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The Thermal Radiation Formula of Planck (1900)

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This so-called normal energy distribution
represents something absolute, and
since the reseach for absolutes
has always appeared to me
to be the highest form
of research, I applied
myself vigorously
to its solution.

Max PLANCK

Abstract

We review the derivation of Planck's Radiation Formula on the light of recent studies in its centenary. We discuss specially the issue of discreteness, Planck's own opinion on his discovery, and the critical analysis on the contribution by Ehrenfest, Einstein, Lorentz, etc. We address also the views of T.S. Kuhn, which conflict with the conventional interpretation that the discontinuity was already found by Planck.

1.– In the year 2000 we celebrated the 100th anniversary of Planck's radiation formula, which opened the scientific world to the quantum. With this opportunity many papers have appeared, dealing with different aspects of the formula, its derivation, the meaning for Planck and for other physicists, the historical context, etc. In this communication we want to recall the origin of the formula on the light of these contributions, and address some questions, old and the new, on the meaning of Planck's achievement.

The fundamental lesson is that the quantum postulate introduces discreteness and therefore justifies atomicity. Namely the old hypothesis of atoms, started with the greeks in the Vth century b.C., reinforced through the work on chemistry in the first part of the XIX century (Proust, Dalton, Prout, Avogadro), utilized heuristically to explain the properties of gases in the second part (Clausius, Maxwell, Boltzmann), became a certainty with the discovery of cathode rays (Plčker), X-rays (Röngten), radiactivity (Becquerel, the Curies), the electron (Thomson) and the nuclear atom (Rutherford), the later already well within the XX century; in a typical paradox of science, the same experiments which proved the existence of atoms also showed, antietymologically, that the atoms were divisible. Now the theory of quanta has determined the structure of atoms, showing why they do exist in the first place.

The historical development of the radiation formula is uncontroversial, and we shall review it here quickly, referring to the many sources which convey a detailed information (see the detailed Bibliography at the end). Next we study the contributions by Planck up to early 1900, when Wien's formula dominated the scene. The two outstanding communications of Planck to the Berlin Academy (19-X-1900, guess of the radiation formula, and 14-XII-1900, statistical justification by introduction of the discrete energy elements) will then ocupy us. The reception of Planck's discovery was cold, not much being published or commented until 1905, and we consider why. We refer then to the papers of Ehrenfest and Einstein, the first to fully realize the big break that Planck's theory supposes; some contributions by Haas, Sommerfeld and Poincaré, and others are briefly referred. Differences between Planck's energy quanta and Einstein's *Lichtquanta* are stresssed, in relation to the indistinguishability issue. We end the historical part by describing what Planck did when he came back to the black body radiation problem in 1910–12, the so-called second theories of Planck.

We endeavour then to comment briefly on some well-known analysis of Planck's achievements by Rosenfeld, Klein, Kuhn and Jost. Several centenary contributions are discussed next, as well as further contributions on the light of open problems and controversial issues. Our paper ends with a final look at the figure of Max Planck.

2.– That a heated body shines is an elementary observation; to understand the dependence of the emitted light (radiation) on the nature and shape of the body, and on the wavelength and the temperature, is the *problem of the heat radiation formula*. Experimentally the temperature ranges were up to 2000 °K, and the vavelength from the near UV up the medium IR.

The question was first addressed by Gustav R. KIRCHHOFF in 1859. He discovered the *universal character* of the radiation law, solving therefore the problem of the dependence of the emitted light on the nature, size and shape of the body; namely for a ordinary body under illumination there are coefficients of absorption a, and reflection r, as well as emission e, but the quotient $e/a = K$, the intensity, Kirchhoff found, is *independent* of the body, if equilibrium is to be achieved. It depends only on wavelength and temperature; Kirchhoff hoped the function $K(\nu, T)$ to have a simple form, as is the case for functions which do not depend on individual properties of bodies. To study the radiation, one approaches a "black" body in which the absorption is maximal by definition ($a = 1$), and studies the radiated intensity as function of wavelength and temperature. Empirically it was clear the warmer the body the greater the total emitted radiation is, and the brightest "colour" shifts to the blue. To realize a black body, one fabricates a hollow cavity (*Hohlraum*) with the walls blackened by lampblack (*negro de humo*), practices a small aperture, heates it up, and analyzes the outgoing radiation with bolometers (measure of intensity) and prisms and gratings (measures of wavelength).

Since 1865 it was accepted that light was electromagnetic radiation (Maxwell), and hence the distribution law should be studied by the thermodynamics of the electromagnetic processes. From the rough experiments performed in the 1870s it was apparent that the total amount radiated grows like the fourth power of the absolute temperature, as first stated by J. STEFAN (1879); if $ud\nu = u(\nu, T)d\nu$ is the differential density of energy of radiation in the hollow cavity at frequency ν and temperature T,

$$\int_0^\infty u(\nu, T)d\nu = \sigma T^4. \tag{1}$$

Here the density is $u = 4pK/c$, with K the previous intensity. The above law (1) was easy to deduce theoretically (L. BOLTZMANN, 1884). In modern terms, it follows at once from dimensional analysis with zero photon mass. Next, it was determined that the wavelength at maximum radiation was inverse with the temperature ($\lambda_{\max} T =$ constant, W. WIEN displacement law, 1893); this is the first example of an adiabatic invariant (Boltzmann), and combined with the Stefan-Boltzmann's result it yielded the law

$$u(\nu, T) = \nu^3 f(\nu/T), \tag{2}$$

reducing the dependence on frequency and temperature to a single *universal* formula on ν/T; notice (2) requires two constants, from dimensional analysis. One cannot go any further with pure thermodynamics and the electromagnetic theory. But by analogy with the velocity distribution formula of Maxwell for gas molecules, Wien *suggested* the concrete form

$$u(\nu, T) = a\nu^3 \exp(-b\nu/T)(3) \tag{3}$$

which became known as *Wien radiation formula* (1896). The law (3) is very natural, and indeed for several years it was thought to fit well with experiments: namely, a power increase (scale invariance) at low ν followed by an exponential damping (a cutoff), typical of many physical processes. The constants a and b should have an universal character, and will play an important role in Planck's interpretation, see later; they were expected, as we said.

