Commutativity of some $p$-normed algebras
with or without involution

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Abstract

The Le Page inequality in a Banach algebra is $\|xy\| \leq \alpha \|yx\|$, for every $x, y \in A$ and some constant $\alpha > 0$. We examine inequalities of Le Page type in $p$-Banach algebras with or without involution. As a consequence, some commutativity results are obtained.

Key words: $p$-Banach algebra, commutativity, generalized involution, hermitian algebra, $C^*$-algebra.

MSC: 46J40, 46H20.

1 Introduction.

Le Page ([5]) considered in a complex Banach algebra $(A, \|\|)$ the following condition $(C): \|xy\| \leq \alpha \|yx\|$, for every $x, y \in A$, and some constant $\alpha > 0$. In the unital case, this condition ensures the commutativity of $A$. The principal ingredient in the proof is Liouville’s theorem for holomorphic functions and the Hahn Banach theorem. This last result is in general not valid in $p$-normed algebras. Here we consider the support pseudo-norm introduced in [9] to examine inequalities of type $(C)$. We show that the condition $(C_1): \|xy\|_p \leq \alpha \|yx\|_p$, for every $x, y \in A$ and some constant $\alpha > 0$, implies that $A/RadA$ is commutative. This makes it possible to obtain some commutativity results. In the unital case with a generalized involution $x \mapsto x^*$, we show that the condition $(C_2): \|x^*y\|_p \leq \alpha \|yx\|_p$, for every $x, y \in A$ and some constant $\alpha > 0$, forces the algebra to be a commutative $C^*$-algebra for a norm equivalent to $\|\|$.

A vector involution $x \mapsto x^*$ ([1]) on a complex algebra $A$ is said to be an involutive anti-morphism if $(xy)^* = x^*y^*$, for every $x, y \in A$. A generalized involution is an algebra...
involution or an involutive anti-morphism. An element \( a \) of \( A \) is said to hermitian (resp., normal) if \( a = a^* \) (resp., \( a^*a = aa^* \)). We designate by \( H(A) \) (resp., \( N(A) \)) the set of hermitian (resp., normal) elements of \( A \). In the unital case, we say that \( a \) is unitary if \( a^*a = aa^* = e \), where \( e \) is the unit element. The set of all unitary elements of \( A \) will be denoted \( U(A) \).

A linear \( p \)-norm on \( A \), \( 0 < p \leq 1 \), is non-negative function \( x \mapsto \|x\|_p \) such that \( \|x\|_p = 0 \) if and only if \( x = 0 \), \( \|x + y\|_p \leq \|x\|_p + \|y\|_p \) and \( \|\lambda x\|_p = |\lambda|^p \|x\|_p \), for all \( x, y \in A \) and \( \lambda \in \mathbb{C} \). By a \( p \)-normed algebra \((A, \|\cdot\|_p)\), we mean an algebra \( A \) endowed with a linear \( p \)-norm \( \|\cdot\|_p \) such that \( \|xy\|_p \leq \|x\|_p \|y\|_p \), for all \( x, y \in A \). In this paper, all \( p \)-normed algebras are not necessarily assumed to be complete. This is different from [10] and [11]. A complete \( p \)-normed algebra is called a \( p \)-Banach algebra. A \( p \)-Banach algebra \((A, \|\cdot\|_p)\) with a generalized involution \( x \mapsto x^* \) is said to be hermitian if the spectrum of every hermitian element is real. Throughout the paper, all algebras considered will be associative and complex. We denote Pták’s function by \( |\cdot| \), that is \( |a|^2 = \rho(a^*a) \), for every \( a \in A \), where \( \rho \) is the spectral radius i.e., \( \rho(a) = \sup \{|z| : z \in \text{Spa} \} \). The center of \( A \) will be denoted \( C(A) \). For \( a, b \in A \), we designate by \([a, b]\) the commutator \( ab - ba \).

2 Condition \((C_1)\)

Our first condition is the analog of Le Page’s inequality in \( p \)-normed \( Q \)-algebras

\[ \|xy\|_p \leq \alpha \|yx\|_p, \text{ for every } x, y \in A \text{ and } \alpha > 0, \]  \((C_1)\)

In this case, we can say the following

**Theorem 2.1** Let \((A, \|\cdot\|_p), 0 < p \leq 1\), be a \( p \)-normed \( Q \)-algebra. If \( A \) satisfies \((C_1)\), then \( A/RadA \) is commutative.

