moreover, for an ordinary polynomial $P_n(z)$ of exact degree n, we define its reverse or reciprocal as $P_n^*(z) = z^n\overline{P_n(1/\bar{z})}$. Concerning the construction and characterization of the quadrature rule (4) satisfying (5) one has the following (see [35] and [27]),

Theorem 3.1 *Set $I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta$, $I_n^\omega(g) = \sum_{j=1}^n \lambda_j g(z_j)$ with $z_j \in \mathbb{T}$, $j = 1,\ldots,n$ and let $\{\varphi_k\}_{k=0}^\infty$ be the sequence of orthonormal (Szegő) polynomials for ω. Then $I_n^\omega(g) = I_\omega(g)$, for all $g \in \Lambda_{-(n-1),n-1}$, if and only if,*

1. *$\{z_j\}_{j=1}^n$ are the zeros of $B_n(z,\tau_n) = \varphi_n(z) + \tau_n\varphi_n^*(z)$ for some $\tau_n \in \mathbb{T}$,*
2. *$\lambda_j = \left(\sum_{k=0}^{n-1} |\varphi_k(z_j)|^2\right)^{-1} > 0$, for all $j = 1,\ldots,n$.*

$I_n^\omega(g)$ as given in Theorem 3.1 is called an n-point Szegő quadrature rule (see [35]) and represents the analog on the unit circle of the Gaussian formulas.

Szegő formulas are also *optimal* in the sense that there can not exist an n-point quadrature formula with nodes on $\mathbb{T}$ exactly integrating any Laurent polynomial either in $\Lambda_{-n,n-1}$ or in $\Lambda_{-(n-1),n}$. However, as mentioned in [46, Section 12], it can be proved that an n-point Szegő formula is exact in $\mathcal{L}_n \subset \Lambda$ such that $\dim(\mathcal{L}_n) = 2n$ and $\Lambda_{-(n-1),n-1} \subset \mathcal{L}_n$ (see [38]). Unlike the Gaussian rules, Szegő formulas are not uniquely determined because of the presence of the arbitrary parameter $\tau_n \in \mathbb{T}$. Thus, given $z_\alpha \in \mathbb{T}$, one can take $\tilde{\tau}_n \in \mathbb{T}$ such that $B_n(z_\alpha,\tilde{\tau}_n) = 0$ where $B_n(z,\tilde{\tau}_n) = \varphi_n(z) + \tilde{\tau}_n\varphi_n^*(z)$. Hence, $\tilde{\tau}_n = -\frac{\varphi_n(z_\alpha)}{\varphi_n^*(z_\alpha)} \in \mathbb{T}$ provides an n-point Szegő quadrature formula with a fixed node $z_\alpha \in \mathbb{T}$ in advance and called a Szegő-Radau quadrature rule.

On the other hand, if $\rho_n(z)$ denotes the monic Szegő polynomial of degree n, one can write (up to a multiplicative factor)

$$B_n(z,\tau_n) = \rho_n(z) + \tau_n\rho_n^*(z).$$

Now, from the recurrence relation for $\{\rho_k\}_{k=0}^\infty$ (see [25], [29], [45, Theorem 11.4.2] or [41, Theorem 1.5.2]),

$$\begin{pmatrix} \rho_{k+1}(z) \\ \rho_{k+1}^*(z) \end{pmatrix} = \begin{pmatrix} z & \delta_{k+1} \\ \overline{\delta_{k+1}}z & 1 \end{pmatrix} \begin{pmatrix} \rho_k(z) \\ \rho_k^*(z) \end{pmatrix}, \quad k = 0,1,\ldots, \tag{6}$$

with $\rho_0(z) = \rho_0^*(z) = 1$, $\delta_0 = 1$ and $\delta_k = \rho_k(0) \in \mathbb{D}$ for all $k \geq 1$ (Verblunsky parameters[1]), then for $\tau_n \in \mathbb{T}$ it follows that,

$$B_n(z, \tau_n) = \rho_n(z) + \tau_n \rho_n^*(z) = C_n[z\rho_{n-1}(z) + \tilde{\tau}_n \rho_{n-1}^*(z)], \quad C_n \neq 0 \text{ and } \tilde{\tau}_n \in \mathbb{T}.$$

Thus, to generate an n-point Szegő formula, we take $\tau_n \in \mathbb{T}$ and consider the zeros of $B_n(z, \tau_n)$ that essentially depends on the parameters $\delta_0, \delta_1, \ldots, \delta_{n-1}$ and τ_n. More precisely, define the matrix

$$H_n(\tau_n) = D_n^{-1/2} \begin{pmatrix} -\delta_1 & -\delta_2 & \cdots & -\delta_{n-1} & -\tau_n \\ \sigma_1^2 & -\overline{\delta_1}\delta_2 & \cdots & -\overline{\delta_1}\delta_{n-1} & -\overline{\delta_1}\tau_n \\ 0 & \sigma_2^2 & \cdots & \overline{\delta_2}\delta_{n-1} & -\overline{\delta_2}\tau_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{n-1}^2 & -\overline{\delta_{n-1}}\tau_n \end{pmatrix} D_n^{1/2}, \tag{7}$$

where $\sigma_k = \sqrt{1 - |\delta_k|^2} \in (0, 1]$, $k = 1, 2, \ldots n$ and $D_n = \operatorname{diag}[\gamma_0, \ldots, \gamma_{n-1}] \in \mathbb{R}^{n \times n}$ with $\gamma_0 = 1$, $\gamma_k = \gamma_{k-1}\sigma_k^2 > 0$, $k = 1, \ldots, n-1$ and $\tau_n \in \mathbb{T}$. Under these conditions one has ([30]-[31]),

Theorem 3.2 *$H_n(\tau)$ given in (7) is an unreduced unitary upper Hessenberg matrix for all $\tau \in \mathbb{T}$, so that its eigenvalues $\{z_j\}_{j=1}^n$ which are distinct and of unit magnitude are the zeros of $B_n(z, \tau) = \rho_n(z) + \tau\rho_n^*(z)$ or equivalently, the nodes of the n-point Szegő formula for the parameter τ. Furthermore, the square of the first component of the eigenvector of unit length associated with z_j yields the weight λ_j.*

