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AV-bundles, Lie algebroid theory and the inhomogeneous cosymplectic formulation of the dynamics in jet manifolds

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To my friend Pepin Cariñena on the ocassion of his 60th birthday

Abstract

In this paper, we develop a cosymplectic inhomogeneous formulation for a (regular) Lagragian system whose Lagrangian is a section of an AV-bundle Z^1 over the evolution space and such that Z^1 satisfies certain properties. The Lie algebroid theory is used. This general construction is applied to a particular example: Newtonian mechanics in a Newtonian space-time.

Key words: *AV-bundles, Lie algebroids, cosymplectic geometry, inhomogeneous formulation, Lagrangian dynamics, Hamiltonian dynamics, Newtonian Mechanics.*

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1 Introduction

The most natural geometric framework for studying mechanical systems is a fibred manifold $\pi : E \to \mathbb{R}$. In fact, E is the configuration manifold and the 1-jet manifold $J^1\pi$ of 1-jets of local sections of π is the evolution space. The Lagrangian function will be a real C^∞-function $L : J^1\pi \to \mathbb{R}$ defined on $J^1\pi$ and in the particular case when L is regular the corresponding Hamiltonian section $h : V^*\pi \to T^*E$ is a section of the canonical projection $\mu : T^*E \to V^*\pi$, where $V\pi$ is the vertical bundle to π. Moreover, one may construct a cosymplectic structure on $J^1\pi$ (respectively, $V^*\pi$) and the solutions of the Euler-Lagrange equations (respectively, the Hamilton equations) are the integral curves of the corresponding Reeb vector field (see [9, 10]; see also [1, 3]). Note that $J^1\pi$ is an affine bundle over E modelled on the vector bundle $V\pi \to E$ and that $\mu : T^*E \to V^*\pi$ is a AV-bundle (in the terminology of [5]). So, the affine character is present in the theory (we recall that AV-bundles were introduced in [5] as affine line bundles which are modelled on trivial vector lines bundles).

On the other hand, there are some physical theories where we find difficulties when we interpret the Lagrangian as a real function on $J^1\pi$. For instance, in the standard geometric inhomogeneous formulation of Newtonian Mechanics in a Newtonian space-time. In fact, in this formulation there is a strong dependence on the inertial frame chosen. Different Lagrangian (and different Hamiltonians) are used for different inertial frames. In [6], a nice frame independent inhomogeneous (homogeneous) formulation of analytical mechanics in Newtonian space-time is presented. The AV-differential geometry is widely used. In fact, the inhomogenous (homogeneous) Lagrangian is interpreted as a section of a certain AV-bundle.

The aim of this Note is to develop a cosymplectic inhomogeneous formulation for a (regular) Lagrangian system whose Lagrangian is a section l of an AV-bundle Z^1 over $J^1\pi$, Z^1 satisfying certain properties. For this purpose, the Lie algebroid theory will be used. The resultant general construction may be applied to the particular example which was discussed in [6] and, as consequence, we obtain a cosymplectic inhomogeneous formulation of Newtonian Mechanics in a Newtonian space-time. In addition, in the particular case when the AV-bundle Z^1 is trivial then the section l is a Lagrangian function on $J^1\pi$ and one recovers some classical results about the standard cosymplectic inhomogeneous formulation of time-dependent Mechanics.

The Note is structured as follows. In Section 2, we recall some definitions and results about Lie algebroids and linear Poisson structures, AV-bundles and some geometrical structures on $J^1\pi$. In Section 3, we discuss the inhomogeneous cosymplectic formulation of the Lagrangian (Hamiltonian) dynamics on jet manifolds and its relation with the AV differential geometry and the Lie algebroid theory. Finally, in Section 4 we apply our results to a particular example: Newtonian mechanics in a Newtonian space-time.

2 Preliminaries

2.1 Lie algebroids and linear Poisson structures

Let A be a vector bundle of rank m over the manifold E of dimension n and $\tau_A : A \to E$ be the vector bundle projection. Denote by $\Gamma(\tau_A)$ the $C^\infty(E)$-module of sections of $\tau_A : A \to E$. A *Lie algebroid structure* $([\![\cdot,\cdot]\!]_A, \rho_A)$ on A is a Lie bracket on the space $\Gamma(\tau_A)$ and a bundle map $\rho_A : A \to TE$, called the *anchor map,* such that if we also denote by $\rho_A : \Gamma(\tau_A) \to \mathfrak{X}(E)$ the homomorphism of $C^\infty(E)$-modules induced by the anchor map then $[\![X, fY]\!]_A = f[\![X,Y]\!]_A + \rho_A(X)(f)Y$, for $X, Y \in \Gamma(\tau_A)$ and $f \in C^\infty(E)$. The triple $(A, [\![\cdot,\cdot]\!]_A, \rho_A)$ is called a Lie algebroid over E (see [11]). In such a case, the anchor map $\rho_A : \Gamma(\tau_A) \to \mathfrak{X}(E)$ is a homomorphism between the Lie algebras $(\Gamma(\tau_A), [\![\cdot,\cdot]\!]_A)$ and $(\mathfrak{X}(E), [\cdot,\cdot])$. A natural example of Lie algebroid is the tangent bundle TE of a manifold E.

If $([\![\cdot,\cdot]\!]_A, \rho_A)$ is a Lie algebroid structure on a vector bundle $\tau_A : A \to E$ then the dual bundle $\tau_A^* : A^* \to E$ admits a linear Poisson structure Π_{A^*}. Moreover, if $\{\cdot,\cdot\}_{A^*}$ is the Poisson bracket associated with Π_{A^*} then $\{\cdot,\cdot\}_{A^*}$ is characterized by the following relations

$$\{f \circ \tau_A^*, g \circ \tau_A^*\}_{A^*} = 0, \quad \{\hat{X}, g \circ \tau_A^*\}_{A^*} = \rho_A(X)(g) \circ \tau_A^*, \quad \{\hat{X}, \hat{Y}\}_{A^*} = \widehat{[\![X,Y]\!]}_A, \quad (2.1)$$

for $X, Y \in \Gamma(\tau_A)$ and $f, g \in C^\infty(E)$. Here, $\hat{X}$ denotes the linear function on A^* induced by X. In the particular case when A is the tangent bundle to E then the linear Poisson structure of $A^* = T^*E$ is just the canonical symplectic structure on T^*E (see [2]).