**Proof.** Without loss of generality, we may suppose \( A \) complete for the inequality \((C_1)\) extends to the completion \( \hat{A} \) and \( A \cap \text{Rad} \hat{A} \subset \text{Rad} A \) for \( A \) is a \( Q \)-algebra. Moreover, considering \( A/RadA \) instead of \( A \), we may suppose \( A \) semi-simple. For any \( x \in A \), put

\[ \|x\| = \inf \sum_{i=1}^{n} \|x_i\|_p^{\frac{1}{p}}, \]

where the infimum is taken over all decompositions of \( x = \sum_{i=1}^{n} x_i, x_i \in A \). By [9], \( \|\cdot\| \) is a submultiplicative semi-norm on \( A \). Moreover, we have

\[ \rho(x) \leq \|x\| \leq \|x\|_p^{\frac{1}{p}}, \text{ \ \forall x \in A.} \]  \((1)\)

It follows from \((C_1)\), that

\[ \|xy\| \leq \alpha^{\frac{1}{p}} \|yx\|_p^{\frac{1}{p}}, \text{ \ \forall x, y \in A.} \]  \((2)\)
Now if $A$ is not unital, consider its unitization $A^1 = A \oplus C$. For $a, b, c \in A$, consider the map $f$ defined, on $C$, by
\[ f(\lambda) = (\exp(\lambda a)) bc \exp(-\lambda a). \]

One checks that, for any $\varphi$ in the topological dual of $(A, \|\cdot\|)$, $\varphi \circ f$ is holomorphic. It is also bounded by (2). By Liouville’s theorem $\varphi \circ f$ is constant, and so the coefficient of $\lambda$ in the power series expansion of $\varphi \circ f$ is zero, i.e., $\varphi([a, bc]) = 0$. By Hahn-Banach theorem, we obtain $\| [a, bc] \| = 0$. It follows from (1) that $\rho([a, bc] y) = 0$, for every $y \in A$. Whence $[a, bc] \in \text{Rad}A$ since $\text{Rad}A = \{ x \in A / \rho(x) = 0, \forall y \in A \}$. This shows that $A^2 = \{ xy/x, y \in A \}$ is contained in the centre of $A$. Thus, for any integer $n > 0$ and all $x, y \in A$, we obtain $(xy)^n = x^n y^n$. Using the fact that $\rho(x)^p = \lim_n \| x^n \|_p^\frac{1}{n}$, we deduce that the spectral radius is submultiplicative on $A$ and hence the set of quasi-nilpotent elements of $A$ coincides with the radical of $A$. Now, for every $x$ and $y$ in $A$, we have $(xy - yx)^2 = 0$. Whence $xy - yx \in \text{Rad}A = \{ 0 \}$ and so $A$ is commutative.

Let us notice that if $\| x \|_p \leq \alpha \rho(x)^p$, for every $x \in A$, then $C_1$ is verified. Whence the following classical result.

**Corollary 2.1** Let $(A, \|\cdot\|_p)$, $0 < p \leq 1$, be a $p$-normed $Q$-algebra such that $\| x \|_p \leq \alpha \rho(x)^p$, for every $x \in A$ and some $\alpha > 0$. Then $A/\text{Rad}A$ is commutative.

The analog of a G. Niestegge’s result obtained in [7] is the following

**Corollary 2.2** Let $(A, \|\cdot\|_p)$, $0 < p \leq 1$, be a $p$-normed $Q$-algebra such that $\| xy + y \|_p \leq \alpha \| yx + y \|_p$, for every $x, y \in A$ and some $\alpha > 0$. Then $A/\text{Rad}A$ is commutative.

**Proof.** The inequality in hypotheses is equivalent to $\| xy \|_p \leq \alpha \| yx \|_p$, for every $x \in A^1 = A \oplus C$ and $y \in A$.

Let $(A, \|\cdot\|_p)$, $0 < p \leq 1$, be a $p$-normed $Q$-algebra satisfying $(C_1)$. Then we have Theorem 2.1 with $\text{Ker} \|\cdot\|$ in place of $\text{Rad}A$, where $\text{Ker} \|\cdot\| = \{ x \in A : \| x \| = 0 \}$. If moreover the topological dual of $(A, \|\cdot\|)$ separates points on $A$, then $\text{Ker} \|\cdot\| = \{ 0 \}$ and therefore $(C_1)$ ensures the commutativity of $A$.