Finally, in a similar way as done when dealing with the estimation of $\int_{-1}^{+1} f(x)\sigma(x)dx$ so that the points ± 1 are taken as nodes in a quadrature formula (Gauss-Lobatto rule), suppose z_α and z_β on $\mathbb{T}$ such that $z_\alpha \neq z_\beta$ and take $n > 2$. Then (see [5], [34]) there exist (and they can be easily computed) complex numbers $\tilde{\delta}_{n+1} \in \mathbb{D}$ and $\tilde{\tau}_n \in \mathbb{T}$ such that z_α and z_β are zeros of

$$\tilde{B}_n(z) = z\tilde{\rho}_{n-1}(z) + \tilde{\tau}_n \tilde{\rho}_{n-1}^*(z) \quad \text{with} \quad \tilde{\rho}_{n-1}^*(z) = z\rho_{n-2}(z) + \tilde{\delta}_{n-1}\rho_{n-2}^*(z). \tag{8}$$

Thus, if we denote by $z_1, \ldots, z_{n-2}, z_\alpha$ and z_β the zeros of $\tilde{B}_n(z)$ we have that $z_j \in \mathbb{T}$, $z_j \neq z_k$ if $j \neq k$ and $z_j \notin \{z_\alpha, z_\beta\}$, $1 \leq j, k \leq n-2$. Furthermore, there exist positive weights A, B and λ_j, $j = 1, \ldots n-2$ such that,

$$\tilde{I}_n^\omega(g) = Ag(z_\alpha) + Bg(z_\beta) + \sum_{j=1}^{n-2} \lambda_j g(z_j) = I_\omega(g), \text{ for all } g \in \Lambda_{-(n-2),n-2}. \tag{9}$$

[1]There are at least four other terms: Szegő, reflection, Schur and Geronimus parameters, see [41, Chapter 1.5].

$\tilde{I}_n^{\omega}(g)$ in (9), and that could be incidentally exact in $\Lambda_{-(n-1),n-1}$, provided that $\tilde{\delta}_{n-1} = \delta_{n-1}$, is called an n-point Szegő-Lobatto formula for ω with prescribed nodes z_α and z_β.

Szegő, Szegő-Radau and Szegő-Lobatto formulas will be sometimes referred as Szegő-type quadrature rules whose computation is the aim of this paper, under some special conditions on the weight ω as described on the next section.

4 Symmetric weight functions

In this section we will be concerned with Szegő-type rules associated with a symmetric weight function ω on $[-\pi, \pi]$, that is, $\omega(-\theta) = \omega(\theta)$, $\theta \in [-\pi, \pi]$. Setting (trigonometric moments)

$$\mu_k = \int_{-\pi}^{\pi} e^{-ik\theta}\omega(\theta)d\theta, \quad k = 0, \pm 1, \pm 2, \dots \tag{10}$$

(recall that we are assuming that $\mu_0 = 1$) and considering the sequence $\{\delta_k\}_{k=0}^{\infty}$ of Verblunsky parameters, then it follows,

Lemma 4.1 *The following statements are all equivalent:*

1. *ω is a symmetric weight function on $[-\pi, \pi]$.*
2. *The Toeplitz matrices associated with ω are symmetric, i.e. $\mu_{-k} = \mu_k$ for all $k \in \mathbb{Z}$.*
3. *The trigonometric moments are real, i.e. $\mu_k \in \mathbb{R}$ for all $k \in \mathbb{Z}$.*
4. *The Verblunsky parameters δ_k lie in $(-1, 1)$ for all $k \geq 1$.*

As for the quadratures, it will be convenient to give the following

Definition 4.2 *Let ω be a symmetric weight function on $[-\pi, \pi]$. Then, an n-point Szegő formula $I_n^{\omega}(g) = \sum_{j=1}^{n} \lambda_j g(z_j)$ for $I_\omega(g)$ is said to be symmetric if the nodes are real or appear on $\mathbb{T}$ in complex conjugate pairs.*

Now we are concerned with the characterization and computation of symmetric Szegő formulas, if there exist. For this purpose, from Lemma 4.1 it should be taken into account that the sequence of monic Szegő polynomials $\{\rho_k\}_{k=0}^{\infty}$ has real coefficients and hence, it holds that (see e.g. [6]):

Proposition 4.3 *Let ω be a symmetric weight function on $[-\pi, \pi]$. Then,*

1. *An n-point Szegő formula $I_n^{\omega}(g) = \sum_{j=1}^{n} \lambda_j g(z_j)$ generated by $B_n(z, \tau_n) = \rho_n(z) + \tau_n \rho_n^*(z)$ is symmetric, if and only if, $\tau_n \in \{\pm 1\}$.*
2. *Let $I_n^{\omega}(g) = \sum_{j=1}^{n} \lambda_j g(z_j)$ be an n-point Szegő formula for $I_\omega(g)$ and suppose that $z_j = \overline{z_k}$ for some j and k, $1 \leq j, k \leq n$. Then, $\lambda_j = \lambda_k$.*

From Proposition 4.3, we see that when dealing with symmetric rules, their computation essentially reduces to one half. In this respect, computation will be carried out by passing to the interval [-1,1] having in mind the following,

Proposition 4.4 *ω is a symmetric weight function on $[-\pi,\pi]$, if and only if, there exists a weight function σ on $[-1,1]$ such that $\omega(\theta)=\sigma(\cos\theta)|\sin\theta|$. Furthermore, it holds*

$$\int_{-1}^{+1} f(x)\sigma(x)dx = \frac{1}{2}\int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta, \quad g(e^{i\theta}) = f\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right).$$

A connection between quadrature formulas for ω and σ on $[-\pi,\pi]$ and $[-1,1]$ respectively is shown in the following (see [6] and also [17]):

Proposition 4.5 *Take $r,s\in\{0,1\}$ and consider $n-r-s$ distinct nodes $\{x_j^{r,s}\}_{j=1}^{n-r-s}$ on $(-1,1)$ along with the n real numbers $A_+^{r,s}$, $A_-^{r,s}$ and $\{A_j^{r,s}\}_{j=1}^{n-r-s}$. Set $x_j^{r,s}=\cos\theta_j^{r,s}$, $\theta_j^{r,s}\in(0,\pi)$ and define $z_j^{r,s}=e^{i\theta_j^{r,s}}$, $z_{n-r-s+j}^{r,s}=\overline{z_j^{r,s}}$ and $\lambda_j^{r,s}=\lambda_{n+j-r-s}^{r,s}=A_j^{r,s}$, $1\le j\le n-r-s$. Then, the following statements are equivalent:*