On the other hand, if Π_{A^*} is a linear Poisson structure on the vector bundle $\tau_A^* : A^* \to E$ then Π_{A^*} induces a Lie algebroid structure $([\![\cdot,\cdot]\!]_A, \rho_A)$ on the vector bundle $\tau_A : A \to E$ which is given by (2.1) (see [2]).

Finally, if $([\![\cdot,\cdot]\!]_A, \rho_A)$ and $([\![\cdot,\cdot]\!]_{A'}, \rho_{A'})$ are Lie algebroid structures on the vector bundles $\tau_A : A \to E$ and $\tau_{A'} : A' \to E$ and $\Phi : A \to A'$ is a vector bundle morphism (over the identity of E) between A and A' then Φ is a *Lie algebroid morphism* (that is, $\Phi[\![X,Y]\!]_A = [\![\Phi X, \Phi Y]\!]_{A'}$ and $\rho_{A'}(\Phi X) = \rho_A(X)$, for $X, Y \in \Gamma(\tau_A)$) if and only if the dual map $\Phi^* : (A')^* \to A^*$ is a *Poisson morphism* (that is, $\{f \circ \Phi^*, g \circ \Phi^*\}_{(A')^*} = \{f, g\}_{A^*} \circ \Phi^*$, for $f, g \in C^\infty(A^*)$).

2.2 AV-bundles

Let $\tau_Z : Z \to M$ be an affine bundle of rank 1 over a manifold M modelled on the trivial vector bundle $\tau_{M\times\mathbb{R}} : M \times \mathbb{R} \to M$, that is, $\tau_Z : Z \to M$ is an *AV-bundle* in the terminology of [4]. Then, we have an action of $\mathbb{R}$ on each fiber of Z. This action induces a vector field X_Z on Z which is vertical with respect to the projection $\tau_Z : Z \to M$.

On the other hand, there exists a one-to-one correspondence between the space of sections of $\tau_Z : Z \to M$, $\Gamma(\tau_Z)$, and the set $\{F_l \in C^\infty(Z)/X_Z(F_l) = 1\}$. In fact, if $l \in \Gamma(\tau_Z)$ and (x^i, s) are local fibred coordinates on Z such that $X_Z = \dfrac{\partial}{\partial s}$ then l may be considered a local function L on M, $x^i \to L(x^i)$, and the function F_l on Z is locally given by $F_l(x^i, s) = L(x^i) + s$ (for more details, see [4]).

2.3 Some geometrical structures on $J^1\pi$

Let E be an (n+1)-dimensional fibred manifold over $\mathbb{R}$, i.e., there exists a surjective submersion $\pi : E \to \mathbb{R}$. We denote by $J^1\pi$ the 1-jet manifold of local sections of π, namely

$$J^1\pi = \{j_t^1\phi/\phi : U \subseteq \mathbb{R} \to E, \pi \circ \phi = id_U, U \text{ open neighbourhood of } t\}.$$

If (t, q^A) are fibred coordinates on E, then $J^1\pi$ has local coordinates (t, q^A, v^A). In fact, if $\phi(s) = (s, \phi^A(s))$, $s \in U$, then $j_t^1\phi$ has coordinates $(t, \phi^A(t), \dfrac{d\phi^A}{ds}(t))$. Therefore, $J^1\pi$ has dimension $2n+1$ and it is a fibred manifold over E and $\mathbb{R}$ with canonical projections $\pi_{1,0}$: $J^1\pi \to E$ and $\pi_1 : J^1\pi \to \mathbb{R}$, respectively. In local coordinates we have $\pi_{1,0}(t, q^A, v_A) = (t, q^A)$ and $\pi_1(t, q^A, v^A) = t$. We define a canonical embedding $i : J^1\pi \to TE$ as follows $i(j^1_t\phi) = \dot{\phi}(t)$, where $\dot{\phi}(t) \in T_{\phi(t)}E$ is the tangent vector at t of the curve $s \to \phi(s)$. If we take local coordinates (t, q^A, τ, τ^A) on TE, we have $i(t, q^A, v^A) = (t, q^A, 1, v^A)$. Now, denote by η_E the 1-form on E given by $\eta_E = \pi^*(dt)$ and by $V\pi$ the vertical bundle of $\pi : E \to \mathbb{R}$. It follows that

$$J^1\pi \cong i(J^1\pi) = \{v \in TE/\eta_E(v) = 1\}, \qquad V\pi = \{v \in TE/\eta_E(v) = 0\}.$$

Thus, $J^1\pi$ is an affine subbundle (over E) of the vector bundle $\tau_{TE} : TE \to E$ which is modelled over the vector subbundle $\tau_{V\pi} : V\pi \to E$. Note that the dual bundle $(J^1\pi)^+$ to $J^1\pi$ is isomorphic to the cotangent bundle T^*M to M. So, the bidual bundle to $J^1\pi$ may be identified with the tangent bundle TE of E.

