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On the well-posedness for the coupling of multidimensional quasilinear diffusion-transport equations

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This paper is dedicated to Prof. Monique Madaune Tort.

Abstract

This paper deals with the coupling of a quasilinear parabolic problem with a first order hyperbolic one in a multidimensional bounded domain Ω. In a region Ω_p a diffusion-advection-reaction type equation is set while in the complementary $\Omega_h \equiv \Omega \backslash \Omega_p$, only advection-reaction terms are taken into account. Suitable transmission conditions along the interface $\partial\Omega_p \cap \partial\Omega_h$ are required. We select a weak solution characterized by an entropy inequality on the whole domain. This solution is given by a vanishing viscosity method.

Keywords: Coupling problem; degenerate parabolic-hyperbolic equation; entropy solution.

1 Introduction

We are interested in a coupling of a quasilinear parabolic equation with an hyperbolic first-order one in a bounded domain Ω of $\mathbb{R}^n$; $n \geq 1$. The main motivation for considering this problem is the study of infiltration processes in an heterogeneous porous media. For instance, in a stratified subsoil made up of layers with different geological characteristics, the effects of diffusivity may be negligible in some layers. Such a coupled problem occurs also in fluid-dynamical theory for viscous-compressible flows around a rigid profile so that near this profile the viscosity effects have to be taken into account while at a distance they can be neglected. Another example arises in heat transfer studies as mentioned in [8].

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We consider the case of two layers, that is sufficient. Then, the geometrical configuration is such that:

$\overline{\Omega} = \overline{\Omega_h} \cup \overline{\Omega_p}$; Ω_h and Ω_p are two disjoint bounded domains with Lipschitz boundaries denoted by $\Gamma_l = \partial\Omega_l$, $l \in \{h,p\}$ and $\Gamma_{hp} = \Gamma_h \cap \Gamma_p$ is such that $\mathcal{H}^{n-1}(\overline{\Gamma_{hp}} \cap (\overline{\Gamma_l \backslash \Gamma_{hp}})) = 0$ where for q in $[0, n+1]$, $\mathcal{H}^q$ is the q-dimensional Hausdorff measure over $\mathbb{R}^{n+1}$. For l in $\{h,p\}$, ν_l is the outward normal unit vector defined $\mathcal{H}^{n-1}$-a.e. on Γ_l.

The interface is denoted by $\Sigma_{hp} =]0,T[\times\Gamma_{hp}$. At last, $Q =]0,T[\times\Omega$ and for l in $\{h,p\}$, $Q_l =]0,T[\times\Omega_l$, $\Sigma_l =]0,T[\times\Gamma_l$.

Now, due to a combination of conservation laws and Darcy's law, the physical model is described as follows:
For any positive and finite real T, find a measurable and bounded function u on Q such that,

$$\begin{array}{rcll} \partial_t u - \sum_{i=1}^{n} \partial_{x_i}(f(u)\partial_{x_i}P) + g(t,x,u) & = & 0 & \text{in } Q_h, \\ \partial_t u - \sum_{i=1}^{n} \partial_{x_i}(f(u)\partial_{x_i}P) + g(t,x,u) & = & \Delta\phi(u) & \text{in } Q_p, \\ u & = & 0 & \text{on }]0,T[\times\partial\Omega, \\ u(0,.) & = & u_0 & \text{on } \Omega. \end{array} \tag{1}$$

Then, suitable conditions on u across the interface Σ_{hp} must be added. As for the linear problem studied by F. Gastaldi and *al.* in [8] or for the one dimensional nonlinear problem studied by G. Aguilar and *al.* in [3], these transmission conditions include the continuity property of the flux through the interface formally written here as:

$$-f(u)\nabla P.\nu_h = (\nabla\phi(u) + f(u)\nabla P).\nu_p \text{ on } \Sigma_{hp}. \tag{2}$$

Moreover, the transmission conditions involve a property on the (dis)continuity of the function u *via* an entropy condition.

Let us mention that this problem has already been studied by the authors for a nondecreasing flux function f when $\nabla P.\nu_h \leq 0$, $\mathcal{H}^{n-1}$-a.e. on Γ_{hp} in [1] and when $\nabla P.\nu_h$ has a constant sign $\mathcal{H}^{n-1}$ a.e. on Γ_{hp} in [2].

1.1 Assumptions and notations

- The datum P is a known stationary function that belongs to $W^{2,\infty}(\Omega)$ and such that $\Delta P = 0$ which is not restrictive as soon as (1) includes some reaction terms.

- The reaction function g belongs to $W^{1,\infty}(]0,T[\times\Omega\times\mathbb{R})$ and we set

$$M'_g = \operatorname*{ess\,sup}_{(t,x,u)\in]0,T[\times\Omega\times\mathbb{R}} |\partial_u g(t,x,u)| \text{ and } M_0 = \operatorname*{ess\,sup}_{]0,T[\times\Omega} |g(t,x,0)|.$$

- The initial data u_0 belong to $L^\infty(\Omega)$. Thus we can define the nondecreasing time-depending function

$$M : t \in [0,T] \to M(t) = \|u_0\|_{L^\infty(\Omega)} e^{M'_g t} + M_0 \frac{e^{M'_g t} - 1}{M'_g}.$$

To simplify we write $M = M(T)$.

Now, we assume local hypotheses on f and ϕ.