1. *$I_{n;(r,s)}^{\sigma}(f)=rA_+^{r,s}f(1)+sA_-^{r,s}f(-1)+\sum_{j=1}^{n-r-s}A_j^{r,s}f(x_j^{r,s})=I_\sigma(f)$, for all $f\in\mathcal{P}_N$.*
2. *$I_{2n-r-s}^{\omega}(g)=2[rA_+^{r,s}g(1)+sA_-^{r,s}g(-1)]+\sum_{j=1}^{2(n-r-s)}\lambda_j^{r,s}g(z_j^{r,s})=I_\omega(g)$, for all $g\in\Lambda_{-N,N}$.*

Now, from Theorem 2.2 one sees that as $N=2n-1-r-s$ the following results in:

a) As $r=s=0$, then $N=2n-1$ and therefore $I_{n;(0,0)}^{\sigma}(f)=\sum_{j=1}^{n}A_j^{0,0}f(x_j^{0,0})$ coincides with the n-point Gaussian formula for σ yielding the following $2n$-point quadrature rule for ω:

$$I_{2n}^{\omega}(g)=\sum_{j=1}^{2n}\lambda_j^{0,0}g(z_j^{0,0})=I_\omega(g), \text{ for all } g\in\Lambda_{-(2n-1),2n-1},$$

which is clearly a $2n$-point symmetric Szegő rule. Hence, the nodes $\{z_j^{0,0}\}_{j=1}^{2n}$ are the zeros of $B_{2n}(z,\tau_{2n})=\rho_{2n}(z)+\tau_{2n}\rho_{2n}^*(z)$ with $\tau_{2n}\in\{\pm1\}$. Since $\rho_{2n}(-1)=\rho_{2n}^*(-1)$ it follows that $\tau_{2n}=1$ i.e. $\{z_j^{0,0}\}_{j=1}^{2n}$ are the zeros of $B_{2n}(z,1)=\rho_{2n}(z)+\rho_{2n}^*(z)$. In short, we have:

$$z_j^{0,0}=x_j^{0,0}+i\sqrt{1-(x_j^{0,0})^2} \text{ and } \lambda_j^{0,0}=A_j^{0,0} \text{ for all } j=1,\dots n,$$

where $\{x_j^{0,0}\}_{j=1}^{n}$ are the zeros of the n-th orthogonal polynomial for σ and $\{A_j^{0,0}\}_{j=1}^{n}$ the corresponding Christoffel numbers of order n for σ.

b) As $r=1$ and $s=0$, then $N=2n-2$ and $I_{n;(1,0)}^{\sigma}(f)=A_+^{1,0}f(1)+\sum_{j=1}^{n-1}A_j^{1,0}f(x_j^{1,0})$ represents the n-point Gauss-Radau formula for σ with a fixed node at $x=1$, giving rise to the following $(2n-1)$-point rule for ω:

$$I_{2n-1}^{\omega}(g)=2A_+^{1,0}g(1)+\sum_{j=1}^{n-1}\lambda_j^{1,0}[g(z_j^{1,0})+g(\overline{z_j^{1,0}})]=I_\omega(g), \text{ for all } g\in\Lambda_{-(2n-2),2n-2}.$$

Hence, we have again a $(2n-1)$-point symmetric Szegő formula for $I_\omega(g)$ whose nodes are the zeros of $B_{2n-1}(z,-1) = \rho_{2n-1}(z) - \rho^*_{2n-1}(z)$. Now it follows,

$$z_j^{1,0} = x_j^{1,0} + i\sqrt{1-(x_j^{1,0})^2} \;\text{ and }\; \lambda_j^{1,0} = A_j^{1,0} = \frac{\tilde{A}_j^{1,0}}{1-x_j^{1,0}} \;\text{ for all }\; j = 1,\dots n-1, \quad (11)$$

where $\{x_j^{1,0}\}_{j=1}^{n-1}$ are the zeros of the $(n-1)$-orthogonal polynomial for $\sigma_{1,0}(x) = (1-x)\sigma(x)$ and $\tilde{A}_j^{1,0}$, $j = 1,\dots,n-1$ its corresponding Christoffel numbers of order $n-1$. Moreover, since $I_{2n-1}(1) = I_\omega(1) = 1$ it follows,

$$A_+^{1,0} = \frac{1}{2} - \sum_{j=1}^{n-1} \lambda_j^{1,0}.$$

c) As $r = 0$, $s = 1$, then $N = 2n-2$ and similarly to the previous case, $I^\sigma_{n;(0,1)}(f) = A_-^{0,1} f(1) + \sum_{j=1}^{n-1} \lambda_j^{0,1} f(x_j^{0,1})$ represents the n-point Gauss-Radau formula for σ with a fixed node at $x = -1$ and yielding

$$I^\omega_{2n-1}(g) = 2A_-^{0,1} g(-1) + \sum_{j=1}^{n-1} \lambda_j^{0,1}[g(z_j^{0,1}) + g(\overline{z_j^{0,1}})] = I_\omega(g), \text{ for all } g \in \Lambda_{-(2n-2),2n-2},$$

that represents a $(2n-1)$-point symmetric Szegő formula for $I_\omega(g)$ whose nodes are the zeros $B_{2n-1}(z,1) = \rho_{2n-1}(z) + \rho^*_{2n-1}(z)$. Again, we have:

$$z_j^{0,1} = x_j^{0,1} + i\sqrt{1-(x_j^{0,1})^2} \;\text{ and }\; \lambda_j^{0,1} = A_j^{0,1} = \frac{\tilde{A}_j^{0,1}}{1+x_j^{0,1}} \;\text{ for all }\; j = 1,\dots n-1,$$

where $\{x_j^{0,1}\}_{j=1}^{n-1}$ are the zeros of the $(n-1)$-th orthogonal polynomial for $\sigma_{0,1}(x) = (1+x)\sigma(x)$ and $\tilde{A}_j^{0,1}$, $j = 1,\dots,n-1$ its corresponding Christoffel numbers of order $n-1$. In a similar way,

$$A_-^{1,0} = \frac{1}{2} - \sum_{j=1}^{n-1} \lambda_j^{0,1}.$$

d) As $r = s = 1$, then $N = 2n-3$ and we see that $I^\sigma_{n;(1,1)}(f) = A_+^{1,1} f(1) + A_-^{1,1} f(-1) + \sum_{j=1}^{n-2} A_j^{1,1} f(x_j^{1,1})$ represents the n-point Gauss-Lobatto formula for $I_\sigma(f)$ and giving rise to the following quadrature rule for $I_\omega(g)$:

$$I^\omega_{2n-2}(g) = 2[A_+^{1,1} g(1) + A_-^{1,1} g(-1)] + \sum_{j=1}^{n-2} \lambda_j^{1,1}[g(z_j^{1,1}) + g(\overline{z_j^{1,1}})].$$