On the other hand, there exists a canonical endomorphism $\tilde{S}$ of $TJ^1\pi$ which is called the *vertical endomorphism.* $\tilde{S}$ is a vector field of type $(1, 1)$ on $J^1\pi$ defined as follows. If $\tilde{X} \in T_{j^1_t\phi}(J^1\pi)$ then $(T\pi_{1,0})(\tilde{X}) - T\phi((T\pi_1)(\tilde{X})) \in (V\pi)_{\phi(t)}$ and we define

$$\tilde{S}\tilde{X} = ((T\pi_{1,0})(\tilde{X}) - T\phi((T\pi_1)(\tilde{X})))^v_{j^1_t\phi},$$

where ${}^v_{j^1_t\phi} : (V\pi)_{\phi(t)} \to T_{j^1_t\phi}(J^1\pi)$ denotes the vertical lift. The local expression of $\tilde{S}$ is

$$\tilde{S} = (dq^A - v^A dt) \otimes \frac{\partial}{\partial v^A}.$$

A vector field ξ on $J^1\pi$ is a *non-autonomous second order differential equation* (NSODE for simplicity) if $\tilde{S}(\xi) = 0$ and $\eta(\xi) = 1$, η being the 1-form on $J^1\pi$ given by $\eta = (\pi_1)^*(dt)$. The vector field ξ is a NSODE if and only if it has the following local expression

$$\xi(t, q^A, v^A) = \frac{\partial}{\partial t} + v^A \frac{\partial}{\partial q^A} + \xi^A \frac{\partial}{\partial v^A}.$$

A local section ϕ is $\pi : E \to \mathbb{R}$ is an *integral section* of a NSODE ξ if the 1-jet prolongation $j^1\phi$ of ϕ to $J^1\pi$ is an integral curve of ξ. Thus, $t \to \phi(t) = (t, \phi^A(t))$ is an integral section of ξ if and only if it satisfies the following system of non-autonomous differential equations of second order

$$\frac{d^2\phi^A}{dt^2} = \xi^A(t, \phi^B, \frac{d\phi^B}{dt}), \qquad \frac{d\phi^A}{dt} = v^A.$$

It should be remarked that an integral curve γ of a NSODE ξ is necessarily a 1-jet prolongation, say $\gamma = j^1\phi$, where ϕ is an integral section of ξ (for more details, see [13]).

3 AV-bundles, Lie algebroid theory and the inhomogeneous cosymplectic formulation of the Lagrangian (Hamiltonian) dynamics

Let $\pi : E \to \mathbb{R}$ be a fibration from a manifold E of dimension $n+1$ on $\mathbb{R}$. Suppose that $\zeta_{Z^1} : Z^1 \to E$ is an affine bundle modelled on the vector bundle $\zeta_{V^1} : V^1 \to E$ of rank $n+1$. Assume also that $\tau_{Z^1} : Z^1 \to J^1\pi$ is an epimorphism of affine bundles, that $\tau_{V^1} : V^1 \to V\pi$ is the corresponding epimorphism of vector bundles and that $e_0 : E \to V^1$ is a section of $\zeta_{V^1} : V^1 \to E$ such that $\ker \tau_{V^1}(y) =< e_0(y) >$, for all $y \in E$. Now, let $\zeta_Z : Z \to E$ be the bidual bundle to $\zeta_{Z^1} : Z^1 \to E$. Then, the epimorphism of affine bundles $\tau_{Z^1} : Z^1 \to J^1\pi$ induces an epimorphism of vector bundles $\tau_Z : Z \to TE$ and $\ker \tau_Z(y) =< e_0(y) >$, for all $y \in E$ (we recall that the bidual bundle to $J^1\pi$ is isomorphic to the tangent bundle of E). Moreover, it is clear that $\tau_{Z^1} : Z^1 \to J^1\pi$ and $\tau_Z : Z \to TE$ are AV-bundles.

On the other hand, if $i^1 : V^1 \to Z$ is the canonical inclusion then, since the pair $(Z, i^1 \circ e_0)$ is an special vector bundle over E (in the terminology of [4]), one may consider the affine dual bundle of Z as the affine subbundle $Z^\ddagger$ of Z^* defined by

$$Z^\ddagger = \{\varphi \in Z^* / (\widehat{i^1 \circ e_0})(\varphi) = 1\}.$$

$Z^\ddagger$ is an affine bundle modelled on the vector bundle $\tau_{T^*E} : T^*E \to E$. As we know, T^*E admits a canonical symplectic structure.

Next, we will analyse a particular class of affine symplectic structures on $Z^\ddagger$.

Let $\Omega_{Z^\ddagger}$ be an affine symplectic structure on $Z^\ddagger$. In other words, $\Omega_{Z^\ddagger}$ is a closed nondegenerate 2-form on $Z^\ddagger$ and the Poisson bracket of two affine functions on $Z^\ddagger$ is an affine function. Then, using some results which were proved in [7] (see Corollary 3.9 in [7]), we deduce that $\Omega_{Z^\ddagger}$ induces a linear Poisson structure Π_{Z^*} on Z^* such that

$$\ker \Pi_{Z^*} =< d(\widehat{i^1 \circ e_0}) > . \tag{3.1}$$

Conversely, if Π_{Z^*} is a linear Poisson structure on Z^* and (3.1) holds then, using again Corollary 3.9 in [7], we have that Π_{Z^*} restricts to a nondegenerate affine Poisson structure on $Z^\ddagger$. In other words, Π_{Z^*} induces an affine symplectic structure on $Z^\ddagger$.

Thus, we conclude that there exists a one-to-one correspondence between affine symplectic structures on $Z^\ddagger$ and linear Poisson structures on Z^* such that (3.1) holds.

Now, we will consider Lie algebroid structures $([\![\cdot,\cdot]\!]_Z, \rho_Z)$ on the vector bundle $\zeta_Z : Z \to E$ such that:

(C1) The map $\tau_Z : Z \to TE$ is an epimorphism of Lie algebroids (over the identity of E) and

(C2) The section $i^1 \circ e_0$ is a central element in the Lie algebra $(\Gamma(\zeta_Z), [\![\cdot,\cdot]\!]_Z)$.