- The flux function f is a Lipschitzian function on $[-M, M]$ with constant M'_f and such that $f(0) = 0$. To express the boundary conditions on the frontier of the hyperbolic area, we introduce as in [9, 10] the function $\mathcal{F}$ defined on $[-M, M]^3$ by:

  $$\mathcal{F}(a,b,c) = \frac{1}{2}\{\text{sgn}(a-b)(f(a)-f(b))-\text{sgn}(c-b)(f(c)-f(b))+\text{sgn}(a-c)(f(a)-f(c))\}.$$

- ϕ is an increasing Lipschitz-continuous function on $[-M, M]$ such that $\phi(0) = 0$.

- For any positive real μ, sgn_μ is the Lipschitz-approximation of the function "sgn" defined by:

  $$\forall \tau \in [0, +\infty[\,,\ \text{sgn}_\mu(\tau) = \min\left(\frac{\tau}{\mu}, 1\right) \text{ and } \text{sgn}_\mu(-\tau) = -\text{sgn}_\mu(\tau).$$

- Throughout this work, σ (resp. $\bar{\sigma}$) denotes the variable on Σ_l (resp. Γ_l), $l \in \{h, hp, p\}$. This way, for any t of $[0, T]$, $\sigma = (t, \bar{\sigma})$.

- For any real a and b, $\mathcal{I}(a, b) = [min(a, b); max(a, b)]$.

1.2 Functional spaces

- In the sequel, $W(0, T)$ is the Hilbert space

  $$W(0,T) \equiv \{v \in L^2(0,T; H_0^1(\Omega)); \partial_t v \in L^2(0,T; H^{-1}(\Omega))\}$$

  equipped with the norm $\|w\|_{W(0,T)} = \left(\|\partial_t w\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|\nabla w\|^2_{L^2(Q)^n}\right)^{1/2}$ and V is the Hilbert space

  $$V = \{v \in H^1(\Omega_p), v = 0 \ \ a.e. \text{ on } \Gamma_p \backslash \Gamma_{hp}\}$$

  equipped with the norm $\|v\|_V = \|\nabla v\|_{L^2(\Omega_p)^n}$. We denote $\langle ., . \rangle$ the pairing between V and V'.

- $BV(\mathcal{O})$ with $\mathcal{O} = \Omega_h$ or $\mathcal{O} = Q_h$ is the space of summable functions v with bounded total variation on $\mathcal{O}$ where the total variation is given by

  $$TV_{\mathcal{O}}(v) = \sup\left\{\int_{\mathcal{O}} v(x) div\Phi(x)dx,\ \Phi \in \mathcal{D}(\mathcal{O})^q,\ \|\Phi\|_{L^\infty(\mathcal{O})^q} \leq 1\right\}$$

  where q is the dimension of the open set $\mathcal{O}$. Moreover, we denote by γv the trace on Γ_{hp} or Σ_{hp} of a function v belonging to $BV(\mathcal{O})$.

This work is organized as follows: the concept of a weak entropy solution to (1)-(2) is defined in Section 2, 2.1 through an entropy inequality in the whole domain, the boundary conditions on the outer frontier of the hyperbolic area being expressed by referring to [9, 10].

This global formulation contains two local formulations: one on the hyperbolic area, stated in the paragraph 2.2, and one in the parabolic domain, stated in the paragraph 2.3. We also highlight some interface conditions along Σ_{hp} in Section 3, 3.1, proper to ensure the uniqueness of a weak entropy solution to (1)-(2), in the paragraph 3.2. The section 4 is devoted to the existence property to (1)-(2) through the vanishing viscosity method.

2 The Entropy Formulation

2.1 Weak entropy solution

The definition of a weak entropy solution to (1)-(2) has to include an entropy criterion in Q_h where the quasilinear first-order hyperbolic operator is set. This way, the problem (1)-(2) can be viewed as an evolutional problem for a quasilinear parabolic equation that *strongly degenerates* in a fixed subdomain Q_h of Q. As in [1, 2, 3], we propose a weak formulation through a global entropy inequality in the whole Q, the latter giving rise to a variational equality in the parabolic domain, to an entropy inequality in the hyperbolic one and to interface conditions along Σ_{hp}. To establish these ones, it will be convenient to start by introducing a global entropy formulation using mollified entropy pairs. For this purpose, we set for any convex function η in $W^{2,\infty}_{loc}(\mathbb{R})$ and any (w, k) in $[-M, M]^2$

$$I_{(\eta)}(w,k) = \int_k^w \eta'(\phi(\tau) - \phi(k))\, d\tau \text{ and } q_{(\eta)}(w,k) = \int_{\phi(k)}^{\phi(x)} \eta''(\tau - \phi(k))(f \circ \phi^{-1})(\tau)\, d\tau.$$

Definition 1 *A function u is a weak entropy solution to the coupling problem* (1)-(2) *if*

- $u \in L^\infty(Q)$ *with values in* $[-M, M]$*,* M *being defined Section 1.1,* $\phi(u) \in L^2(0, T; V)$ *and :*

- $\forall \varphi \in \mathcal{D}(Q)$*,* $\varphi \geq 0$*,* $\forall k \in [-M, M]$*,* $\forall \eta \in W^{2,\infty}_{loc}(\mathbb{R})$*,*

$$\int_Q I_{(\eta)}(u,k)\partial_t\varphi dx dt - \int_{Q_p} \eta'(\phi(u) - \phi(k))\nabla\phi(u).\nabla\varphi\, dx\, dt$$
$$- \int_Q \left(f(u)\eta'(\phi(u) - \phi(k)) - q_{(\eta)}(u,k)\right) \nabla P.\nabla\varphi\, dx\, dt - \int_Q \eta'(\phi(u) - \phi(k))g(t,x,u)\varphi\, dx\, dt \geq 0. \tag{3}$$