Since $I^\omega_{2n-2}(g) = I_\omega(g)$, for all $g \in \Lambda_{-(2n-3),2n-3}$, we see that it represents again a $(2n-2)$-point symmetric Szegő formula and its nodes are the zeros of $B_{2n-2}(z,-1) = \rho_{2n-2}(z) - \rho^*_{2n-2}(z)$. Now,

$$z_j^{1,1} = x_j^{1,1} + i\sqrt{1-(x_j^{1,1})^2} \;\text{ and }\; \lambda_j^{1,1} = A_j^{1,1} = \frac{\tilde{A}_j^{1,1}}{1-(x_j^{1,1})^2} \;\text{ for all }\; j = 1,\dots n-2,$$

where $\{x_j^{1,1}\}_{j=1}^{n-2}$ are the zeros of the $(n-2)$-th orthogonal polynomial for $\sigma_{1,1}(x) = (1-x^2)\sigma(x)$ and $\tilde{A}_j^{1,1}$, $j = 1, \ldots, n-2$ the corresponding Christoffel numbers of order $n-2$. As for the remaining weights $A_+^{1,1}$ and $A_-^{1,1}$ since the quadrature formula exactly integrates $g(z) = 1$ and $g(z) = z$, it follows a system of two equations in the unknowns $A_+^{1,1}$ and $A_-^{1,1}$ which can be explicitly solved, yielding

$$2A_+^{1,1} = \frac{1-\mu_{-1}}{2} - \sum_{j=1}^{n-2} \lambda_j^{1,1}\left(1 - \Re(z_j^{1,1})\right) \text{ and } 2A_-^{1,1} = \frac{1+\mu_{-1}}{2} - \sum_{j=1}^{n-2} \lambda_j^{1,1}\left(1 + \Re(z_j^{1,1})\right).$$

As a conclusion, we can say that when dealing with a symmetric weight function ω on $[-\pi, \pi]$ the computation of any n-point symmetric Szegő rule reduces to an eigenvalue problem for a Jacobi matrix of dimension $E[n/2]$ associated with the weight functions $\sigma_{r,s}(x) = (1-x)^r(1+x)^s\sigma(x)$ on $[-1,1]$, where $E[x]$ denotes the integer part of x, and $r, s \in \{0,1\}$ while σ is such that $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$.

Finally, let us analyze the computation of a symmetric Szegő-Lobatto formula, if there exists, when ω is symmetric and we have fixed in advance two nodes on $\mathbb{T}$, say z_α and z_β which are complex conjugate. As already seen, for $n > 2$ there exist positive numbers A, B and λ_j, $j = 1, \ldots, n-2$ along with $n-2$ distinct nodes $z_1, \ldots, z_n$ on $\mathbb{T}$ such that $z_j \notin \{z_\alpha, z_\beta\}$, $j = 1, \ldots, n-2$ and so that

$$\tilde{I}_n^\omega(g) = Ag(z_\alpha)+Bg(z_\beta)+\sum_{j=1}^{n-2} \lambda_j g(z_j) = I_\omega(g), \text{ for all } g \in \Lambda_{-(n-2),n-2} \quad \text{(Szegő-Lobatto formula).} \tag{12}$$

In this case and as shown in [34], the parameter $\tilde{\delta}_{n-1}$ in formula (8) can be taken real, i.e. $\tilde{\delta}_{n-1} \in (-1,1)$ so that $\tilde{\tau}_n \in \{\pm 1\}$. Hence, the Szegő-Lobatto formula (12) is symmetric. Actually (12) is an n-point symmetric Szegő formula for a new symmetric weight function $\tilde{\omega}(\theta)$ whose first $n-1$ Verblunsky parameters are $\delta_1, \ldots, \delta_{n-2}$ and $\tilde{\delta}_{n-1}$. Therefore, computation reduces to the first situation but now replacing $\omega(\theta)$ by $\tilde{\omega}(\theta)$ and $\sigma(x)$ by $\tilde{\sigma}(x)$ such that $\tilde{\omega}(\theta) = \tilde{\sigma}(\cos(\theta))|\sin(\theta)|$.

For instance, once fixed z_α and z_β on $\mathbb{T}$ such that $z_\beta = \overline{z_\alpha}$, suppose n even, say $n = 2m$ and set $x_\alpha = \Re(z_\alpha)$. Since in formula (8), $\tilde{\delta}_{n-1}$ is real and $\tilde{\tau}_n \in \{\pm 1\}$, suppose that $\tilde{\tau}_n = 1$ i.e. the nodes of $\tilde{I}_{2m}^\omega(g)$ given by (12) are the zeros of $\tilde{B}_{2m}(z) = \tilde{B}_{2m}(z,1) = z\rho_{2m-1}(z) + \tilde{\rho}_{2m-1}^*(z)$ which are real or appear in complex conjugate pairs. Since $\tilde{B}_{2m}(\pm 1) \neq 0$, we can write,

$$\tilde{I}_{2m}^\omega(g) = A[g(z_\alpha) + g(\overline{z_\alpha})] + \sum_{j=1}^{m-1} \lambda_j[g(z_j) + g(\overline{z_j})].$$

Setting $x_j = \Re(z_j)$, $j = 1, \ldots, m-1$ we see that $x_\alpha, x_1, \ldots, x_{m-1}$ are the zeros of the m-th orthogonal polynomial for $\tilde{\sigma}(x)$ and $A, \lambda_1, \ldots, \lambda_{m-1}$ the corresponding Christoffel numbers of order m. Furthermore, since in this case we know that $x_\alpha = \Re(z_\alpha)$ is an eigenvalue of the Jacobi matrix, a deflation method could be conveniently used, having in mind that $A = \frac{1}{2} - \sum_{j=1}^{m-1} \lambda_j$. The other three remaining cases, that is, n even and $\tilde{\tau}_n = -1$ and n odd and $\tilde{\tau}_n = \pm 1$ can be treated in a similar way. In short, the computation of the symmetric Szegő-Lobatto formulas leads to the computation of the Gauss-type quadrature rules associated with the new weight function $\tilde{\sigma}(x)$ such that $\tilde{\omega}(\theta) = \tilde{\sigma}(\cos\theta)|\sin\theta|$, $\theta \in [-\pi, \pi]$.