In fact, we will introduce the set $\mathcal{A}_Z$ given by

$$\begin{aligned} \mathcal{A}_Z \;=\; & \{([\![\cdot,\cdot]\!]_Z, \rho_Z) / ([\![\cdot,\cdot]\!]_Z, \rho_Z) \text{ is a Lie algebroid structure on } Z \\ & \quad \text{which satisfies (C1) and (C2)}\}. \end{aligned}$$

On the other hand, we will denote by $\mathcal{S}_Z$ the set defined by

$$\begin{aligned}\mathcal{S}_Z &= \{\Omega_{Z^\ddagger}/\Omega_{Z^\ddagger} \text{ is an affine symplectic structure on } Z^\ddagger \\ &\quad \text{and } \tau_Z^* : T^*E \to Z^* \text{ is a Poisson morphism } \}.\end{aligned}$$

Then, using the above results (see also Section 2.1), we have

Proposition 3.1 *There exists a one-to-one correspondence between the sets $\mathcal{A}_Z$ and $\mathcal{S}_Z$.*

Using the Poincaré Lemma, one may prove the following result.

Proposition 3.2 *If $(\llbracket\cdot,\cdot\rrbracket_Z, \rho_Z)$ is an element of the set $\mathcal{A}_Z$ then the Lie algebroid $(Z, \llbracket\cdot,\cdot\rrbracket_Z, \rho_Z)$ is locally isomorphic to the standard Lie algebroid $\tau_{TE} \circ pr_1 : TE \times \mathbb{R} \to E$.*

We recall that the standard Lie algebroid structure $(\llbracket\cdot,\cdot\rrbracket_{TE\times\mathbb{R}}, \rho_{TE\times\mathbb{R}})$ on the vector bundle $\tau_{TE} \circ pr_1 : TE \times \mathbb{R} \to E$ is given by

$$\llbracket(X, f), (Y, g)\rrbracket_{TE\times\mathbb{R}} = ([X, Y], X(g) - Y(f)), \quad \rho_{TE\times\mathbb{R}}(X, f) = X,$$

for $(X, f), (Y, g) \in \mathfrak{X}(E) \times C^\infty(E)$.

In the rest of this Section, we will assume that $\Omega_{Z^\ddagger}$ is an element of $\mathcal{S}_Z$ or, equivalently, that we have a Lie algebroid structure $(\llbracket\cdot,\cdot\rrbracket_Z, \rho_Z)$ on the vector bundle $\zeta_Z : Z \to E$ which belongs to the set $\mathcal{A}_Z$.

Remark 3.3 Since $\tau_Z^*(\eta_E)$ is a 1-cocycle for the Lie algebroid $(Z, \llbracket\cdot,\cdot\rrbracket_Z, \rho_Z)$, we deduce that Z^1 is a *Lie affgebroid* (see [5, 12]) for the definition of a Lie affgebroid) and that the map $\tau_{Z^1} : Z^1 \to J^1\pi$ is an epimorphism of Lie affgebroids (see [8] for the definition of a morphism of Lie affgebroids). $\diamondsuit$

Next, we consider the affine dual bundle $(V^1)^\ddagger$ of the special vector bundle (V^1, e_0), that is, $(V^1)^\ddagger = \{\psi \in (V^1)^*/\hat{e}_0(\psi) = 1\}$. Then, one may define an epimorphism $\mu : Z^\ddagger \to (V^1)^\ddagger$ between the affine bundles $Z^\ddagger \to E$ and $(V^1)^\ddagger \to E$ given by

$$\mu(\varphi) = \varphi_{|V^1_y}, \text{ for } \varphi \in Z^\ddagger_y \text{ and } y \in E.$$

Now, we will obtain the local expressions of the 2-form $\Omega_{Z^\ddagger}$ and the projection $\mu : Z^\ddagger \to (V^1)^\ddagger$.

Using Proposition 3.2, we may choose local coordinates (t, q^A, v^A) on $J^1\pi$ as in Section 2.3 and a local basis $\{e, e_A, e_0\}$ of $\Gamma(\zeta_Z)$ such that $\tau_Z(e) = \frac{\partial}{\partial t}, \tau_Z(e_A) = \frac{\partial}{\partial q^A}$ and $\llbracket e, e_A\rrbracket_Z = \llbracket e_A, e_B\rrbracket_Z = 0$, for all A and B (note that e is a local section of the affine bundle $\zeta_{Z^1} : Z^1 \to E$). Thus, we have the corresponding local coordinates (t, q^A, v^A, v^0) on V^1 and Z^1 and the dual local coordinates (t, q^A, p_A, p_0) on $(V^1)^*$. We also may consider the corresponding local coordinates (t, q^A, v, v^A, v^0) on Z and the dual local coordinates (t, q^A, p, p_A, p_0) on

Z^*. Moreover, the local equation defining $Z^\ddagger$ (respectively, $(V_1)^\ddagger$) as an affine subbundle of Z^* (respectively, $(V^1)^*$) is $p_0 = 1$. Therefore, (t, q^A, p, p_A) (respectively, (t, q^A, p_A)) are local coordinates on $Z^\ddagger$ (respectively, $(V^1)^\ddagger$). Finally, using the above coordinates, we deduce that

$$\Omega_{Z^\ddagger} = dq^A \wedge dp_A + dt \wedge dp, \quad \mu(t, q^A, p, p_A) = (t, q^A, p_A). \tag{3.2}$$

Remark 3.4 Suppose that Z^1 is the trivial affine bundle $J^1\pi \times \mathbb{R}$, that $\tau_{Z^1} : J^1\pi \times \mathbb{R} \to J^1\pi$ is the canonical projection onto the first factor, that $e_0 : E \to V\pi \times \mathbb{R}$ is the section given by $e_0(y) = (0_y, 1)$, for $y \in E$ and that $([\![\cdot,\cdot]\!]_Z, \rho_Z)$ is the standard Lie algebroid structure on the vector bundle $TE \times \mathbb{R} \to E$. Then, $Z^\ddagger$ and $(V^1)^\ddagger$ may be identified with T^*E and $V^*\pi$, respectively, and, under these identifications, $\Omega_{Z^\ddagger}$ is just the canonical symplectic 2-form on T^*E and μ is the canonical projection from T^*E on $V^*\pi$. $\diamondsuit$

3.1 The Lagrangian formalism

3.1.1 Poincaré-Cartan 2-form and Legendre transformation

Suppose that $l : J^1\pi \to Z^1$ is a section of the projection $\tau_{Z^1} : Z^1 \to J^1\pi$. l will be called the *affine Lagrangian for the inhomogeneous formulation of the dynamics* (independent on the choice of the inertial frame).