However, Wien's law (3) implies that for very high T the density goes to constant with T, which is not very physical: one should expect the energy density to grow without limit with increasing temperature; this problem does not arise in the case of Maxwell distribution, which refers to velocities, not density. Indeed, the refined experiments carried out in Berlin since 1900 mainly by Rubens and Kurlbaum proved that the density, for very low frequency (equivalent to large T), is proportional to T.

For simple derivations of formulas in this paragraph see the Appendix. Best secondary sources for this period are [Born 46], [Sanchez-Ron 01], [Kuhn 78] and [Jammer 66].

3.– Now enters Max PLANCK (*Kiel 1858; †Göttingen 1947). He was an expert on thermodynamics, very much impressed by the absolute things, like the (first) law of conservation of energy, stated along 1840-50 by Joule, Kelvin, Helmholtz and others, and also by the second law, the increase of entropy, discovered by his admired R. Clausius first in 1850, later by Kelvin (1853). At the time, the mechanical theory of heat was accepted, that is, heat is just another form of energy, not an *entelechia*, the *flogiston*; the idea of reducing physics to mechanics dominated. For Planck, mechanics represented the best way to understand physics (and chemistry), and he sought a *mechanical* explanation of the second law, understood as an exact law of nature, on the same footing as the energy conservation law. By "mechanical", Planck did not mean the atomistic point of view, but continuum mechanics; in fact, for a long time he considered the atomic hypothesis something irrelevant (if not nocive) for the second principle, because in the kinetic theory of gases the second law is not absolute (Maxwell demon). He stands between Boltzmann, in an extreme, who always put atomicity first, and the energeticists (Ostwald, etc.), who negated atoms (as Mach did), pretending to reduce all phenomena to different forms of energy, on the other. The mechanical model he had in mind was rather close to the continuum aether of electromagnetism, and Planck sought to prove the second law from continuum mechanics, in particular the irreversible approach to equilibrium.

This is the route which took him to the *Wärmestrahlung*: he thought the radiant energy of a heated body to be an ideal system to prove the entropy increase as consequence of *conservative* laws. He also hoped, in the process, to find "Kirchhoff function", that is, the radiation formula. The heroic struggle of Planck, and his final defeat, is a paradigmatic example of how an investigation doomed to fail could lead, if pursued intelligently and with honesty, to a fundamental, but completely unexpected discovery; by failure it is meant here that the second law in Planck's form $\Delta S \geq 0$ for $t > 0$ (with S the entropy) could *not* be proven from the blackbody radiation theory *alone* any more that it could not be proven in the kinetic theory of gases without the *Stosszahlansatz* (molecular disorder) of Boltzmann.

In five papers, 1897-1899 Planck tried to explain the origin of irreversibility in the physics of thermal radiation. As the material in the cavity is irrelevant (Kirchhoff), he considers an oscillator, imitating the resonators used by Hertz, constituted by a vibrating dipole $q\boldsymbol{r}$ in presence of an electromagnetic field $\mathcal{E}$; the dipole radiates energy at the rate

$$P = -dE/dt = 2q^2/3c^3 \, \|\boldsymbol{r}''\|^2 \tag{4}$$

and the differential equation for the dipole amplitude $\boldsymbol{r}$ is

$$m\boldsymbol{r}'' + k\boldsymbol{r} - \gamma\boldsymbol{r}''' = q\mathcal{E} \tag{5}$$

where m is the mass, k the oscillator constant, q the electric charge, $\mathcal{E} = \mathcal{E}(t)$ the external electric field, and the cubic term is due to the radiation damping, with $\gamma = 2q^2/3c^3$. The damping is conservative, because energy is not lost to the whole system, it is just transformed into radiant energy (in contrast to damping by friction, in which case is transformed into heat). The damping is small, however, hence in zeroth order $\boldsymbol{r}'' = -(k/m)\boldsymbol{r}$ and Eq. (5) is

$$K\boldsymbol{f} + (2K/3Lc^3)\boldsymbol{f}' + L\boldsymbol{f}'' = \mathcal{E} \tag{6}$$

where $\boldsymbol{f} = q\boldsymbol{r}$ is the dipole moment, $K = k/q^2$ and $L = m/q^2$. Neither (5) nor (6) by themselves are invariant under time reversal (because the odd derivative terms change under $t \to -t$), and this is why and where Planck hoped to reach irreversibility. Eq. (6) is easy to solve for a given field $\mathcal{E}$, giving a transient plus a sustained wave (see Appendix).

However, the via towards irreversibility is really closed, as Boltzmann pointed out inmediately: the whole system of equations for the resonator plus the e.m. field is time-reversal invariant, and the transformation of the incoming e.m. plane wave contained in $\mathcal{E}$ into the scattered (radiated) outgoing spherical wave does not represent an irreversible change: the theory allows perfectly well an incoming spherical wave "scattered" into the outgoing plane wave: grudgingly, Planck had to admit that irreversibility, in the *Wärmestrahlung* as in the kinetic theory only obtains by explicit exclusion of some improbable situations: the molecular disorder of Boltzmann has here as counterpart the "natural radiation hypothesis" of Planck, imitating the former. The natural radiation is the one for which there is no correlation between the phases of the different Fourier components of $\mathcal{E}$. This is the defeat of Planck: irreversibility does not come from conservative mechanics unless extra hypothesis; Planck admitted this around early 1898: the validity of the second law for the thermal radiation will be, like in Boltzmann's H-theorem, not absolute.