5 The connection with Jacobi matrices

As seen in the previous section, given a symmetric weight function ω on $[-\pi, \pi]$, we can compute its Szegő-type quadrature formulas in terms of the Gauss-type rules for a weight function σ on $[-1, 1]$ such that $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$ so that to carry on an efficient computation we need the corresponding Jacobi matrices associated with σ. However, the initial available information that we have on the weight function ω are its trigonometric moments (10) and only in very few cases the Szegő polynomials are explicitly known. Here it should be recalled that the basic information to compute Szegő quadrature formulas are the Verblunsky parameters, $\delta_0 = 1$ and $\delta_k = \rho_k(0)$ for all $k = 1, 2, \ldots$. Thus, starting from the trigonometric moments, the Verblunsky parameters can be efficiently computed by means of Levinson algorithm, consisting in the implementation of the Szegő recurrence (6) (see [37]). An alternative procedure called split Levinson algorithm was derived in [19] when ω is symmetric. Indeed, the latter routine computes the corresponding para-orthogonal polynomials $B_n(z, \pm 1)$ (and hence, the Verblunsky parameters) with half the work of the computation saved. Also, a well known map from trigonometric moments to Verblunsky coefficients is Schur's algorithm (see e.g. [33]). Now the question is: given the Verblunsky parameters $\{\delta_k\}_{k=0}^{\infty}$ for ω, how can the Jacobi matrix for σ be computed? The answer can be found in the so-called Geronimus relations (see [24]).

Theorem 5.1 *Let ω be a symmetric weight function on $[-\pi, \pi]$ and σ the weight function on $[-1, 1]$ related to ω by $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$. Let $\{\delta_k\}_{k=0}^{\infty}$ be the sequence of Verblunsky parameters for ω and $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be the coefficients of the Jacobi matrix (2) for σ. Then, the following holds:*

$$2a_n = \sqrt{(1 - \delta_{2n})(1 - \delta_{2n-1}^2)(1 + \delta_{2n-2})}, \quad n \geq 1 \quad \text{and} \quad 2b_n = \delta_{2n-1}(1-\delta_{2n}) - \delta_{2n+1}(1+\delta_{2n}), \quad n \geq 0. \tag{13}$$

Example 5.2 *As an illustration of Theorem 5.1, let us consider the weight function ω on $[-\pi, \pi]$ associated with the Poisson Kernel, namely $\omega(\theta) = |z + \gamma|^{-2}$ where $\gamma \in (-1, 1)$ and $z = e^{i\theta}$. Since γ is real, ω is clearly an even function. In this case, it is well known (see e.g. [45, Theorem 11.2]) that $\rho_n(z) = z^{n-1}(z + \gamma)$ for all $n \geq 1$ and hence $\delta_0 = 1$, $\delta_1 = \gamma$, and $\delta_k = 0$ for all $k \geq 2$. Thus, from (13) it results that,*

$$a_1 = \frac{1}{2}\sqrt{2(1 - \gamma^2)}, \quad b_0 = -\gamma \quad b_1 = \frac{\gamma}{2}, \quad a_k = 1 \quad \text{and} \quad b_k = 0, \text{ for all } k \geq 2.$$

In general, for our purposes the calculations can be arranged as follows. Suppose that the Verblunsky parameters $\{\delta_k\}_{k=0}^{\infty}$ for the symmetric weight function ω on $[-\pi,\pi]$ are known and set

$$\sigma_{r,s}(x) = (1-x)^r(1+x)^s\sigma(x), \quad r,s\in\{0,1\}, \quad x\in[-1,1],$$

where $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$.

Let $\mathcal{J}^{r,s}$ denote the Jacobi matrix associated with $\sigma_{r,s}(x)$. Thus, $\mathcal{J}^{0,0}$ is the Jacobi matrix for $\sigma(x)$ whose entries are directly given by Theorem 5.1. Set the LU decomposition $\mathcal{J}^{0,0} = L^{0,0}U^{0,0}$. Then, by (3) we have that $\mathcal{J}^{1,0} = U^{0,0}L^{0,0}+I$ and $\mathcal{J}^{0,1} = U^{0,0}L^{0,0}-I$. Finally, by considering the LU decomposition of $\mathcal{J}^{1,0}$, that is $\mathcal{J}^{1,0} = L^{1,0}U^{1,0}$, then $\mathcal{J}^{1,1} = U^{1,0}L^{1,0}-I$. Once the Jacobi matrices $\mathcal{J}^{r,s}$ have been determined, the computation of the symmetric Szegő-type quadratures for ω or equivalently the Gauss-type rules for σ is a straightforward task.

However, as for the computation of the Jacobi matrices $\mathcal{J}^{r,s}$, with $r,s\in\{0,1\}$, we might also think of the following alternative approach. Indeed, for $z=e^{i\theta}$ and $r,s\in\{0,1\}$, set

$$\omega_{r,s}(\theta) = \sigma_{r,s}(\cos\theta)|\sin\theta| = (1-\cos\theta)^r(1+\cos\theta)^s\omega(\theta) = \frac{1}{2^{r+s}}|z+r|^2|z+s|^2\omega(\theta),$$

and consider the trigonometric moments $\mu_k^{r,s} = \int_{-\pi}^{\pi} e^{-ik\theta}\omega_{r,s}(\theta)d\theta$. Then, it can be checked for all $k=0,1,\dots$ that

$$\mu_k^{r,s} = \frac{1}{2^{r+s}}\left\{[(r-s)^2+1+r^2s^2]\mu_k + [s(1+r^2)-r(1+s^2)](\mu_{k-1}+\mu_{k+1}) - rs(\mu_{k-2}+\mu_{k+2})\right\}. \tag{14}$$