If T^+E is the open subset of TE defined by $T^+E = \{v \in TE / \eta_E(v) > 0\}$ then l may be extended to a section $l^+ : T^+E \to Z$ of $\tau_Z : Z \to TE$ (over T^+E) given by

$$l^+(v) = \eta_E(v) i_{Z^1}(l(i^{-1}(\frac{v}{\eta_E(v)}))), \text{ for } v \in T^+E,$$

where $i : J^1\pi \to TE$ and $i_{Z^1} : Z^1 \to Z$ are the canonical inclusions. Note that $\dfrac{v}{\eta_E(v)} \in i(J^1\pi)$.

l^+ will be called the *affine Lagrangian for the homogeneous formulation of the dynamics* (independent on the choice of the inertial frame).

Since $\tau_Z : Z \to TE$ is an AV-bundle one may consider the vector field X_Z on Z induced by the action of $\mathbb{R}$ on Z and the real function $F_{l^+} : \tau_Z^{-1}(T^+E) \to \mathbb{R}$ induced by the section $l^+ : T^+E \to Z$. We have that $X_Z(F_{l^+}) = 1$ and, thus, we may define the map $Leg_l : J^1\pi \to Z^\ddagger$ given by

$$Leg_l(j^1_t\phi)(z') = \frac{d}{ds}_{|s=0} F_{l^+}(z + sz'),$$

for $z, z' \in Z_{\phi(t)}$, with $\tau_Z(z) = i(j^1_t\phi)$.

The map Leg_l is called *the extended Legendre transformation* associated with l.

The *Poincaré-Cartan* 2*-form* associated with l is the 2-form Ω_l on $J^1\pi$ given by $\Omega_l = Leg_l^*(\Omega_{Z^\ddagger})$. The *Legendre transformation* associated with l is the map $leg_l : J^1\pi \to (V^1)^\ddagger$ defined by $leg_l = \mu \circ Leg_l$.

If we choose local coordinates as above such that the local expression of l is

$$l(t, q^A, v^A) = (t, q^A, v^A, L(t, q^A, v^A))$$

then

$$l^+(t, q^A, \dot{t}, \dot{q}^A) = (t, q^A, \dot{t}, \dot{q}^A, \dot{t}L(t, q^A, \frac{\dot{q}^A}{\dot{t}}), \text{ for } \dot{t} > 0,$$

$$F_{l^+}(t, q^A, v, v^A, v^0) = vL(t, q^A, \frac{v^A}{v}) + v^0, \text{ for } v > 0$$

and

$$\begin{array}{rcl} Leg_l(t, q^A, v^A) & = & (t, q^A, L - v^A \dfrac{\partial L}{\partial v^A}, \dfrac{\partial L}{\partial v^A}), \\ leg_l(t, q^A, v^A) & = & (t, q^A, \dfrac{\partial L}{\partial v^A}), \\ \Omega_L(t, q^A, v^A) & = & (\dfrac{\partial^2 L}{\partial t \partial v^A} + v^B \dfrac{\partial^2 L}{\partial q^B \partial v^A} - \dfrac{\partial L}{\partial q^A}) w^A \wedge dt \\ & & - \dfrac{\partial^2 L}{\partial v^B \partial q^A} w^A \wedge w^B + \dfrac{\partial^2 L}{\partial v^A \partial v^B} w^A \wedge dv^B, \end{array} \tag{3.3}$$

where $w^A = dq^A - v^A dt$.

Remark 3.5 Under the same hypotheses as in Remark 3.4, the lagrangian section l may be considered as a Lagrangian function $L : J^1\pi \to \mathbb{R}$ and the 2-form Ω_l on $J^1\pi$ and the map $leg_l : J^1\pi \to (V^1)^\ddagger \cong V^*\pi$ are just the standard Poincaré-Cartan 2-form and the standard Legendre transformation associated with L. $\diamondsuit$

3.1.2 Euler-Lagrange equations and regular Lagrangians

If $\phi : I \subseteq \mathbb{R} \to E$ is a section of the projection $\pi : E \to \mathbb{R}$ then one may consider the 1-jet prolongation of ϕ, $j^1\phi : I \subseteq \mathbb{R} \to J^1\pi$ and its tangent lift $\frac{d(j^1\phi)}{dt} : I \subseteq \mathbb{R} \to TJ^1\pi$. The curve ϕ is a solution of the *Euler-Lagrange equations* for l if and only if

$$i_{\frac{d}{dt}(j^1\phi)} \Omega_l(j^1_t\phi) = 0, \quad \text{ for all } t.$$

If (t, q^A, v^A) are local coordinates on $J^1\pi$, $l(t, q^A, v^A) = (t, q^A, v^A, L(t, q^A, v^A))$ and $\phi(t) = (t, q^A(t))$ then, using (3.3), we deduce that ϕ is a solution of the Euler-Lagrange equations for l if and only if

$$\frac{d}{dt}(\frac{\partial L}{\partial v^A}) - \frac{\partial L}{\partial q^A} = 0, \quad v^A = \frac{dq^A}{dt}, \text{ for all } A.$$

We may assume that the above curves ϕ are the integral sections of a NSODE ξ. In such a case, ξ satisfies the following equation $i_\xi \Omega_l = 0$. In other words, we may reformulate geometrically our problem as search for a vector field ξ on $J^1\pi$ satisfying the following conditions

$$i_\xi \Omega_l = 0, \quad i_\xi \eta = 1, \quad \tilde{S}\xi = 0,$$

where η is the 1-form on $J^1\pi$ defined by $\eta = (\pi_1)^*(dt)$ and $\tilde{S}$ is the vertical endomorphism.