- $\forall \zeta \in L^1(\Sigma_h \backslash \Sigma_{hp})$*,* $\zeta \geq 0$*,* $\forall k \in [-M, M]$*,*

$$\operatorname*{ess\,lim}_{\tau \to 0^-} \int_{\Sigma_h \backslash \Sigma_{hp}} \mathcal{F}(u(\sigma + \tau\nu_h), 0, k)\nabla P(\bar{\sigma}).\nu_h \zeta d\mathcal{H}^n \leq 0, \tag{4}$$

- $$\operatorname*{ess\,lim}_{t \to 0^+} \int_\Omega |u(t,x) - u_0(x)| dx = 0. \tag{5}$$

Remark 1 *For any u in $L^\infty(Q)$ with values in $[-M,M]$, any φ in $\mathcal{D}(Q)$, $\varphi \geq 0$, and k in $[-M,M]$, the assertion : "(3) is true for any η in $W^{2,\infty}_{loc}(\mathbb{R})$" is equivalent to the entropy inequality written with the standard Kruzhkov entropy pairs:*

$$\int_Q |u-k|\partial_t\varphi\,dx\,dt - \int_{Q_p} \nabla|\phi(u)-\phi(k)|.\nabla\varphi\,dx\,dt - \int_Q \mathrm{sgn}(u-k)(f(u)-f(k))\nabla P.\nabla\varphi\,dx\,dt$$
$$-\int_Q \mathrm{sgn}(u-k)g(t,x,u)\varphi\,dx\,dt \geq 0. \tag{6}$$

2.2 *An entropy inequality on the hyperbolic zone*

We first derive from (3) and (4) an entropy inequality in the hyperbolic domain. Indeed,

Proposition 1 *Let u be a weak entropy solution to* (1)-(2)*. Then for any real k in $[-M,M]$ and any φ of $\mathcal{D}(]0,T[\times\mathbb{R}^n)$, $\varphi \geq 0$,*

$$\begin{aligned}
&\int_{Q_h} (|u-k|\partial_t\varphi - \mathrm{sgn}(u-k)(f(u)-f(k))\nabla P.\nabla\varphi - \mathrm{sgn}\,(u-k)\,g(t,x,u)\varphi)\ dx\,dt\\
&\geq -\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}} \mathrm{sgn}(u(\sigma+\tau\nu_h)-k)(f(u(\sigma+\tau\nu_h))-f(k))\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n\\
&-\int_{\Sigma_h\setminus\Sigma_{hp}} \mathrm{sgn}(k)f(k)\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n\\
&+\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_h\setminus\Sigma_{hp}} \mathrm{sgn}(u(\sigma+\tau\nu_h))f(u(\sigma+\tau\nu_h))\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n.
\end{aligned} \tag{7}$$

Proof - In (3) choose, for any positive μ and for any real τ:

$$\eta'(\tau) = \mathrm{sgn}_\mu(\tau).$$

By taking the limit when μ goes to 0^+ thanks to the Dominated Convergence Theorem, it comes that for φ in $\mathcal{D}(Q_h)$, $\varphi \geq 0$,

$$\int_{Q_h} (|u-k|\partial_t\varphi - \mathrm{sgn}(u-k)(f(u)-f(k))\nabla P.\nabla\varphi - \mathrm{sgn}(u-k)g(t,x,u)\varphi)\ dx\,dt \geq 0. \tag{8}$$

By referring to F.Otto's works in [9, 10], we deduce from (8) that for any real k in $[-M,M]$ and β in $L^1(\Sigma_h)$,

$$\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_h} \mathrm{sgn}(u(\sigma+\tau\nu_h)-k)(f(u(\sigma+\tau\nu_h))-f(k))\nabla P(\bar\sigma).\nu_h\beta(\sigma)d\mathcal{H}^n \text{ exists.} \tag{9}$$

Then, it results from (8) (see [9, 10]) that, for any real k in $[-M,M]$ and φ in $\mathcal{D}(]0,T[\times\mathbb{R}^n)$, $\varphi \geq 0$,

$$\begin{aligned}
&\int_{Q_h} (|u-k|\partial_t\varphi - \mathrm{sgn}(u-k)(f(u)-f(k))\nabla P.\nabla\varphi - \mathrm{sgn}(u-k)g(t,x,u)\varphi)\ dx\,dt\\
\geq\ & -\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_h} \mathrm{sgn}(u(\sigma+\tau\nu_h)-k)(f(u(\sigma+\tau\nu_h))-f(k))\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n.
\end{aligned}$$

To conclude we split the frontier of Ω_h into Γ_{hp} and $\Gamma_h\setminus\Gamma_{hp}$ and we use the boundary condition (4) on $\Sigma_h\setminus\Sigma_{hp}$. ■

2.3 A variational equality on the parabolic zone

We give now some information on the regularity for $\partial_t u$ in Q_p and we derive from (3) a variational equality satisfied by any weak entropy solution u to the coupling problem (1)-(2).