However, the route to obtain the universal Kirchhoff formula remains open: the situation at equilibrium is easier to describe that the approach to it. The next important result of Planck is to admit equilibrium between the resonator (by now called oscillator) and the bathing radiation, and then to relate the radiated power (energy per unit time) that is, the loss of oscillator energy, to energy absorption from the field. At the end, there is a simple relation between the radiation energy density $u = u(\nu, T)$ and the elementary oscillator energy U; it is

$$u = 8\pi\nu^2/c^3\, U, \tag{7}$$

a fundamental result in Planck's research. The simplest way to understand (7) is by dimensional analysis, as U is an energy, but u is an energy density: $(\nu/c)^2 d(\nu/c)$ is a differential inverse volume; $2 \times 4\pi$ comes from polarization and angular integration.

To solve for the radiation formula, Planck needs to know the mean oscillator energy U in a thermal bath at T. To find it, Planck takes an indirect route, retorts to thermodynamics. Instead of guessing $U = U(\nu, T)$, he did guess the entropy dependence on U, and his Ansatz (obtained by working backwards from (3)) was a relation between the oscillator entropy S and U:

$$R \int -(\partial^2 S/\partial U^2)^{-1} = \alpha U \tag{8}$$

as the simplest (and, he first thought, unique) possibility. For $\alpha > 0$, this ensures entropy should increase with time (we do not show here the arguments, but see later), in agreement with the second law. Together with $dS = dU/T$, (8) leads easily to the form

$$u(\nu, T) = \alpha\nu^3 \exp(-b\nu/T). \tag{9}$$

That is, the law of Wien! We are already in early 1900. Because the universal character of the Kirchhoff function, the actual formula (9) is universal, and so are the constants a and b; but at the time, two other universal constants were known, the velocity of light c and G, Newton's constant. So, we have a perfectly natural system of units, defined by natural primary phenomena, not anthropomorphically: Planck is exultant and claims that extraterrestian people would have the same constants, too; for a conservative man like Planck was, this is an astonishing affirmation! Later Planck realized that only a is really new (see later; b should be related to a and to the constant of gases R and ultimately to Boltzmann's constant k), and this is better: centimeter, gram and second, defined humanely, are traded by G, c and a (eventually proportional to h). It is remarkable that the universal feature of the Planck's constant h was realized *before* h was connected with discontinuity (!), a point insufficiently emphasized by *cognoscenti*; even [Kuhn 78] to whom we follow in part, does not attach too much significance to it. The point is, already in the Wien displacement law $\lambda_{\max} T$ = constant, there is a new universal constant. To complicate matters more, at the time it was not known that b embodies also Boltzmann constant, never written as such by Boltzmann, by the way.

4.– That was nice, as Wien's formula was thought to be correct at the time. In fact too nice: as mentioned, the experiments carried out along 1900 at the longest infrared waves available, up to 60 μ, showed inequivocally that u is linear in T for T large, as it was physically reasonable (see above). It is astonishing what a simple modification of (8) brings the experiment data in order: Planck admits simultaneously that (8) is not unique, and realizes that in the limit $u \propto T$ one should have (Cfr. (8)) $R \propto U^2$, so the simplest interpolation formula (with both $\alpha, \beta > 0$)

$$R = \alpha U + \beta U^2 \tag{10}$$

leads, by the same steps as before, to the formula

$$u(\nu, T) = a\nu^3/c^3\left[\exp(b\nu/T) - 1\right] - 1 \qquad \text{(Planck, 19-X-1900)} \tag{11}$$

in perfect agreement then and always with the refined experiments, with the same constants as before: this is why we insist that h (or rather b), by itself, is introduced already at the Wien's formula level. To see the accuracy of (11) we just recall that the ever-pervading cosmic microwave background radiation fits to the formula (11) for $T = 2.726$ °K at nearly the millionth level precision (COBE device, 1993).

"Never in the history of physics was there such an inconspicuous mathematical interpolation with such far-reaching physical and philosophical consequences" [Jammer 66, p.18].

As Pais has remarked [Pais 82, 19a], had Planck stopped with (11), he would have always be remembered as the man who found the radiation formula, and would have a place among the greats. The fact that he went on, to supply a theoretical support for (11) is a measure of his greatness. It represented for him the biggest effort in his life. He had to yield to Boltzmann again ("an act of desesperation"), this time, against his most intimate convictions, that is, to the statistical considerations which have had enable Boltmann to prove the H-theorem. The reason, or rather the lack of, is that statistical considerations were the only ones, known around 1900, to calculate entropies, by the probabilistic formula chiseled in Boltzmann's tomb in Vienna $S = k \log W$, with W probability (*Wahrsheinlichkeit*).

Planck follows a combinatorial method traced from Boltzmann, including the apparently innocent discretization of energy, $E = \varepsilon, 2\varepsilon, 3\varepsilon, \ldots$, but the crucial point is that to obtain the expected result (11), contrary to Boltzmann, he cannot take $\varepsilon \to 0$, because then only the $u \propto T$ limit obtains (this again follows today from the zero mass of the photon). The combinatorics is simple and the final result is

$$u(\nu, T) = 8\pi\nu^2/c^3\, h\nu/[\exp(h\nu/kT) - 1] \qquad \text{(Planck, 14-XII-1900)} \tag{12}$$

with the necessary identification $\varepsilon = h\nu$ to satisfy the displacement law. We have restored to today's accepted constants k and h; it turns out that $k = R/L$, the ratio of the gas constant to Avogadro-Loschmidt number. Planck christened k as Boltzmann's constant; he also referred to his discrete $\varepsilon, 2\varepsilon, \ldots$, as *energy quanta*. Planck wrote h for *Hilfsgrösse*; this is what he first though of it (I owe this remark to M. F. Rañada (Zaragoza)).

This is how the *quantum* entered first time in physics, to remain for ever, altough it is true that Planck did not pay, at the time, too much importance to it. By nearly universal consent, the date December 14, 1900 is considered the birthday of quantum theory, and a century later it has been duly celebrated throughout the world.