The affine Lagrangian section l is said to be *regular* if the pair (Ω_l, η) is a cosymplectic structure on $J^1\pi$, that is, $\eta \wedge \Omega_l^n = \eta \wedge \Omega_l \wedge \ldots^{(n} \ldots \wedge \Omega_l$ is a volume form on $J^1\pi$.

From (3.3), it follows that l is regular if and only if for each system of local coordinates (t, q^A, v^A) on $J^1\pi$ we have that the matrix $(\dfrac{\partial^2 L}{\partial v^A \partial v^B})$ is regular.

If l is regular then there exists a unique solution ξ_l of the equations $i_{\xi_l}\Omega_l = 0$ and $i_{\xi_l}\eta = 1$. In fact, ξ_l is the Reeb vector field of the cosymplectic structure (Ω_l, η). In addition, using (3.3), we deduce that ξ_l is a NSODE, that is, $\tilde{S}\xi_l = 0$. Therefore, the integral sections of ξ_l are just the solutions of Euler-Lagrange equations for l. ξ_l is called the *Euler-Lagrange vector field* associated with l.

Remark 3.6 Under the same hypotheses as in Remark 3.4, the regular affine Lagrangian section l may considered as a regular Lagrangian function $L : J^1\pi \to \mathbb{R}$ and ξ_l is the standard Euler-Lagrange vector field associated with L. $\diamondsuit$

3.2 The Hamiltonian formalism

The spaces $Z^\ddagger$ and $(V^1)^\ddagger$ are affine bundles over E modelled on the vector bundles

$$< (i^1 \circ e_0) >^0 = \{\varphi \in Z^* / (\widehat{i^1 \circ e_0})(\varphi) = 0\}$$

and

$$< e_0 >^0 = \{\psi \in (V^1)^* / \widehat{e}_0(\psi) = 0\},$$

respectively. Moreover, the map $\mu : Z^\ddagger \to (V^1)^\ddagger$ is an epimorphism of affine bundles and the corresponding epimorphism of vector bundles $\mu^l : < i^1 \circ e_0 >^0 \to < e_0 >^0$ is given by

$$\mu^l(\varphi) = \varphi_{|V^1_y}, \text{ for } \varphi \in Z^*_y \text{ and } y \in E.$$

Note that $\ker \mu^l_y = < \tau^*_Z(\eta_E)(y) >$, for all $y \in E$. Thus, $\mu : Z^\ddagger \to (V^1)^\ddagger$ is an AV-bundle (this will be the bundle of the Hamiltonian section).

On the other hand, from (3.3), it follows that a Lagrangian section $l : J^1\pi \to Z^1$ is regular if and only if the Legendre transformation $leg_l : J^1\pi \to (V^1)^\ddagger$ is a local diffeomorphism.

Next, we will assume that l is *hyperregular,* that is, the map $leg_l : J^1\pi \to (V_1)^\ddagger$ is a global diffeomorphism. Then, one may consider the section $h : (V^1)^\ddagger \to Z^\ddagger$ of the AV-bundle $\mu : Z^\ddagger \to (V^1)^\ddagger$ given by $h = Leg_l \circ leg_l^{-1}$. h is the *Hamiltonian section.*

Now, we will introduce the 2-form Ω_h on $(V^1)^\ddagger$ defined by

$$\Omega_h = h^*(\Omega_{Z^\ddagger}), \tag{3.4}$$

$\Omega_{Z^\ddagger}$ being the symplectic 2-form on $Z^\ddagger$. Suppose that (t, q^A, p, p_A) and (t, q^A, p_A) are local coordinates on $Z^\ddagger$ and $(V^1)^\ddagger$ and that the local expression of the Hamiltonian section h is

$$h(t, q^A, p_A) = (t, q^A, -H(t, q^A, p_A), p_A).$$

Then, using (3.2) and (3.4), we deduce that

$$\Omega_h = dq^A \wedge dp_A + dH \wedge dt. \tag{3.5}$$

Let $\pi_1^\ddagger : V_1^\ddagger \to \mathbb{R}$ be the canonical projection and $\eta_1^\ddagger$ be the 1-form on $V_1^\ddagger$ given by $\eta_1^\ddagger = (\pi_1^\ddagger)^*(dt)$. From (3.5), it follows that the pair $(\Omega_h, \eta_1^\ddagger)$ is a cosymplectic structure on $(V^1)^\ddagger$, that is, $\eta_1^\ddagger \wedge \Omega_h^n = \eta_1^\ddagger \wedge \Omega_h \wedge \dots^{(n} \dots \wedge \Omega_h$ is a volume form on $(V^1)^\ddagger$, $d\eta_1^\ddagger = 0$ and $d\Omega_h = 0$. Thus, we may consider the Reeb vector field ξ_h which is characterized by the conditions

$$i_{\xi_h}\Omega_h = 0, \quad i_{\xi_h}\eta_1^\ddagger = 1.$$

Using (3.5), we have that the local expression of ξ_h is

$$\xi_h = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_A}\frac{\partial}{\partial q^A} - \frac{\partial H}{\partial q^A}\frac{\partial}{\partial p_A},$$

and, therefore, the integral curves of ξ_h satisfy the *Hamilton equations*

$$\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}, \quad \frac{dp_A}{dt} = -\frac{\partial H}{\partial q^A}, \quad \text{for all } A.$$

ξ_h is called the *Hamiltonian vector field* associated with the Hamiltonian section h.