Proposition 2 *Let u be a weak entropy solution to the coupling problem (1)-(2). Then $\partial_t u$ belongs to $L^2(0,T;V')$. Furthermore, for any φ in $L^2(0,T;V)$,*

$$\begin{aligned}&\int_0^T \langle \partial_t u, \varphi\rangle dt + \int_{Q_p} \nabla\phi(u).\nabla\varphi\, dx\, dt + \int_{Q_p} f(u)\nabla P.\nabla\varphi\, dx\, dt + \int_{Q_p} g(t,x,u)\varphi\, dx\, dt \\ &+ \operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_{hp}} f(u(\sigma+\tau\nu_h))\nabla P(\bar\sigma).\nu_h\varphi d\mathcal{H}^n = 0.\end{aligned} \tag{10}$$

Proof. This proposition is proved in [1] (Proposition 3.4) by starting with (3). ■

3 The Uniqueness Property

We prove the uniqueness property in the class of weak entropy solutions satisfying the *strong trace property*:

$$\exists u^{hp} \in L^1(\Sigma_{hp}),\ \operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_{hp}} |u(\sigma+\tau\nu_h) - u^{hp}(\sigma)| d\mathcal{H}^n = 0. \tag{11}$$

In this framework for any weak entropy solution u satisfying (11), we can establish that along the interface u satisfies two transmission conditions; the first one corresponds to (2) and the second one is an entropy-type inequality. Indeed :

3.1 About the transmission conditions along Σ_{hp}

Proposition 3 *Let u be a weak entropy solution to* (1)-(2) *satisfying* (11). *Then, for $\mathcal{H}^n$-a.e. σ in Σ_{hp},*

$$\forall k \in \mathcal{I}(u(\sigma), u^{hp}(\sigma)),\ \operatorname{sgn}(u(\sigma) - u^{hp}(\sigma))(f(u^{hp}(\sigma)) - f(k))\nabla P(\bar\sigma).\nu_h(\bar\sigma) \geq 0, \tag{12}$$

where $u(\sigma) = \phi^{-1}(\phi(u)(\sigma))$ and $\phi(u)(\sigma)$ is the trace on Σ_{hp} of $\phi(u)_{|Q_p}$.

Proof. Let $(\rho_\delta)_{\delta>0}$ be a sequence of $\mathcal{C}^1(\overline{\Omega})$, such that

$$\begin{aligned}&\forall\delta>0,\ 0\leq\rho_\delta\leq 1,\ \rho_\delta(x)=1 \text{ if } x\in\Gamma_{hp}\\ &\forall\delta>0,\ \rho_\delta(x)=0 \text{ if } x\in\Omega,\ dist(x,\Gamma_{hp})\geq\delta \text{ and } \|\delta\nabla\rho_\delta\|_\infty \text{ is bounded,}\\ &\forall x\in\Omega\backslash\Gamma_{hp},\ \rho_\delta(x)\to 0 \text{ when } \delta\to 0^+.\end{aligned}$$

This way, from (10) and (11) it comes that: $\forall\varphi\in L^2(0,T,V)$,

$$\begin{aligned}&\int_0^T \langle\partial_t u, \varphi\rho_\delta\rangle dt + \int_Q \{(\nabla\phi(u) + f(u)\nabla P).\nabla\varphi + +g(t,x,u)\varphi\}\,\rho_\delta\, dx\, dt\\ &+\int_{Q_p} \nabla\phi(u).\nabla\rho_\delta\varphi\, dx\, dt + \int_{Q_p} f(u)\nabla P.\nabla\rho_\delta\varphi\, dx\, dt\\ &+\int_{\Sigma_{hp}} f(u^{hp}(\sigma))\nabla P(\bar\sigma).\nu_h(\bar\sigma)\varphi(\sigma)d\mathcal{H}^n = 0.\end{aligned} \tag{13}$$

Now, we take $\varphi = \operatorname{sgn}_\mu(\phi(u) - \phi(k))\psi_{|Q_p}$ where ψ belongs to $H_0^1(Q)$, $\psi \geq 0$. In order to take the limit with respect to δ for the first term in the left-hand side, we use an integration by parts formula based on a convexity inequality (see e.g. [7], the Mignot-Bamberger Lemma). This way,

$$\int_0^T \langle \partial_t u, \varphi\rho_\delta \rangle dt = -\int_0^T \int_\Omega I_\mu(u,k)\rho_\delta \partial_t \psi \, dx \, dt,$$

where

$$I_\mu(u,k) = \int_k^u \operatorname{sgn}_\mu(\phi(\tau) - \phi(k)) \, d\tau. \tag{14}$$

So clearly,

$$\lim_{\delta \to 0^+} \int_0^T \langle \partial_t u, \varphi\rho_\delta \rangle dt = 0.$$