5.– Historically it is very clear that Planck had introduced the finite energy elements ε as Boltzmann did, to calculate entropies from denumerable entities: he already saw that, contrary to Boltzmann, the size of the energy element cannot be taken to be zero, and to this he attributed his discovery. Hence he did realize he had found something important, and because of this our point of view is closer to that of Mehra-Rechenberg [Mehra 82] than to [Kuhn 78], that he indeed realized the new discovery, although did not perceived, yet, that he had discovered the discontinuity, and in this point we think Kuhn is right. He also saw that h had dimensions of action, and the identification of k allowed him to compute Avogadro's number L and the elementary charge e, by taking R, the gas constant, and F, the Faraday, as known at the time, and computing k and h from the black-body fit. With the scale molar vs molecular settled (i.e., L known) and also the elementary electric charge, Planck was to be henceforth a devoted atomist [Heilbron 86, p. 23]. Only E. Mach, among the notable, remained antiatomist until his very death in 1916.

The reception of Planck's formula and theory was cold. Out of stressing the beautiful experimental fit, people were not very keen with the obscure reasonings of Planck, and the black body physics was a pretty isolated corner of the general physical research (much centered, at the time, in radiactivity, the photoeffect and X-rays). One should add that the sheer number of researchers in physics in the world was then perhaps a hundredth of today's. Still, Planck's formula was quoted in several german and british sources at the time [Kuhn 78].

Rayleigh had pointed out in June, 1900, that classical theory would predict $U = kT$ for the oscillator energy (for it $\langle E_{\rm kin}\rangle = \langle E_{\rm pot}\rangle$, each with $kT/2$), from the equipartition theorem, and by 1905 he, Einstein and Jeans were firm that the exact prediction of classical physics for the Kirchhoff function had to be

$$u(\nu, T) = 8\pi(\nu^2/c^3)kT, \tag{13}$$

absurd, as no maximum in ν implied divergence for the total radiation at any temperature, the (later) UV catastrophe of Ehrenfest. The formula (13) became to be known as the Rayleigh-Jeans law (R-J). Notice the two former constants conspire to leave only one, k: classical physics was absolutele unable to produce the maximum of the radiation formula, because this requires two *disentangled* constants! The same conclusion (13) was reached by Lorentz in 1908 (Rome lecture), when the electrons, by then secure componentes of elementary matter, took the place of the imaginary oscillators. It is not sufficiently emphasized that turn-of-century theoretical physics was not only getting away from *experientia* (specific heats, black body radiation, motion of the aether: the "clouds" of Kelvin), but just becoming inconsistent; another paradox was discovered by J. W. Gibbs, with mixtures of nearly, but not quite, equal molecular species.

By 1912 irrefutable proofs of the need for "quantization" were provided by Einstein (1905), Lorentz (Wolfskehl Götttingen lectures, 1910), Poincaré (1912 paper, [Prentis 95]), etc., see [Hermann 69]. As for Planck, he remained in an uneasy position; for a long time he tried to find room for h in the framework of classical physics. He even thought the burden will be buried in the atomic particles (electrons etc.) which started to proliferate around 1900. Then he abandoned, and around 1910-12 he developed the so-called second radiation theory. Planck was very german and conservative: he wanted the quantum to carry the less possible damage to classical physics; also he was 42 in 1900. One important point, though, appeared in the first edition of his *Vorlesungen* (1906) [Planck 06]. Introducing phase space, an idea he seems to borrow from Gibbs, he concluded on the expression $\varepsilon = h\nu$ not from "correspondence" with the displacement law, but from equal-area ellipses of the oscillator. This opened Planck's eyes to the meaning of h more than anything else, and he henceforth refers to h as "the elementary quantum of action" (*elementares Wirkungsquantum*). More on that later.

Paul EHRENFEST was a singular figure in physics in the first third of the XX century; he was to be a critical mind for the quantum theory, and enjoyed particular friendship with both Einstein and Bohr. He realized also, around 1905/6, that Planck had tacitly quantized the ellipses, and that that was the real novelty. Ehrenfest also studied carefully the approach to equilibrium (randomization) of the natural radiation, see the long discussion in [Kuhn 78, pp. 152-169]; the treatment in [Mehra 82, Vol I, pt. 1], is much shorter. He also worried about the counting of states, and the difference, for light quanta (see next), between Planck's formula and the Wien limit. See also [Navarro 03].

The article of Albert EINSTEIN in 1905 dealing with the light quantum hypothesis is so well known that we shall be brief. As regards the black body radiation, the main point is that the Wien limit of the Planck's formula is recovered supposing that radiation is composed of *corpuscular* grains of energy $\varepsilon = h\nu$, because the computable entropy function coincides with that obtained by Planck to justify, in early 1900, Wien's formula. Of course, there more reasons to introduce the *Lichtquanta*, as Einstein called them; the photoelectric effect is the best known, but the original (and best, we think) reason is to cure the asymmetry between matter and radiation if the later is continuous (Einstein points here clearly to the UV catastrophe); this magistral idea of Einstein, of denouncing unphysical asymetries in classical physics was already used, equally succesfully, in the starting paragraphs of the special relativity paper, also in his *Annus Mirabilis* of 1905: the coil vs. magnet motion *Irrlehre*, as pre-relativity theory predicted different results according which is moving.

So Einstein's light quanta reproduce the low-density (Wien) limit of Planck's formula; otherwise Einstein was very careful not to endow the (future) photons with too much ontology; in particular, until 1917 no momentum was asigned ($p = h\nu/c$), altough momentum fluctuations did appear in a 1909 paper. Question arises (today), if light is really corpuscular, why he (E.) did not obtain the correct (Planck) formula? There is some discussion for this in the literature, old and modern; we feel that Eistein was unable to do it: he could not, in early 1905, obtain the correct radiation formula from the bare "photon" hypothesis, so he published just the Wien "approximation" [1]. We shall use the term *photons* (G.N. Lewis, 1.926) as a commodity for light quanta.