On the other hand, it is clear that $leg_l^*(\Omega_h) = \Omega_l$ and $leg_l^*(\eta_1^\ddagger) = \eta$. Consequently, the Legendre transformation leg_l is a cosymplectomorphism between the cosymplectic manifolds $(J^1\pi, \Omega_l, \eta)$ and $((V^1)^\ddagger, \Omega_h, \eta_1^\ddagger)$. So, the Euler Lagrange vector field ξ_l and the Hamiltonian vector field ξ_h are leg_l-related. This implies that if $\phi : I \subseteq \mathbb{R} \to E$ is a solution of the Euler-Lagrange equations for l then $\gamma = leg_l \circ j^1\phi$ is a solution of the Hamilton equations for h. Conversely, if $\gamma : I \subseteq \mathbb{R} \to (V^1)^\ddagger$ is a solution of the Hamilton equations for h then $leg_l^{-1} \circ \gamma = j^1\phi$, where ϕ is a solution of the Euler-Lagrange equations for l.

Remark 3.7 Under the same hypotheses as in Remark 3.4, the Hamiltonian section may be considered as a section $h : V^*\pi \to T^*E$ of the canonical projection $\mu : T^*E \to V^*\pi$ and, under this identification, the pair $(\Omega_h, \eta_1^\ddagger)$ is a cosymplectic structure on $V^*\pi$ and ξ_h is the Reeb vector field of $(\Omega_h, \eta_1^\ddagger)$. $\diamondsuit$

4 An example

In order to illustrate the results obtained in Section 3 we will consider an example which was discussed in [6].

The *Newtonian space-time* is a system (E, τ, g), where E is a four-dimensional affine space with the model vector space V, τ is a non-zero element of V^* and $g : E^0 \to (E^0)^*$ is an scalar product on $E^0 = \ker \tau$.

We will denote by E^1 the affine subspace of V given by $E^1 = \{u \in V / \tau(u) = 1\}$ and for each $u \in E^1$ we will introduce the linear epimorphism $i_u : V \to E^0$ defined by $i_u(v) = v - \tau(v)u$. An element u of E^1 may be interpreted as an *inertial reference frame.*

The space-time E is fibred over the time $T = E/E^0$ which is an affine space of dimension 1 modelled on $\mathbb{R}$. So, the fibration $\pi : E \to T$ is just the canonical projection.

Note that

$$TE \cong E \times V, \quad J^1\pi \cong E \times E^1, \quad V\pi \cong E \times E^0.$$

Now, for each $u \in E^1$, we will consider the inhomogeneous Lagrangian function $L_u : J^1\pi \cong E \times E^1 \to \mathbb{R}$ given by

$$L_u(y, w) = \frac{m}{2} g(w - u)(w - u) - \varphi(y),$$

where $\varphi : E \to \mathbb{R}$ is a potential.

The Lagrangian function L_u is hyperregular. Thus, in order to obtain the well-known equations of motion, one may apply the classical Lagrangian (Hamiltonian) inhomogeneous formalism of the dynamics. These geometrical constructions will depend on the inertial reference frame u. However, we can develop an inhomogeneous formulation of the dynamics independent on the choice of the inertial frame as follows (see [6]).

If u and u' are two inertial reference frames then we deduce that

$$L_u(y, w) - L_{u'}(y, w) = m\sigma(u', u)(w), \quad \text{for } (y, w) \in E \times E^1,$$

where $\sigma : E^1 \times E^1 \to V^*$ is the map defined by

$$\sigma(u', u)(v) = g(u' - u)(i_{\frac{u+u'}{2}}(v)), \text{ for } v \in V.$$

This result suggests to consider the equivalence relation $\sim$ on the set $E^1 \times V \times \mathbb{R}$ defined by

$$(u, v, r) \sim (u', v', r') \Leftrightarrow v = v' \text{ and } r = r' + m\sigma(u', u)(v).$$

It follows that the quotient set $W = (E^1 \times V \times \mathbb{R})/\sim$ is a real vector space with $w_0 = [(u, 0, 0)]$ as the zero vector and $w_1 = [(u, 0, 1)] \neq 0$, u being an arbitrary element of E^1. Moreover, one may prove that $W/ < w_1 > \cong V$ and, therefore, we have a canonical projection $\tau_W : W \to V$ (see [6]). Thus, it is clear that $W^1 = \tau_W^{-1}(E^1)$ is an affine space modelled on the vector space $W^0 = \tau_W^{-1}(E^0)$. We will denote by $\tau_{W^1} : W^1 \to E^1$ and by $\tau_{W^0} : W^0 \to E^0$ the canonical projections.

Then, in this particular example, the affine bundle Z^1 (in Section 3) is just the trivial affine bundle $\zeta_{Z^1} : Z^1 = E \times W^1 \to E$ which is modelled on the trivial vector bundle

$\zeta_{V_1} : V^1 = E \times W^0 \to E$. The bidual bundle to Z^1 is the trivial vector bundle $\zeta_Z : Z = E \times W \to E$.

The projections $\tau_Z : Z = E \times W \to TE \cong E \times V, \tau_{Z^1} : Z^1 = E \times W^1 \to J^1\pi \cong E \times E^1$ and $\tau_{V^1} : V^1 = E \times W^0 \to V\pi \cong E \times E^0$ are just the product maps $Id \times \tau_W, Id \times \tau_{W_1}$ and $Id \times \tau_{W^0}$, respectively. The section $e_0 : E \to V^1 = E \times W^0$ of $\zeta_{V^1} : V^1 = E \times W^0 \to E$ is given by $e_0(y) = (y, w_1)$, for all $y \in E$. On the other hand, the affine bundle $Z^\ddagger \to E$ is isomorphic to the trivial affine bundle $pr_1 : E \times P \to E$, where P is the quotient affine space $P = (E^1 \times V^*)/\sim^1$ and $\sim^1$ is the equivalence relation on $E^1 \times V^*$ defined by