Then, for the second term in the second line of (13), as $f \circ \phi^{-1}$ is continuous, thanks to the properties of the sequence $(\rho_\delta)_{\delta>0}$, we can assert that for any positive μ,

$$\lim_{\delta \to 0^+} \int_{Q_p} f(u)\nabla P.\nabla\rho_\delta \, \varphi \, dx \, dt = -\int_{\Sigma_{hp}} f(u(\sigma))\nabla P(\bar\sigma).\nu_h(\bar\sigma)\operatorname{sgn}_\mu(\phi(u)(\sigma) - \phi(k))\psi(\sigma) d\mathcal{H}^n,$$

and therefore it results from (13) that $\displaystyle\lim_{\delta \to 0^+} \int_{Q_p} \operatorname{sgn}_\mu(\phi(u) - \phi(k))\psi\nabla\phi(u).\nabla\rho_\delta \, dx \, dt$ exists and

$$\begin{aligned}
&\lim_{\delta \to 0^+} \int_{Q_p} \operatorname{sgn}_\mu(\phi(u) - \phi(k))\psi\nabla\phi(u).\nabla\rho_\delta \, dx \, dt \\
&= -\int_{\Sigma_{hp}} (f(u^{hp}(\sigma)) - f(u(\sigma)))\nabla P(\bar\sigma).\nu_h(\bar\sigma)\operatorname{sgn}_\mu(\phi(u)(\sigma) - \phi(k))\psi(\sigma) d\mathcal{H}^n.
\end{aligned} \tag{15}$$

The equality (15) means in a sense that the flux through the interface is continuous.

Now, let us come back to (3), written with $\eta = \operatorname{sgn}_\mu$, $\mu > 0$, and $\varphi = \psi\rho_\delta$, $\psi \geq 0$, ψ in $H_0^1(Q)$ - which is always possible thanks to a density argument. It comes that for any real k in $[-M, M]$,

$$\begin{aligned}
&\int_Q I_\mu(u,k)\rho_\delta \partial_t\psi \, dx \, dt - \int_{Q_p} \operatorname{sgn}_\mu(\phi(u) - \phi(k))\rho_\delta \nabla\phi(u).\nabla\psi \, dx \, dt \\
&- \int_{Q_p} \operatorname{sgn}_\mu(\phi(u) - \phi(k))\psi\nabla\phi(u).\nabla\rho_\delta \, dx \, dt \\
&- \int_Q \left\{(f(u)\operatorname{sgn}_\mu(\phi(u) - \phi(k)) - q_\mu(u,k))\nabla P.\nabla\psi + \operatorname{sgn}_\mu(\phi(u) - \phi(k))g(t,x,u)\psi\right\} \rho_\delta \, dx \, dt \\
&- \int_Q \left\{(f(u)\operatorname{sgn}_\mu(\phi(u) - \phi(k)) - q_\mu(u,k))\nabla P.\nabla\rho_\delta\right\} \psi \, dx \, dt \geq 0,
\end{aligned} \tag{16}$$

where I_μ is given by (14) and

$$q_\mu(u,k) = \int_{\phi(k)}^{\phi(u)} \operatorname{sgn}'_\mu(\tau - \phi(k))(f \circ \phi^{-1})(\tau) \, d\tau.$$

In order to pass to the limit when δ goes to 0^+ in (16), we use (15) for the second line and we split the fourth one into an integral over Q_h and an integral over Q_p; then we refer to (11). It

comes that:
$\forall k \in [-M, M], \forall \psi \in H_0^1(Q), \psi \geq 0, \forall \mu > 0,$

$$\int_{\Sigma_{hp}} (\text{sgn}_\mu(\phi(u(\sigma)) - \phi(k))(f(u^{hp}(\sigma)) - f(u(\sigma)))\nabla P(\bar{\sigma}).\nu_h(\bar{\sigma})\psi(\sigma)d\mathcal{H}^n$$
$$+\int_{\Sigma_{hp}} \left\{(f(u(\sigma))\text{sgn}_\mu(\phi(u(\sigma)) - \phi(k)) - q_\mu(u(\sigma), k))\right\} \nabla P(\bar{\sigma}).\nu_h(\bar{\sigma})\psi(\sigma)d\mathcal{H}^n$$
$$-\int_{\Sigma_{hp}} \left\{(f(u^{hp}(\sigma))\text{sgn}_\mu(\phi(u^{hp}(\sigma)) - \phi(k)) - q_\mu(u^{hp}(\sigma), k))\right\} \nabla P(\bar{\sigma}).\nu_h(\bar{\sigma})\psi(\sigma)d\mathcal{H}^n \geq 0.$$

So, for any positive μ and any real k in $[-M, M]$, and $\mathcal{H}^n$ a.e. on Σ_{hp},

$$(\text{sgn}_\mu(\phi(u(\sigma)) - \phi(k))(f(u^{hp}(\sigma)) - f(u(\sigma)))\nabla P(\bar{\sigma}).\nu_h(\bar{\sigma})$$
$$+\left\{f(u(\sigma))\text{sgn}_\mu(\phi(u(\sigma)) - \phi(k)) - q_\mu(u(\sigma), k)\right\} \nabla P(\bar{\sigma}).\nu_h(\bar{\sigma})$$
$$-\left\{f(u^{hp}(\sigma))\text{sgn}_\mu(\phi(u^{hp}(\sigma)) - \phi(k)) - q_\mu(u^{hp}(\sigma), k)\right\} \nabla P(\bar{\sigma}).\nu_h(\bar{\sigma}) \geq 0.$$

And when μ goes to 0^+, $q_\mu(w, k)$ goes to $f(k)\text{sgn}(w-k)$ for all reals w and k in $[-M, M]$, ϕ being increasing. So the μ-limit provides that for $\mathcal{H}^n$-a.e. σ on Σ_{hp} and for any real k in $[-M, M]$:

$$(f(u^{hp}(\sigma)) - f(u(\sigma)))\text{sgn}(u(\sigma) - k)\nabla P(\bar{\sigma}).\nu_h(\bar{\sigma})$$
$$+\left(\text{sgn}(u(\sigma) - k)(f(u(\sigma)) - f(k)) - \text{sgn}(u^{hp}(\sigma) - k)(f(u^{hp}(\sigma)) - f(k))\right) \nabla P(\bar{\sigma}).\nu_h(\bar{\sigma}) \geq 0.$$