### 3 Condition $(C_2)$

In this section, we look at the condition
\[ \| x^* y \|_p \leq \alpha \| yx \|_p, \text{ for every } x, y \in A \text{ and } \alpha > 0 \quad (C_2) \]

In a unital Banach algebra, we obtained in [2] that the condition $(C_2)$ forces the algebra to be a $C^*$-algebra. In the non-unital case, it implies that the algebra is hermitian. In the
$p$-Banach algebras with a generalized involution condition $(C_2)$ appears also to be strong as the following result shows.

**Theorem 3.1** Let $(A, \| \cdot \|_p)$, $0 < p \leq 1$, be a unital $p$-Banach algebra with a generalized involution $x \mapsto x^*$. If $A$ satisfies $(C_2)$, then

1) There exists a positive constant $M$ such that

$$\| h^2 \|_p \leq M \rho(h^2)^p, \text{ for every } h \in H(A).$$

2) $A$ is a commutative $C^*$-algebra for a norm equivalent to $\| \cdot \|_p$.

**Proof.** 1) Let $h \in H(A)$ such that $\rho(h) < 1$. Then $\rho(h^2) < 1$. By an analog result of Ford’s square root lemma, there is $k \in H(A)$ such that $k^2 = e - h^2$ and $hk = kh$, where $e$ is the unit element. Let $u = h + ik$. It is easy to show that $u \in U(A)$ and $u^2 = e + 2iku$.

We have

$$\| ku \|_p = \frac{1}{2p} \| u^2 - e \|_p \leq \frac{\alpha + 1}{2p} \| e \|_p$$

Therefore

$$\| h^2 \|_p = \| e - k^2 \|_p \leq \| e \|_p + \| (ku)(u^*k) \|_p \leq \| e \|_p + \alpha \| ku \|_p^2 \leq M$$

where

$$M = \| e \|_p \left[ 1 + \frac{\alpha (\alpha + 1)}{2p} \| e \|_p \right].$$

Hence for an arbitrary $h \in H(A)$, we have $\| h^2 \|_p \leq M \rho(h^2)^p$.

2) It follows from condition $(C_2)$ that $\| u^2 \|_p \leq \alpha \| e \|_p$, for every $u \in U(A)$. We obtain

$$\rho(u) = \sqrt{\rho(u^2)} \leq \| u^2 \|_p^{\frac{1}{2p}} \leq (\alpha \| e \|_p)^{\frac{1}{2p}}.$$ 

By standard arguments ([8]), one shows that the algebra is hermitian. Let $h \in H(A)$ such that $\rho(h) < 1$. There is $k \in H(A)$ such that $k^2 = e - h$ and $hk = kh$. Then, by 1) there exists a constant $M > 0$ such that $\| k^2 \|_p \leq M \rho(k^2)^p$. Whence

$$\| h \|_p = \| e - k^2 \|_p \leq \| e \|_p + M \rho(e - h)^p \leq \| e \|_p + 2^pM$$

Now for an arbitrary $h \in H(A)$, we have $\| h \|_p \leq c \rho(h)^p$, where $c = \| e \|_p + 2^pM$. Given $x \in RadA$, we have $h = \frac{1}{2}(x + x^*) \in RadA$ and $k = \frac{1}{2}(x - x^*) \in RadA$. This implies that $A$ is semi-simple. We consider first the case where $x \mapsto x^*$ is an algebra involution. Since $A$ is hermitian and semi-simple, we show as in the Banach case ([8]) that Pták’s function $|.|$ is an algebra norm. The inequality $\| h \|_p \leq c \rho(h)^p$, for every $h \in H(A)$, implies that, for $x = h + ik \in A$, where $h, k \in H(A)$, we have

$$\| x \|_p \leq c (\| h \|_p^p + | k |^p) \leq 2c | x |^p.$$
Moreover
\[ |x|^{2p} \leq \|x^* x\|_p \leq \alpha^2 \|x\|_p^2. \]