$$(u, p) \sim^1 (u', p') \Leftrightarrow p = p' + m\sigma(u', u)$$

(see [6]). Note that P is an affine space modelled on V^*. In addition, the affine bundle $(V^1)^\ddagger \to E$ is isomorphic to the trivial affine bundle $pr_1 : E \times P_0 \to E$, where P_0 is the quotient affine space $P_0 = E' \times (E^0)^*/\sim^0$ and $\sim^0$ is the equivalence relation on $E^1 \times (E^0)^*$ defined by

$$(u, p_0) \sim^0 (u', p_0') \Leftrightarrow p_0 = p_0' + mg(u' - u)$$

Now, for each $u \in E^1$ and $v \in V$, we may consider the section $s_{(u,v)}$ of the vector bundle $\zeta_Z : Z = E \times W \to E$ given by $s_{(u,v)}(y) = (y, [(u, v, 0)])$, for all $y \in E$. We remark that if $\{v_i\}$ is a basis of V then $\{s_{(u,v_i)}, e_0\}$ is a global basis of $\Gamma(\zeta_Z)$. So, one may introduce a Lie algebroid structure $([\![\cdot,\cdot]\!]_Z, \rho_Z)$ on the vector bundle $\zeta_Z : Z = E \times W \to E$ which is characterized as follows

$$[\![s_{(u,v)}, s_{(u',v')}]\!]_Z = [\![s_{(u,v)}, e_0]\!]_Z = 0, \text{ for } u, u' \in E^1 \text{ and } v, v' \in V,$$

and $\rho_Z(s_{(u,v)})$ and $\rho_Z(e_0)$ are the vector fields on E defined by

$$\begin{aligned} \rho_Z(s_{(u,v)}) : E \to TE \cong E \times V, &\qquad y \in E \to \rho_Z(s_{(u,v)})(y) = (y, v) \in E \times V, \\ \rho_Z(e_0) : E \to TE \cong E \times V, &\qquad y \in E \to \rho_Z(e_0)(y) = (y, 0) \in E \times V. \end{aligned}$$

On the other hand, the Lagrangian functions L_u, $u \in E^1$, define a section $l : J^1\pi \cong E \times E^1 \to Z^1 = E \times W^1$ of the projection $\tau_{Z^1} : Z^1 = E \times W^1 \to J^1\pi \cong E \times E^1$ as follows

$$l(y, w) = (y, [(u, w, L_u(y, w))]), \text{ for } (y, w) \in E \times E^1.$$

l is the *affine Lagrangian for the inhomogeneous formulation of the dynamics* (see [6]).

The affine Lagrangian l is hyperregular. Furthermore, if $\Omega_l : T(J^1\pi) \times_{J^1\pi} T(J^1\pi) \cong E \times E^1 \times V \times E^0 \times V \times E^0 \to \mathbb{R}$ is the Poincaré-Cartan 2-form, $leg_l : J^1\pi \cong E \times E^1 \to (V^1)^\ddagger \cong E \times P_0$ is the Legendre transformation and $\xi_l : J^1\pi \cong E \times E^1 \to TJ^1\pi \cong$

$E\times E'\times V\times V^0$ is the Euler-Lagrange vector field associated with l then we obtain that

$$\begin{array}{rcl}\Omega_l(y,w,\dot y,\dot w,\dot y',\dot w') & = & \Omega_{L_u}(y,w,\dot y,\dot w,\dot y',\dot w') = m\{g(i_u(\dot y))(\dot w') - g(\dot w)(i_u(\dot y'))\}\\ & & +\tau(\dot y)\{(d_s\varphi(y))(i_u(\dot y')) - mg(\dot w')(w-u)\}\\ & & -\tau(\dot y')\{(d_s\varphi(y))(i_u(\dot y)) - mg(\dot w)(w-u)\},\\ leg_l(y,w) & = & (y,[(u,mg(w-u))]),\\ \xi_l(y,w) & = & \xi_{L_u}(y,w) = (y,w;w,\frac{1}{m}g^{-1}(d_s\varphi(y))),\end{array}$$

for $(y,w)\in E\times E^1$, $(\dot y,\dot w),(\dot y',\dot w')\in V\times E^0$. Here, $d_s\varphi$ denotes the vertical differential of φ with respect to the projection π (thus, $d_s\varphi(y)\in(E^0)^*$).

Finally, we conclude that the Hamiltonian section $h_l:(V^1)^\ddagger\cong E\times P_0\to Z^\ddagger\cong E\times P$, the 2-form $\Omega_{h_l}:T(V^1)^\ddagger\times_{(V^1)^\ddagger}T(V^1)^\ddagger\cong E\times P_0\times V\times(E^0)^*\times V\times(E^0)^*\to\mathbb{R}$ and the Hamiltonian vector field $\xi_{h_l}:(V^1)^\ddagger\cong E\times P_0\to T(V^1)^\ddagger\cong E\times P_0\times V\times(E^0)^*$ are given by

$$\begin{array}{rcl}h_l(y,[(u,p)]) & = & (y,[(u,p\circ i_u-(\frac{1}{2m}p(g^{-1}(p))+\varphi(y))\tau)]),\\ \Omega_{h_l}(y,[(u,p)];\dot y,\dot p,\dot y',\dot p') & = & i_u(\dot y)(\dot p') - \dot p(i_u(\dot y')) + \tau(\dot y')\{(d_s\varphi(y))(i_u(\dot y)) + \frac{1}{m}\dot p(g^{-1}(p))\}\\ & & -\tau(\dot y)\{(d_s\varphi(y))(i_u(\dot y')) + \frac{1}{m}\dot p'(g^{-1}(p))\},\\ \xi_{h_l}(y,[(u,p)]) & = & (y,p;u+\frac{1}{m}g^{-1}(p),-d_s\varphi(y)),\end{array}$$

for $(y,p)\in E\times(E^0)^*, u\in E^1$ and $(\dot y,\dot p),(\dot y',\dot p')\in V\times(E^0)^*$.

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