Thus, starting from the trigonometric moments μ_k for ω we can compute the moments $\mu_k^{r,s}$ for $\omega_{r,s}$ by (14) and then from here, making use of the Levinson´s algorithm, the corresponding Verblunsky parameters $\delta_k^{r,s}$ for $\omega_{r,s}(\theta)$ can be computed. Finally, from Theorem 5.1 we deduce the Jacobi matrices $\mathcal{J}^{r,s}$. However, since the computations to generate the coefficients $\delta_k^{1,0}$ or $\delta_k^{0,1}$ can not be stored, in general, to compute $\delta_k^{1,1}$ this way seems to be much more expensive and longer. Even in the case where we dispose of the Verblunsky parameters $\delta_k^{0,0} = \delta_k$, $k=0,1,\dots$ for ω and although we can deduce an explicit relation between the sequences $\{\delta_k\}_0^{\infty}$ and $\{\delta_k^{r,s}\}_0^{\infty}$ with $r,s\in\{0,1\}$ (see [23]), this process involves so many calculations that it does not seem advisable. For instance, for all $n\geq 1$ it holds that,

$$\delta_n^{1,0} = \frac{\rho_{n+1}(1)\rho_n(1)}{k_nK_n(1,1)} - \delta_{n+1},$$

$\{\rho_k\}_{k=0}^{\infty}$ being the sequence of monic Szegő polynomial for ω, $k_n = \|\rho_n\|_{\omega}^2 = \prod_{j=1}^{n}(1-|\delta_j|^2)$ and where $K_n(x,y)$ denotes the reproducing kernel for $\mathcal{P}_n$ with respect to the inner product induced by ω, that is

$$K_n(x,y) = \sum_{j=0}^{n}\frac{\rho_j(x)\overline{\rho_j(y)}}{k_j}.$$

From the relation (see [41, pp. 57-58])

$$\rho_n(1) = \sqrt{k_n} \prod_{j=1}^{n} \sqrt{\frac{1+\delta_j}{1-\delta_j}}, \quad n \geq 1,$$

an explicit connection between $\{\delta_n^{1,0}\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ can be stated.

6 Numerical experiments involving Rogers-Szegő polynomials

As it is known, most of the examples about symmetric weight functions ω on $[-\pi, \pi]$ considered in the literature directly arise from weight functions σ on $[-1, 1]$ after making the change of variable $x = \cos\theta$ so that $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$. Thus, when some kind of information on the weight function σ in terms of moments or Jacobi parameters is available, the computation of the symmetric Szegő-type quadrature formulas for ω reduces to the computation of the Gauss-type quadrature for σ and very little has to be done. Hence, the interest appears when we have the usual available information on ω but little or no information on σ, apart from the above connection between both weight functions. This is the case of the symmetric weight function ω giving rise to the sequence of the so-called Rogers-Szegő polynomials, which will be used to illustrate the approach presented in the previous sections with some numerical experiments. This weight function is the "wrapped" Gaussian measure given by

$$\omega(\theta) = \omega(q;\theta) = \frac{1}{\sqrt{2\pi \log(1/q)}} \sum_{j=-\infty}^{+\infty} \exp\left(\frac{-(\theta - 2\pi j)^2}{2\log(1/q)}\right), \quad 0 < q < 1. \qquad (15)$$

Properties of Rogers-Szegő polynomials, the family of orthogonal polynomials on $\mathbb{T}$ with respect to ω given by (15) have been recently studied; see e.g. [41, Chapter 1.6] and its references along with [16]. In spite of the rather special shape of ω, it is surprising we have explicit information about the familiar parameters characterizing ω. For instance, the sequence of trigonometric moments is given by (see e.g. [41, Chapter 1.6]) $\mu_n = q^{\frac{n^2}{2}}$ for all $n \geq 0$ and the Verblunsky parameters by

$$\delta_n = (-1)^n q^{\frac{n}{2}},\ n = 0, 1, \ldots. \qquad (16)$$

Even more, the family of monic Rogers-Szegő polynomials is explicitly given by

$$\rho_n(z) = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{n-j}{2}} z^j,$$

where $\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{(n)_q}{(j)_q (n-j)_q}$ (q-binomial coefficient) with $(n)_q = (1-q)(1-q^2)\cdots(1-q^n)$. Observe that ω is clearly symmetric; hence there exists a weight function σ on $[-1, 1]$ such that $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$, but nothing is known about σ. The aim of this section is to perform some illustrative numerical experiments concerning the computation of symmetric Szegő-type quadratures for ω given by (15) and making use of (16) along the results of Section 5. These quadratures are of a great interest when dealing with the computation of integrals of the form $\int_{-\infty}^{\infty} f(x)e^{-\gamma x^2}dx$, with $\gamma > 0$ and f a 2π-periodic function (see [16]).

From (16) it follows that the Hessenberg matrices defined in (7) for the weight function $\omega(\theta)$ given by (15) have the form

$$H_n(\tau_n) = D_n^{-1/2} \begin{bmatrix} q^{1/2} & -q & \cdots & (-1)^n q^{(n-1)/2} & -\tau_n \\ 1-q & q^{3/2} & \cdots & (-1)^{n+1} q^{n/2} & q^{1/2}\tau_n \\ 0 & 1-q^2 & \cdots & (-1)^n q^{(n+1)/2} & -q\tau_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-q^{n-1} & (-1)^n q^{(n-1)/2}\tau_n \end{bmatrix} D_n^{1/2},$$

where $D_n = \operatorname{diag}[1, (1)_q, (2)_q, \ldots, (n-1)_q]$ and $\tau_n \in \mathbb{T}$. From Theorem 5.1, the coefficients of the Jacobi matrices for $\sigma(x)$ such that $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$ are given by,

$$a_n = \frac{1}{2}\sqrt{(1-q^n)(1-q^{2n-1})(1+q^{n-1})}, \quad n \geq 1 \quad \text{and} \quad b_n = \frac{1}{2}q^{n-\frac{1}{2}}\left(q^{n+1}+q^n+q-1\right), \quad n \geq 0.$$

Thus, we can compute the nodes and weights of the corresponding n-point symmetric Szegő rule for different values of n either via Hessenberg or via Jacobi matrices. A comparison has been made concerning the computational time in seconds required in the solution of both eigenvalue problems in MAPLE® 9.5[2] with 30 digits by using an standard routine and fixing $q = 0.2$ and $\tau_n = 1$. The results are displayed on Table 1 and they clearly show the advantage of Jacobi over Hessenberg. However, it should be recalled here that a whole variety of practical eigenvalue computation algorithms for unitary Hessenberg matrices has already been developed in the literature; see e.g. [2], [8]-[9], [20], [32] and [43].