There two different arguments, given today, for that oddity: first, the "duality" wave-particle. Namely Einstein himself proved in 1909, that the fluctuation formula for the energy

$$\Delta E^2 \int \left\langle (E - \langle E \rangle)^2 \right\rangle \tag{14}$$

applied to Planck's formula produces straightforwardly

$$\Delta E^2 = h\nu \langle E \rangle + c^3/8\pi\nu^2 \langle E \rangle^2 \tag{15}$$

(per unit volume and frequency range), where $\langle E \rangle = U$ is the mean oscillator energy.

This is a very remarkable formula. The split in it is identical as the split $R = \alpha U + \beta U^2$, the starting point of Planck's phenomenological deduction of his formula, Cfr. (10) above. This has been noticed already, see e.g. [Hermann 69, p. 59]. Therefore, the linear part in (15) corresponds to the Wien (corpuscular!) limit, the quadratic part to the R–J (wavelike!?) limit.

[1] I thank L. Navarro (Barcelona) for a discussion of this point

Einstein himself adopted that interpretation, and this is one of the reasons, we reckon, for the widespread propagation of the idea of the particle-wave duality; Einstein stated explicitely that in view of this, he expected "the next development in theoretical physics will provide a theory of light to be interpreted as a kind of fusion of the wave and emission [corpuscular] theories" [Pais 82, 21a]. We believe this is a misleading idea, and shall comment upon it later.

The second argument contradicts the former. It is this: the correct Planck's formula can be obtained, from the purely corpuscular point of view, *taking in account the photon indistinguishability*. This was done by Einstein in 1906, by Debye in 1910 and by Bose in 1924, without mentioning, none of the three, indistinguishability! The effect is dissimulated by a different, peculiar counting. For example, taking the light as composed of "molecules" of energy $nh\nu$ as separated entities, one arrives easily (see Appendix) to the correct radiation formula; this is the Einstein derivation; in this (correct!) approach, the split in (15) has to be interpreted just contrary to usual: for low densities, the correlation efect, which is quantal, as it comes from identical particles, does not show up; so the Wien limit of the radiation formula is "classical", for corpuscles, as uncorrelated massless particles, whereas the R-J limit is the pure quantum part!! Please notice this is counterintuitive: one expects quantum effects to be noticeable for low T, contrary to our case here; this is again an effect of zero mass of the photons. As for Debye's calculation, it is formally the same as Einsteins's, but the prefactor $8\pi\nu^2/c^3$ is taken as the number of resonance modes in the cavity, plus the Planck energy quantization: so Debye gets rid of the oscillator altogether; for the derivation see [Born 46, VII-1]. The derivation by Bose (1924) also disposes of the fictitious oscillators.

The identity of particles provides a nonclassical interdependence of photons which mimics the ondulatory properties. But this was never understood by Planck, Einstein Debye or Bose (although Plancks gets close to justify a good counting with identical particles; see [Rosenfeld 36]); when Einstein introduced the Bose-Einstein statistics in 1925 he notes the interdependence, but just says it is mysterious. The thing only became clear as a corolary of Heisenberg's and Dirac's first treatment of identity of particles in the new quantum mechanics, in 1926; it is a beautiful example of the power of the healthy positivistic attitude in science: interchange of identical particles is unobservable, therefore the theory should abide by it; in fact, Leibnitz already thought along these lines.

It follows that the wave-particle duality interpretation must be false! Indeed, this much is stated in [Bach 89]. He claims that Einstein made a mistake in his "dual" (1909) interpretation of (15). Bach concludes that two terms contributing independently to the fluctuation formula is the wrong inference from probability theory, and that Einstein interpretation is false; but it will take us too afar to delve in this, to which we want to come back in the future. So we content ourselves here to add some comments on the attempts to understand the Einstein-Wien (independent) photons vs. the Einstein-Planck (interdependent) photons in the old quantum theory. This point was considered by Ehrenfest, Natanson, Wolfke, Krutkow and de Broglie, in 1910-23; in particular the latter aimed to a particle description of interference, a seemingly impossible task, but understood today through Feynman's path integral formulation of quantum mechanics. A thorough study has been published recently [Perez-Canals 02] and we refer to it. The historical controversy is well told in [Jammer 66, pp. 50-52], in [Mehra 82, I-2 p. 559] and in [Whittaker 10, pp.102-104]; the later is one of the few sources, as far as we are aware, to emphasize explicitely that a purely corpuscular theory of light can produce Planck's formula. This is with hindsight, of course!

Among the other contributors to quantum theory up to 1912, we just mention two. A. Haas was the first (1910) on thinking of a conection between the quantum, and atomic discreteness: he wanted to explain the quantum of action in terms of the atoms; he found a relation between the radius of the Thomson atom and the quantum of action; for a delightful exposition see [Hermann 69, pp. 91 ff.]; Haas'theory is a direct antecedent of Bohr's atom, see [Heilbron 69]. A. SOMMERFELD rightly pointed out [Hermann Ch. 6], in the Karlsruhe and Solvay 1911 conferences, that it was the other way around: it is the existence, stability and excitation of atoms which should be understood in terms of Planck's constant. Other important point was realized by several people (Planck, Einstein, ...): the dimensions of the quantum of action h is the same as that of e^2/c; so some attemps were made to relate the quantum of action h to the "quantum" of electricity e. Today, the mystery of the value of the fine-structure constant $\alpha = e^2/hc \approx (137)^{-1}$ is still with us ....