Hence \(|\cdot|\) is equivalent to \(\|\cdot\|_p\) and \((A, |\cdot|)\) is a C*-algebra. To see that \(A\) is commutative, observe that \(|a|^2 \leq \mu |a|^2\), for some \(\mu > 0\) and every \(a \in A\). Then, by induction \(|a^{2^n}| \geq \left(\frac{1}{\mu}\right)^{2^n-1} |a|^{2^n}\). Whence \(\rho(a) \geq \frac{1}{\mu} |a|\) which implies commutativity by Corollary 2.2. Suppose now that \(x \mapsto x^*\) is an involutive anti-morphism. We will show that the algebra \(A\) is commutative. In this case, \(H(A)\) is a real \(p\)-Banach algebra. Moreover \(\text{Rad}(H(A)) = \{0\}\), since \(\|h\|_p \leq c \rho(h)^p\), for every \(h \in H(A)\). Hence \(H(A)\) is a real semi-simple \(p\)-Banach algebra in which every square is quasi-invertible for \(A\) is hermitian. By an analog result, in \(p\)-Banach algebra, of theorem 4.8 of Kaplansky ([4]), the algebra \(H(A)\) is commutative. This completes the proof.

The trivial case of any Banach space with the product zero and any generalized involution shows that the existence of a unit is essential in thorem 3.1 ([2]). In the case \(A\) is not unital, we have the following

**Theorem 3.2** Let \((A, \|\cdot\|_p)\), \(0 < p \leq 1\), be a non unital \(p\)-Banach algebra with a generalized involution \(x \mapsto x^*\) satisfying \((C_2)\). Then

1) \(A\) is hermitian.

2) If \(x \mapsto x^*\) is an algebra involution, then \(A/\text{Rad}A\) is commutative.

**Proof.** 1) For every normal element \(a \in A\), condition \((C_2)\) implies \(\rho(a)^{2p} \leq \alpha^2 \left(\|aa^*\|_p\right)^{\frac{1}{2^p-1}}\).

We obtain \(\rho(a) \leq |a|\). As in [8], one shows that \(A\) is hermitian. To see 2), observe that \((C_2)\) implies \(\|xy\|_p \leq M \|yx\|_p\), for a \(M > 0\). Whence the conclusion by Theorem 2.1.

**4 Condition \((C_3)\)**

Our last condition is
\[ \|x^* y^*\|_p \leq \alpha \|yx\|_p, \text{ for every } x, y \in A \text{ and } \alpha > 0. \] (C3)

If \(x \mapsto x^*\) is a continuous algebra involutive, then the condition \((C_3)\) is satisfied. If \(x \mapsto x^*\) is a continuous involutive anti-morphism, \((C_3)\) implies Le Page condition \(\|xy\|_p \leq \alpha^2 \|yx\|_p\). If \(x \mapsto x^*\) is only a linear involution, we have the following

**Theorem 4.1** Let \((A, \|\cdot\|_p)\), \(0 < p \leq 1\), be a unital \(p\)-Banach algebra and \(x \mapsto x^*\) be a continuous linear involution such that \(e^* = e\). If \(A\) satisfies \((C_3)\), then \((ab)^* = b^* a^*\) in \(A/\text{Rad}A\).
Proof. For \( a, b \in A \) and \( c \in A \), consider the map \( f \) defined, on \( C \) and with values in \( A \), by
\[
f(\lambda) = ((\exp(\lambda a)) c^* (b \exp(-\lambda a))^*) .
\]
For any \( \varphi \) in the topological dual of \((A, \|\|)\), \( \varphi \circ f \) is harmonic. It is also bounded by \((C_3)\).
By Liouville’s theorem \( \varphi \circ f \) is constant. Differentiating relative to the real part (or the imaginary part) of \( \lambda \), we have \( \varphi ((ac)^*b^* - c^*(ba)^*) = 0 \). Whence \((ac)^*b^* - c^*(ba)^* \in \text{Rad}A \) by Hahn-Banach theorem and (1), which proves the result.

Remark Under the condition \( \|x^*y^*\|_p \leq \alpha \|xy\|_p \), for every \( x, y \in A \) and some constant \( \alpha > 0 \), we obtain \((ca)^*b^* - c^*(ab)^* \in \text{Rad}A \), for every \( a, b, c \in A \). Then \((ab)^* = a^*b^* \) in \( A/\text{Rad}A \).

Indeed, for \( a, b, c \in A \), consider the map \( g \) defined, on \( C \), by
\[
g(\lambda) = (c \exp(\lambda a))^* ((\exp(-\lambda a)) b^*)
\]
and proceed as for theorem 4.1.

References