The desired relation is obtained by taking k in the interval $\mathcal{I}(u(\sigma), u^{hp}(\sigma))$. ■

3.2 The uniqueness theorem

The uniqueness property of a weak entropy solution to the coupling problem (1)-(2) is proved under local Hölder continuity assumption for $f \circ \phi^{-1}$ and is provided by the following statement:

Theorem 1 *Assume that there exists a positive constant $\mathcal{C}$ and a real θ in $]\frac{1}{2}, 1]$ such that*

$$\forall (v, w) \in [-M, M]^2, |(f \circ \phi^{-1})(v) - (f \circ \phi^{-1})(w)| \leq \mathcal{C}|v - w|^\theta. \tag{17}$$

Let u_1, u_2 be two weak entropy solutions to (1)-(2) *for initial data $u_{0,1}$ and $u_{0,2}$ respectively and such that* (11) *holds. Then, for a.e. t of* $[0, T]$,

$$\int_\Omega |u_1(t, .) - u_2(t, .)|dx \leq e^{M_g' t} \int_\Omega |u_{0,1} - u_{0,2}|dx.$$

3.2.1 Preliminaries

In order to use the method of doubling variables, we introduce a sequence of mollifiers $(W_\delta)_{\delta>0}$ on $\mathbb{R}^{n+1}$ defined by

$$\forall \delta > 0,\ \forall r = (t, x) \in \mathbb{R}^{n+1},\ W_\delta(r) = \varpi_\delta(t) \prod_{i=1}^n \varpi_\delta(x_i),$$

where $(\varpi_\delta)_{\delta>0}$ is a standard sequence of mollifiers on $\mathbb{R}$. We use classical results on the Lebesgue set of a summable function on Q and a similar property on the whole boundary proved in [11]:

Lemma 1 *Let v and w be in $L^\infty(Q_h)$ such that (8) and (11) hold. Then for any continuous function φ on $\overline{Q_h}$,*

$$\lim_{\delta\to 0^+}\int_{Q_h}\int_{\Sigma_h\setminus\Sigma_{hp}} \operatorname{sgn}(v(r))f(v(r))\nabla P(\bar\sigma).\nu_h(\bar\sigma)\varphi(\frac{\tilde\sigma+r}{2})W_\delta(\tilde\sigma-r)d\mathcal{H}^n_{\tilde\sigma}dr$$
$$=\frac{1}{2}\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_h\setminus\Sigma_{hp}} \operatorname{sgn}(v(\sigma+\tau\nu_h))f(v(\sigma+\tau\nu_h))\nabla P(\bar\sigma)\nu_h(\bar\sigma)\varphi(\sigma)d\mathcal{H}^n,$$

$$\lim_{\delta\to 0^+}\int_{Q_h}\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_h\setminus\Sigma_{hp}} \operatorname{sgn}(v(\sigma+\tau\nu_h))f(v(\sigma+\tau\nu_h))\nabla P(\bar\sigma).\nu_h(\bar\sigma)\varphi(\frac{\sigma+\tilde r}{2})W_\delta(\sigma-\tilde r)d\mathcal{H}^n_\sigma d\tilde r$$
$$=\frac{1}{2}\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_h\setminus\Sigma_{hp}} \operatorname{sgn}(v(\sigma+\tau\nu_h))f(v(\sigma+\tau\nu_h))\nabla P(\bar\sigma).\nu_h(\bar\sigma)\varphi(\sigma)d\mathcal{H}^n,$$

and

$$\lim_{\delta\to 0^+}\int_{Q_h}\int_{\Sigma_{hp}} \operatorname{sgn}(v^{hp}(\sigma)-w(\tilde r))(f(v^{hp}(\sigma))-f(w(\tilde r)))\nabla P(\bar\sigma).\nu_h(\bar\sigma)\varphi(\frac{\sigma+\tilde r}{2})W_\delta(\sigma-\tilde r)d\mathcal{H}^n_\sigma d\tilde r$$
$$=\frac{1}{2}\int_{\Sigma_{hp}} \operatorname{sgn}(v^{hp}(\sigma)-w^{hp}(\sigma))(f(v^{hp}(\sigma))-f(w^{hp}(\sigma)))\nabla P(\bar\sigma).\nu_h(\bar\sigma)\varphi(\sigma)d\mathcal{H}^n$$

where v^{hp} (resp.w^{hp}) is defined by (11) for v (resp. w).

3.2.2 Proof of Theorem 1

(i) We first compare the two solutions u_1 and u_2 in the parabolic zone. The lack of regularity of the time partial derivative of any weak entropy solution to (1)-(2) requires a doubling of the time variable.