Finally, let us recall that Szegő rules depend on a parameter $\tau \in \mathbb{T}$ so that when $\tau = \pm 1$, then symmetric formulas appear which can be efficiently computed via Jacobi matrices. If we take $\tau \in \mathbb{T}\backslash\{\pm 1\}$, the corresponding Szegő formulas are not symmetric anymore and its computation must be done by means of the Hessenberg matrices (7). With an illustrative character, an estimation of the integral $\int_{-\pi}^{\pi} g(\theta)\omega(\theta)d\theta$ with $\omega(\theta)$ given by (15) has been made by using n-point Szegő formulas with different values of τ. The absolute errors displayed in the tables below show that the symmetric rules ($\tau = 1$) produce similar results for the choices $\tau = \pm i$.

[2] MAPLE is a registered trademark of Waterloo Maple, Inc.

n	$Jacobi$	$Hessenberg$
100	0.016	0.0234
200	0.047	2.969
300	0.172	11.281
400	0.454	30.45
500	0.937	81.828
600	1.73	172

Table 1.— A comparision in seconds for the computation of an n point symmetric Szegő quadrature formula for the Rogers-Szegő weight function (15) with $q = 0,2$ and $\tau_n = 1$, via Jacobi or Hessenberg matrices.

n	$\tau = 1$	$\tau = i$	$\tau = -i$
6	$7.8219774E-03$	$6.0089760E-03$	$6.0089760E-03$
8	$1.1307038E-03$	$7.9659815E-05$	$7.9659815E-04$
10	$1.2796254E-04$	$8.2581333E-05$	$8.2581333E-05$
12	$1.07083165E-05$	$6.3268992E-06$	$6.3268992E-06$

Table 2.— A comparision of the absolute errors in the computation of an n point Szegő quadrature formula for the Rogers-Szegő weight function (15) with $q = 0,9$, $g(\theta) = (\cos\theta)^{19}$ and different values of τ.

Acknowledgments

This work is partially supported by Dirección General de Programas y Transferencia de Conocimiento, Ministerio de Ciencia e Innovación of Spain under grant MTM 2008-06671. A. Bultheel acknowledges financial support from the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. F. Perdomo-Pío has been partially supported by Grant of Agencia Canaria de Investigación, Innovación y Sociedad de la Información del Gobierno de Canarias.

References

[1] **M. Alfaro and J.M. Montaner**, *On five-diagonal Toeplitz matrices and orthogonal polynomials on the unit circle*, Numer. Algor. 10(1-2) (1995) 137-154.

[2] **G.S. Ammar, L. Reichel and D.C. Sorensen**, *An implementation of a divide and conquer algorithm for the unitary eigenproblem*, ACM Trans. Math. Software 18(3) (1992) 292-307.

n	$\tau = 1$	$\tau = i$	$\tau = -i$
6	$4.1917414E-06$	$1.8719070E-06$	$3.0695894E-06$
8	$1.2989586E-07$	$6.3949537E-08$	$8.1546108E-08$
10	$3.9410378E-09$	$2.0120779E-09$	$2.27096253E-09$
12	$1.1791645E-10$	$6.11319328E-11$	$6.4948880E-11$

Table 3.— A comparision of the absolute errors in the computation of an n point Szegő quadrature formula for the Rogers-Szegő weight function (15) with $q = 0,5$, $g(\theta) = (\sin\theta + 3)^{-1}$ and different values of τ.

n	$\tau = 1$	$\tau = i$	$\tau = -i$
6	$7.5024567E-06$	$3.8286286E-05$	$3.8340752E-05$
8	$2.1913111E-07$	$1.1277238E-06$	$1.1280409E-06$
10	$6.4403534E-09$	$3.3200865E-08$	$3.3202737E-08$
12	$1.8952515E-10$	$9.7736602E-10$	$9.7737363E-10$

Table 4.— A comparision of the absolute errors in the computation of an n point Szegő quadrature formula for the Rogers-Szegő weight function (15) with $q = 0,2$, $g(\theta) = \frac{\cos\theta}{\sin\theta+3}$ and different values of τ.

[3] **E. Berriochoa, A. Cachafeiro and F. Marcellán**, *Differential properties for Sobolev orthogonality on the unit circle*, J. Comp. Appl. Math. 133 (2001) 231-239.

[4] **M.I. Bueno and F. Marcellán**, *Darboux transformation and perturbation of linear functionals*, Lin. Alg. Appl. 384 (2004), 215-242.

[5] **A. Bultheel, L. Daruis and P. González-Vera**, *Quadrature formulas on the unit circle with prescribed nodes and maximal domain of validity*, J. Comp. Appl. Math. 231(2) (2009) 948-963.

[6] **A. Bultheel, L. Daruis and P. González-Vera**, *A connection between quadrature formulas on the unit circle and the interval* $[-1, 1]$, Appl. Numer. Math. 132 (2001) 1-14.

[7] **A. Bultheel and R. Cruz-Barroso and M. Van Barel**, *On Gauss-type quadrature formulas with prescribed nodes anywhere on the real line.* Calcolo, to appear 2010, doi: 10.1007/s10092-009-0013-x.

[8] **A. Bunste-Gerstner and L. Elsner**, *Schur parameter pencils for the solution of the unitary eigenproblem*, Lin. Alg. Appl. (1992) 741-778.

[9] **A. Bunste-Gerstner and C. He**, *On a Sturm sequence of polynomials for unitary Hessenberg matrices*, SIAM J. Matrix Anal. Appl. 16(4) (1995) 1043-1055.

[10] **A. Cachafeiro and F. Marcellán**, *Orthogonal polynomials of Sobolev type on the unit circle*, J. Approx. Theory 78 (1994) 127-146.

[11] **M.J. Cantero, R. Cruz-Barroso and P. González-Vera**, *A matrix approach to the computation of quadrature formulas on the unit circle*, Appl. Num. Math. 58 (2008) 296-318.