6.– A quick look at Planck's second theory 1910-1912 is in time now, and appropiate. It seems that Planck hit upon it after reading Lorentz's Göttingen lecture, accepting the quanta (quoted in [Klein 66]). Planck has accepted discontinuity, but wants to put it where it makes less harm, another confirmation of his conservative character. He prefers to quantize the oscillator, and leave the radiation, as in the wave theory, continuous; he cannot be blamed for the later: except Einstein and Stark, nobody accepted the *Lichtquanta* yet. Quantization takes place in phase space now, and as such this is a forerunner of the subsequent quantization rules of the Bohr–Sommerfeld atom. In fact, the formula that Planck writes (1911)

$$\int\int dp\, dq = nh \tag{16}$$

has a deep invariant meaning, as we know today, through the symplectic approach to mechanics, because $dp - dq$ is just the symplectic 2–form. More important still was the "average" over the oscillator energies, from which Planck concludes that

$$E = (E_n + E_{n+1})/2 = (n + 1/2)h\nu \tag{17}$$

and the half-quantum makes his first appearance in physics, not to be rediscovered until 1925 . . . .

This second theory of Planck is not very interesting today; for an original interpretation (based in putting discontinuity at the start) see [Kuhn 78, pt. 3].

For lack of space we refrain to gloss over the important role Planck and Einstein played in the development of the third principle of thermodynamics (Nerst) and in the zero-point energy issue.

7.– The mood was not very inclined towards historical studies of the quanta until 1960, and here only a few papers are examined. Some contributions of L. Rosenfeld, a close collaborator of Bohr, (and, according to Pauli, *der Chorknabe des Papstes*) are worth commenting. In his [Rosenfeld 36] paper he recounts carefully Planck's achievement. One nice point he emphasizes is the transition of the variables density u and entropy s, pertaining to the radiation, to the corresponding U and S belonging to the oscillator; in particular, from $\partial s/\partial u = \partial S/\partial U$, and the $s(u)$ relation from Wien's law, Planck concludes the form (8) for $S(U)$, which then uses to justify (9). Is the formula unique? Rosenfeld recalls the important relation

$$\frac{d\Sigma}{dt} = \frac{3}{5}\frac{d^2S}{dU^2}\,\frac{dU}{dt}\,\Delta U \tag{18}$$

for the time variation of the total entropy Σ; as $dU/dt\,\Delta U$ is negative, increasing entropy means that

$$R\int -\left(\frac{d^2S}{dU^2}\right) \geq 0 \tag{19}$$

which is satisfied by $R = \alpha U$, $\alpha > 0$, but by many others as well.

Wien's law is not unique. Rosenfeld, probably rightly, says that Planck stumbled in the form $R = \alpha U + \beta U^2$ not only because the second term would reproduce the experimental finding $u \propto T$ at hight T (low ν) that Rubens reported to Planck, but also because it still gives the "logarithmic" aspect to the function $S = S(U)$ that pleased Planck because similarity with Boltzmann's formulas. Other quantum papers by Rosenfeld in the book referred in [Rosenfeld 36] are worth reading.

M. J. Klein wrote an important paper in 1961 [Klein 61; see also Klein 66]; in fact, one can consider Klein the forerunner of the critical historians of the quantum theory, anteceding Kuhn and Jammer. Here, however, we shall give him just a cursory examination. He deals with two specific points. (i) Did Planck know in October, 1900 about the Rayleigh-Jeans formula, that Rayleigh had published in June, 1900?. Answer: probably yes, but he did not pay attention; Klein gives plausible reasons for both statements; we tend to agree. By the way, the polemic about previous knowledge of a clearly antecedent paper repeats itself in Einstein with respect to the Michelson-Morley experiment, and with Bohr and the Balmer formula. (ii) In what ways did Planck depart from Boltzmann's methods in his statistical calculation? Klein signals a few; Planck calculated the complexions (microstates) of a single macrostate, whereas Botlzmann compared the relative weights of different states, looking for maximalization. In fact, it seems that Planck also in his 14-XII-1900 derivation was again working backwards (because he knew the correct answer), a suggesion from [Roselfeld 36] that Klein accepts. And also the peculiar counting method of Planck is at odds with the distinguishable entities Boltzmann had considered, a point that Ehrenfest forcefully expressed in 1911 and later (compare our comments above).

As his final point, Klein comments on the little interest in Planck's theory up to 1905, and adds an extra reason to the exposed above: many continental theorists, up to 1910, were recalcitrant antiatomists, defending the "energeticism" point of view; among them Ostwald, Mach and Duhem; the battle Botlzmann fought against them (and that perhaps costed him his life) is well documented.

The history of photon statistics is told in [ter Haar 69]. The paper makes good historical points (such as the diverse names attached to k, the role of Ehrenfest, etc.), but the discussion of the photon statistics issue is poor (e.g., ignores the Natanson vs. Krutkow controversy, and also the issue of identity vs. indistinguishability, etc.).

The contribution of Rest Jost in the Einstein Centenary [Jost 79] is the best source for the atomic controversy between Boltzmann and Planck, with Mach as inspiration and Einstein as spectator; the paper is nearly philosophical, and there is no substitute for reading it. Planck started to reject atoms because with them, the proof of the second law was not absolute, and ended in just the opposite: irreversibility leads to the atoms!

Stephen G. Brush is a recognized science historian; in his paper [Brush 02] on cautious revolutionaries, he, of course, looks at Planck. He reminds us of Planck's principle: the new scientific ideas triumph on the long run, because its opponents die, not because they became convinced. Brush also takes part with Kuhn's points of view, and provides powerful arguments (not wholly convincing, we think; for example, Brush insists in Planck's rejection of the photons; but, as we argued, everybody did . . .).

8.– In referring to the papers appearing for the centenary, the one by G. Parisi [Parisi 01] is interesting; he remarks about lack of simple proof of Wien's displacement formula (2); we hope our deduction in the Appendix is simple. Also, he notices that Planck had introduced transition probabilities in this second theory, anteceding Einstein for five years; he also points out the "mistake" made by Bose in his derivation. The best part, we think, is in the final: we learn that Jeans was nearly correct in supposing that thermodynamical equilibrium had not been obtained (it requires a very long time), and also that the correct behaviour of the equilibrium time for small coupling is not known (even today!).