Therefore, let χ be a nonnegative element of $\mathcal{D}(0,T)$. We consider positive reals δ small enough in order that $\alpha_\delta : (\tilde t,t)\to\alpha_\delta(\tilde t,t)=\chi((t+\tilde t)/2)\varpi_\delta\left((t-\tilde t)/2\right)$ belongs to $\mathcal{D}(]0,T[\times]0,T[)$. Then, for $\mu>0$, in (10) written in variables (t,x) for u_1 we consider $\varphi(t,x)=\operatorname{sgn}_\mu(\phi(u_1)(t,x)-\phi(u_2)(\tilde t,x))\alpha_\delta(\tilde t,t)$ and in (10) written in variables $(\tilde t,x)$ for u_2, we consider $\varphi(\tilde t,x)=-\operatorname{sgn}_\mu(\phi(u_1)(t,x)-\phi(u_2)(\tilde t,x))\alpha_\delta(\tilde t,t)$. To simplify the writing, we add a "tilde" superscript to any function in the $\tilde t$ variable. Moreover, thanks to (11) we observe that in (10), for $i=1,2$,

$$\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}} f(u_i(\sigma+\tau\nu_h))\nabla P(\bar\sigma).\nu_h(\bar\sigma)\varphi(\sigma)d\mathcal{H}^n=\int_{\Sigma_{hp}} f(u_i^{hp}(\sigma))\nabla P(\bar\sigma).\nu_h(\bar\sigma)\varphi(\sigma)d\mathcal{H}^n.$$

Then we integrate with respect to the corresponding time variable so that, by adding up, it comes:

$$
\begin{aligned}
&\int_0^T \int_0^T \langle \partial_t u_1 - \partial_{\tilde t} \tilde u_2, \mathrm{sgn}_\mu(\phi(u_1) - \phi(\tilde u_2))\rangle \alpha_\delta \, dt \, d\tilde t \\
&+ \int_{]0,T[\times Q_p} \mathrm{sgn}'_\mu(\phi(u_1) - \phi(\tilde u_2)) |\nabla(\phi(u_1) - \phi(\tilde u_2))|^2 \, \alpha_\delta \, dx \, dt \, d\tilde t \\
&+ \int_{]0,T[\times Q_p} \mathrm{sgn}'_\mu(\phi(u_1) - \phi(\tilde u_2))(f(u_1) - f(\tilde u_2))\nabla P . \nabla(\phi(u_1) - \phi(\tilde u_2))\alpha_\delta \, dx \, dt \, d\tilde t \\
&+ \int_{]0,T[\times Q_p} (g(t,x,u_1) - g(\tilde t, x, \tilde u_2))\mathrm{sgn}_\mu(\phi(u_1) - \phi(\tilde u_2)) \, \alpha_\delta \, dx \, dt \, d\tilde t \\
&= - \int_0^T \int_{\Sigma_{hp}} f(u_1^{hp}(\sigma))\nabla P(\bar\sigma).\nu_h(\bar\sigma)\mathrm{sgn}_\mu(\phi(u_1)(\sigma) - \phi(u_2)(\tilde\sigma))\alpha_\delta \, d\mathcal{H}^n_\sigma \, d\tilde t \\
&+ \int_0^T \int_{\Sigma_{hp}} f(u_2^{hp}(\tilde\sigma))\nabla P(\bar\sigma).\nu_h(\bar\sigma)\mathrm{sgn}_\mu(\phi(u_1)(\sigma) - \phi(u_2)(\tilde\sigma))\alpha_\delta \, d\mathcal{H}^n_{\tilde\sigma} \, dt.
\end{aligned}
\tag{18}
$$

We want to pass to the limit first when μ goes to 0^+ in (18). For the second and third terms in the left-hand side, we argue by using the Cauchy-Scharwz inequality in the third term,

$$
\begin{aligned}
&\int_{Q_p} \mathrm{sgn}'_\mu(\phi(u_1) - \phi(\tilde u_2)) \, |\nabla(\phi(u_1) - \phi(\tilde u_2))|^2 \alpha_\delta \, dx \, dt \\
&+ \int_{Q_p} \mathrm{sgn}'_\mu(\phi(u_1) - \phi(\tilde u_2))(f(u_1) - f(\tilde u_2))\nabla P.\nabla(\phi(u_1) - \phi(\tilde u_2))\alpha_\delta \, dx \, dt \\
&\geq -\frac{1}{2} \int_{Q_p} \mathrm{sgn}'_\mu(\phi(u_1) - \phi(\tilde u_2))(f \circ \phi^{-1}(\phi(u_1)) - f \circ \phi^{-1}(\phi(\tilde u_2)))^2 |\nabla P|^2 \alpha_\delta \, dx \, dt \\
&+ \frac{1}{2} \int_{Q_p} \mathrm{sgn}'_\mu(\phi(u_1) - \phi(\tilde u_2)) \, |\nabla(\phi(u_1) - \phi(\tilde u_2))|^2 \alpha_\delta \, dx \, dt,
\end{aligned}
$$

where the second term in the right-hand side is nonnegative. This way, due to (17) and as P belongs to $W^{1,\infty}(\Omega)$, we establish that there exists a positive constant $\mathcal{C}$ such that

$$
\begin{aligned}
&\int_{Q_p} \mathrm{sgn}'_\mu(\phi(u_1) - \phi(\tilde u_2)) \, |\nabla(\phi(u_1) - \phi(\tilde u_2))|^2 \alpha_\delta \, dx \, dt \\
&+ \int_{Q_p} \mathrm{sgn}'_\mu(\phi(u_1) - \phi(\tilde u_2))(f(u_1) - f(\tilde u_2))\nabla P.\nabla(\phi(u_1) - \phi(\tilde u_2))\alpha_\delta \, dx \, dt \\
&\geq -\mathcal{C} \int_{Q_p} |\phi(u_1) - \phi(\tilde u_2)|^{2\theta} \, \mathrm{sgn}'_\mu(\phi(u_1) - \phi(\tilde u_2))\alpha_\delta \, dx \, dt,
\end{aligned}
$$

and the term in the right-hand side goes to 0 with μ as $\theta > 1/2$.