[12] **M.J. Cantero, L. Moral and L. Velázquez**, *Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle*, Lin. Alg. Appl. 362 (2003) 29-56.

[13] **R. Cruz-Barroso, L. Daruis, P. González-Vera, O. Njåstad**, *Sequences of orthogonal Laurent polynomials, biorthogonality and quadrature formulas on the unit circle*, J. Comp. Appl. Math. 200 (2007) 424-440.

[14] **R. Cruz-Barroso and S. Delvaux**, *Orthogonal Laurent polynomials on the unit circle and snake-shaped matrix factorizations*, J. Approx. Theory 161 (2009) 65-87.

[15] **R. Cruz-Barroso, P. González-Vera and F. Perdomo Pío**, *Orthogonality, interpolation and quadratures on the unit circle and the interval* $[-1, 1]$, J. Comp. Appl. Math. To appear 2010, doi: 10.1016/j.cam.2009.12.021.

[16] **R. Cruz-Barroso, P. González-Vera and F. Perdomo Pío**, *Quadrature formulas associated with Rogers-Szegő polynomials*, Comp. Math. Appl. 57 (2009) 308-323.

[17] **R. Cruz-Barroso, P. González-Vera and F. Perdomo Pío**, *Rational approximants associated with measures supported on the unit circle and the real line*, Appl. Num. Math. To appear 2010.

[18] **R. Cruz-Barroso, P. González-Vera and O. Njåstad**, *On bi-orthogonal systems of trigonometric functions and quadrature formulas for periodic integrands*, Numer. Algor. 44 (2007) 309-333.

[19] **P. Delsarte, Y. Genin**, *The Split Levinson Algorithm*, IEEE Trans Acoust. Spreech, Signal Proc. ASSP-34 (1986) 470-478.

[20] **S. Delvaux and M. Van Barel**, *Eigenvalue computation for unitary rank structured matrices*, J. Comp. Appl. Math. 213(1) (2008) 268-287.

[21] **W. Gautschi**, *Orthogonal Polynomials Computation and Approximation*, Oxford University Press, Oxford (2004).

[22] **W. Gautschi**, *Orthogonal polynomials, quadrature, and approximation: computational methods and software (in matlab)*, In: Orthogonal Polynomials and Special Functions. Lecture Notes in Mathematics 1883, Springer, Berlin (2006) 1-77.

[23] **L.E. Garza Gaona and F. Marcellán**, *Spectral transformations of measures supported on the unit circle and the Szegő transformation*, Numer. Algor. 49 (2008) 169-185.

[24] **Ya.L. Geronimus**, *On the trigonometric moment problem*, Ann. Math. 47(2) (1946) 742-761.

[25] **Ya.L. Geronimus**, *Polynomials orthogonal on a circle and their applications*, Amer. Math. Soc. Transl. 1, Providence, RI (1962) 1-78.

[26] **E. Godoy and F. Marcellán**, *Orthogonal polynomials on the unit circle: Distribution of zeros*, J. Comp. Appl. Math. 37(1-3) (1991) 195-208.

[27] **P. González-Vera, J.C. Santos-León and O. Njåstad**.- *Some results about numerical quadrature on the unit circle*, Adv. Comp. Math. 5 (1996) 297-328.

[28] **G. Golub**, *Some modified matrix eigenvalue problems*, SIAM Rev. 15(2) (1973) 318-334.

[29] **U. Grenander and G. Szegő**, *Toeplitz forms and their applications*, Univ. of California Press, Berkeley 1958, Chelsea, New-York, 2nd edition, 1984.

[30] **W.B. Gragg**, *Positive definite Toeplitz matrices, the Arnoldi process for isometric operators and Gaussian quadrature on the unit circle*, In E.S. Nicholaev Ed., Numer. Meth. Lin. Alg., Moscow University Press, Moscow (1982) 16-32 (in Russian).

[31] **W.B. Gragg**, *Positive definite Toeplitz matrices, the Arnoldi process for isometric operators and Gaussian quadrature on the unit circle*, J. Comp. Appl. Math. 46 (1993) 183-198. This is a slightly revised version of [30].

[32] **W.B. Gragg**, *The QR algorithm for unitary Hessenberg matrices*, J. Comp. Appl. Math. 16 (1986) 1-8.

[33] **C. Jagels and L. Reichel**, *On the construction of Szegő polynomials*, J. Comp. Appl. Math. 46 (1993) 241-254.

[34] **C. Jagels and L. Reichel**, *Szegő-Lobatto quadrature rules*, J. Comp. Appl. Math. 200 (2007) 116-126.

[35] **W.B. Jones, O. Njåstad and W.J. Thron**, *Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle*, Bull. London Math. Soc. 21 (1989) 113-152.

[36] **V.I. Krylov**, *Approximate Calculation of Integrals*, The MacMillan Company, New York, 1962.

[37] **N. Levinson**, *The Wiener RMS (root mean square) error criterion in filter design and prediction*, J.Math. Phys. 25 (1947) 261-278.

[38] **O. Njåstad and J.C. Santos-León**, *Domain of validity of Szegő quadrature formulas*, J. Comp. Appl. Math. 202(2) (2007) 440-449.

[39] **B. Simon**, *CMV matrices: Five years after*, J. Comp. Appl. Math. 208(1) (2007) 120-154.

[40] **B. Simon**, *OPUC on one foot*, Bull. Amer. Math. Soc. 42 (2005) 431-460.

[41] **B. Simon**, *Orthogonal Polynomials on the unit circle, Part 1: Classical Theory*, Amer. Math. Soc. Colloq. Publ. 54.1, Amer. Math. Soc., Providence, RI, 2005.

[42] **B. Simon**, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, Amer. Math. Soc. Colloq. Publ. 54.2, Amer. Math. Soc., Providence, RI, 2005.

[43] **M. Stewart**, *An error analysis of a unitary Hessenberg QR algorithm*, SIAM J. Matrix Anal. Appl. 28(1) (2006) 40-67.

[44] **G. Szegő**, *On bi-orthogonal systems of trigonometric polynomials*, Magyar Tud. Akad. Kutató Int. Közl 8 (1963) 255-273.

[45] **G. Szegő**, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ. Vol 23, Amer. Math. Soc. Providence, R.I. 1975.

[46] **D. Watkins**, *Some perspectives on the eigenvalue problem*, SIAM Review 35(3) (1993) 430-471.