[Rechenberg 00] repeats his plea [Mehra 82] against Kuhn on what discontuity did Planck find; we commented this point above; we disagree that classical physics was not completed until 1905, if only because special relativity meant such a enormous conceptual break (e.g. relative time); it is nice, though, that Rechenberg remind us that Planck saw the action is a relativistic invariant.

[Studart 01] is correct and fairly complete. The combinatorial deduction of the radiation formula is very detailed; it is interesting the suggestion of Planck as a "sonambule" of science in the sense of A. Koestler.

[Sanchez-Ron 00] comments Planck's achievement on the conmemorative issue of the "Revista Española de Física". He reproduces a long part of the letter of Planck to Wood (1931), the best testimony we have on Planck's march towards his quantum. Another spanish author [Zamora 01] is worth looking at for his short but accurate historical presentation. The November 2001 issue of the Bulletin of the Mexican Physical Society is also fully devoted to the celebration [Mex 01].

The triumph (radiation formula) and failure (second law) of Planck is beautifully told in [Straumann 00].

9.– We would like to finish by launching a last look to Planck and his oeuvre. For the dilemmas an upright man like him faced, see [Heilbron 86]. A centennary volume with some partial reprints is [Duck 00]. Planck represents the best of western tradition in science, an archetype which tends to dissapear: the turgid, respected and serious german professor, coming from an academic tradition, like Bohr, Pauli and Heisenberg; Planck was not a genius, and he knew that. His long life represents the zenith and the fall of the german science as no other man does; his personal tragedies would caused doom in anyone else with less stamina. His honesty and sl bonhommie were legendary. With Planck's death in 1947 the 25–centuries domination of European science comes to an end, and the leadership crosses the oceans. Let us hope it will be for good; the new style is already different . . . .

Three final testimonies for our man seem appropiated:

"Very few will remain in the shrine of science, if we eliminate those moved by ambition, calculation, of whatever personal motivations; one of them will be Max Planck"

A. Einstein in [Planck 81]

"The wish to preceive ... the prestablished harmony is the source of the inexhaustible pacience and tenacity that we see in Planck as he strugles with scientific problems, without deviation to simpler and even profitable objetives. Colleagues attribute this attitude to an exceptional strong will and discipline. I believe this is a complete mistake. The emotion provided by these achievements is analogous to the religious experience or to falling in love; the daily effort does not come from design or program, but from a sheer and direct need."

A. Einstein in [Pais 82, 2a]

" The profound purity of his creation, the clarity, depth and and deceptive simplicity of his thinking, and the lifelong nobility of his character and his uncompromised principles are his everlasting monument"

E. C. G. Sudarshan in [Duck 00]

APPENDIX

A. Relation between intensity K and density u:

In general current is density times velocity, $\boldsymbol{j} = u\boldsymbol{v}$; but here intensity K is current flux through the unit sphere, and $v = c$; hence

$$u(\nu, T) = 4\pi K(\nu, T)/c$$

B. The derivation of the Stefan-Boltzmann law:

If heat δQ is added to the cavity, the increase has a part due to the internal energy δU and a part due to the work performed by expansion $p\,\delta V$; this is the first principle:

$$\delta Q = \delta U + p\delta V.$$

Now $U = uV$, where u is the density of energy, function *only* of T, the absolute temperature; for electromagnetic waves, the radiation pressure is $p = u/3$, as deduced by Boltzmann from Maxwell equations (but not checked experimentally until 1902 (Lebedev)). Now $\delta Q/T = dS$, where S, the entropy, is a local function (hence dS is an exact form); therefore, $S = S(T, V)$ and

$$dS = \frac{V}{T}\frac{\partial u}{\partial T} + \frac{4}{3}\frac{u}{T}dV.$$

Exactness implies

$$\frac{\partial((V/T)u'(T))}{\partial V} = \frac{\partial((4/3)(u/T))}{\partial T} = \frac{u'}{T},$$

or

$$\frac{u'}{T} = \frac{4}{3}\left(\frac{u'}{T} - \frac{u}{T^2}\right), \quad \text{or} \quad u(T) = \sigma T^4,$$

which is the Stefan-Boltzmann law.

C. Derivation of the Wien's displacement law

The usual derivation of the displacement law (Wien) is cumbersome: [Parisi 01] complains. The simplest is the following: for adiabatic changes $\delta Q = 0$, hence

$$(4/3)udV + Vdu = 0,$$

or

$$Vu^{3/4} = \text{const.} \quad \text{or} \quad VT^3 = \text{const.}, \quad \text{or} \quad \lambda T = \text{const.},$$

because the increase in $V^{1/3}$is linear with the increase in λ (Doppler effect). This is strictly speaking, Wien's displacement law; hence T enters in $u(\nu, T)$ only in the form T/ν.

Now from the Stefan-Boltzmann result,

$$\int_0^\infty u(\nu, \nu/T) d\nu = \sigma T^4 = T \int_0^\infty u(xT, x) dx,$$

which implies, for u positive,

$$\frac{\partial^4}{\partial y^4} u \left(y \int xT = \nu, x \right) = 0,$$

or $u(y, x) = y^3 \phi(x)$ neglecting lower powers from the S-B result. So

$$u(y, x) = u(\nu, \nu/T) = \nu^3 \phi(\nu/T),$$

which is the usual form of Wien's law.

D.- The forced damped oscillator.

The equation (6) in the x-axis for a single mode Ω of the e.m. field is

$$mx'' + gx' + kx = e\mathcal{E} \cos(\Omega t);$$

written in the operator form $(D - a)(D - b)x = (e\mathcal{E}/m) \cos(\Omega t)$, where a and b are roots of $y^2 + (\gamma/m)y + \omega^2 = 0$, and $D = d/dx$, the equation admits as solution, as $D - a = \exp(at)\, D \exp(-at)$ etc.,

$$x = \text{transient} + C \cos(\Omega t - \theta)$$

where C and θ can be calculated at once; the transient decays with the time constant γ/m. Now the *instantaneous* oscillator energy $E = E(\omega, \Omega)$ is $kx_{\max}^2/2$, $x_{\max} = C$, and hence

$$E = \frac{(e^2\mathcal{E}^2/2m)}{(\Omega - \omega)^2 + (\gamma/m)^2}.$$

Notice how the damping avoids the blow-up for $w \to W$.

Now the energy density for the e.m. field $u = u(\Omega)$ is $(\mathcal{E}^2 + \mathcal{H}^2)/8\pi$, and it is supposed isotropic, i.e. $u = 6\mathcal{E}^2/8\pi$; the *equilibrium* oscillator energy $U(\omega)$ is obtained by integration on Ω. Only the resonance frequency contributes, as γ is very small. The final result is the relation (restoring $\nu = \omega/2\pi$)

$$u(\nu, T) = 8\pi^2\nu^2/c^3\, U(\nu, T),$$

(where T is there just for the ride) written in (7).

E. Planck's Derivation of Wien's and Planck's formula.

From

$$R \int - \left(\frac{\partial^2 S}{\partial U^2} \right)^{-1} = \alpha U$$

we get

$$\frac{\partial S}{\partial U} = -\frac{1}{\alpha}\log U + b,$$

hence,

$$dS = \left(\frac{-\log U}{\alpha} + b\right) dU \int \frac{dU}{T},$$

or

$$U = K\exp(-\alpha/T), \qquad \text{Wien's law in } \ T.$$

Now from

$$R\int -\left(\frac{\partial^2 S}{\partial U^2}\right)^{-1} = \alpha U + \beta U^2,$$

identical calculation gives (the integration constant is fixed)

$$U = K/[\exp(+\alpha/T) - 1], \qquad \text{Planck's law.}$$

Wien's form $\Longrightarrow K, a \propto \nu$.

The full entropy obtains after another integration; write first the full actual equation

$$U(\nu, T) = h\nu/[\exp(h\nu/kT) - 1] \qquad \text{Pl.}$$

and another integration (no constant) gives

$$S(U) = k[\log(1 + U/h\nu)^{1+U/h\nu} - \log(U/h\nu)^{U/h\nu}],$$

which is of the form $A = klogW$, and gave to Planck the idea of a combinatorial approach.

F. Einstein's derivation of Planck's formula.

If the oscillator is quantized, $E(n) = nh\nu$, the mean energy is

$$\langle E\rangle = U = \frac{\partial \log Z}{\partial \beta},$$

with

$$Z = Tr\exp(-\beta H) = \sum_0^\infty x^n, \qquad \text{and} \qquad x = \exp(-\beta h\nu);$$

so $Z(\beta) = (1-x)^{-1}$, and

$$U = \frac{h\nu}{\exp(h\nu/kT) - 1}.$$

BIBLIOGRAPHICAL NOTES

The literature on the black body radiation is overwhelming. There is no question here of quoting the earliest primary sources, as they are difficult to consult, and are called for in most good modern treatises; so here we cite the most important secondary sources.

The early history of the radiation formula is best told in [Kangro 76]. A detailed information on the experimental situation is well described in [Sanchez-Ron 01]. The contribution of Planck is told in many sources; the best simplest in perhaps [Hermann 69], Ch. 1; the encyclopedic work of Mehra and Rechenberg [Mehra 82] deals with Planck in Vol 1, Part 1. The monography by [Kuhn 78] is unsurpassed on the work of Planck and its antecedents; Kuhn's interpretation is however somewhat controversial, see our main text. A centenary book with some reprints is [Duck 00]. The derivations we give on the Appendix are shorter that those given in many places; the most quoted ones are in [Born 46], App. 27. The remark that Wien law leads to the inconsistency u const for T large is in the undergraduate textbook [Cabrera 50], Cap. 34. The famous Vorlesungen of Planck are commented, reproduced and translated in [Planck 06]; there are some changes in the (later) english translation. The biography of Einstein by Pais [Pais 82], VI, 18 & 19 is worth looking at in respect to Planck's work and antecedents.

There are two old masterly expositions of the Old Quantum Theory with chapters of Wärmestrahlung, by [Pauli 26] and by [Rubinowicz 33]; the first follows an unconventional order, Planck's formula is discussed in Sect. 14, but is worth reading (Pauli is a master of exposition); the second is more encyclopedic, as befits to a Sommerfeld's student. There are of course some well known books related to historical studies on the development of quantum theory: [ter Haar 67] devotes a large chapter to Planck and includes an english version of Planck's October and December (1900) fundamental communications. [Whittaker 10] is as meticulous as always, and in his study of Planck uncontroversial. [Hund 67] is remarkable for his emphasis on indistinguishability and statistics, and he rightly stresses the contribution of Natanson and others. There is the monumental japanese work [Taketani 01], with brief and accurate calculations (Vol. II), although the style is peculiar at times. But the best source is perhaps [Jammer 66], both in accuracy and extension and in critical analysis.

Planck's own recollections are given in his Nobel Lecture [Nobel 18] and his Autobiography [Planck 43]. Previous to the centenary there are few articles on Planck's discovery. We have consulted [Rosenfeld 36], [Klein 61], [Klein 66], [ter Haar 69] (on photon statistics). The perils of philosophers dwelling on sheer physical issues are well illustrated in [Agassi 67]. The opposing views of Boltzmann and (young) Planck as regards Atomism are very well expressed in [Jost 79]. Poincaré last contribution (1912) is devoted to a proof of quantum discontinuity, and is glossed in [Prentis 95]; his importance, however, is played down by Kuhn (op. cit.) To know more about Planck directly one should consult the very accesible Autobiography and his Lectures on Thermodynamics [97].

Finally we come to the recent works on ocassion of the centenary. As they are commented upon on the main text, we just select some briefly here. [Parisi 01], [Mex 01], [Sanchez-Ron 00] and [Studart 01].

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