For the first term in the left-hand side of (18), we use the Mignot-Bamberger Lemma (see [7]) to obtain, for a fixed $\tilde t$

$$
\int_0^T \langle \partial_t u_1, \mathrm{sgn}_\mu(\phi(u_1) - \phi(\tilde u_2))\rangle \alpha_\delta dt = - \int_{Q_p} \Big(\int_{\tilde u_2}^{u_1} \mathrm{sgn}_\mu(\phi(\tau) - \phi(\tilde u_2)) d\tau \Big) \partial_t \alpha_\delta \, dx \, dt,
$$

while for a fixed t,

$$
- \int_0^T \langle \partial_{\tilde t} \tilde u_2, \mathrm{sgn}_\mu(\phi(u_1) - \phi(\tilde u_2))\rangle \alpha_\delta d\tilde t = - \int_{Q_p} \Big(\int_{\tilde u_2}^{u_1} \mathrm{sgn}_\mu(\phi(u_1) - \phi(\tau)) d\tau \Big) \partial_{\tilde t} \alpha_\delta \, dx \, d\tilde t,
$$

So, we are able to pass to the limit in (18) when μ goes to 0^+ and it comes

$$
\begin{aligned}
&-\int_{]0,T[\times Q_p} |u_1-\tilde{u}_2|(\partial_t\alpha_\delta+\partial_{\tilde{t}}\alpha_\delta)\,dx\,dt\,d\tilde{t}\\
&\leq \int_{]0,T[\times Q_p} |g(t,x,\tilde{u}_2)-g(\tilde{t},x,\tilde{u}_2)|\alpha_\delta\,dx\,dt\,d\tilde{t}\\
&-\int_0^T\int_{\Sigma_{hp}} (f(u_1^{hp}(\sigma))-f(u_2^{hp}(\tilde{\sigma})))\nabla P(\bar{\sigma}).\nu_h(\bar{\sigma})\mathrm{sgn}(\phi(u_1)-\phi(u_2(\tilde{\sigma})))\alpha_\delta\,d\mathcal{H}^n_\sigma\,d\tilde{t}.
\end{aligned}
$$

Now, we come back to the definition of α_δ to express the sum $\partial_t\alpha_\delta+\partial_{\tilde{t}}\alpha_\delta$. Then we can take the limit with respect to δ through the notion of the Lebesgue's set of a summable function on $]0,T[$. Therefore, as g is Lipschitz-continuous, for any χ in $\mathcal{D}(0,T)$, $\chi\geq 0$,

$$
\begin{aligned}
-\int_{Q_p}|u_1-u_2|\chi'(t)\,dx\,dt &\leq M'_g\int_{Q_p}|u_1-u_2|\chi(t)dxdt\\
&-\int_{\Sigma_{hp}}(f(u_1^{hp})-f(u_2^{hp}))\nabla P.\nu_h\mathrm{sgn}(u_1-u_2)\chi(t)d\mathcal{H}^n,
\end{aligned}
\tag{19}
$$

where we remind that $u_i(\sigma)=\phi^{-1}(\phi(u_i)(\sigma))$ and $\phi(u_i)(\sigma)$ is the trace on Σ_{hp} of $\phi(u_i)_{|Q_p}$.

(ii) Now we work in the hyperbolic domain. We use a doubling method for all the variables. Let ψ be such that $\psi\equiv\chi\zeta$ where χ is a function in $\mathcal{D}(0,T)$, $\chi\geq 0$, as in Part (i) and ζ is in $\mathcal{D}(\mathbb{R}^n)$ such that: $\zeta\geq 0$, $\zeta\equiv 1$ on Q_h. We consider positive reals δ small enough in order that the mapping $(\tilde{t},t)\to\chi((t+\tilde{t})/2)w_\delta\left((t-\tilde{t})/2\right)$ belongs to $\mathcal{D}(]0,T[\times]0,T[)$. Then, for such any positive δ, we define the function Ψ_δ in $]0,T[\times\mathbb{R}^n\times]0,T[\times\mathbb{R}^n$ by $\Psi_\delta(r,\tilde{r})=\chi((t+\tilde{t})/2)\zeta((x+\tilde{x})/2)W_\delta(r-\tilde{r})$.

Due to the proposition 1, the inequality (7) holds for u_1 and u_2. We choose in (7) written for u_1 in variables (t,x),

$$k=\tilde{u}_2\equiv u_2(\tilde{t},\tilde{x})\text{ and }\varphi(t,x)=\Psi_\delta(t,x,\tilde{t},\tilde{x})$$

and in (7) written for u_2 in variables $(\tilde{t},\tilde{x})$,

$$k=u_1(t,x)\text{ and }\varphi(\tilde{t},\tilde{x})=\Psi_\delta(t,x,\tilde{t},\tilde{x}).$$

By integrating over Q_h and adding up, it comes:

$$\begin{aligned}
&-\int_{Q_h\times Q_h}(|u_1-\tilde{u}_2|(\partial_t\Psi_\delta+\partial_{\tilde{t}}\Psi_\delta)-\operatorname{sgn}(u_1-\tilde{u}_2)(f(u_1)-f(\tilde{u}_2))(\nabla P.\nabla_x\Psi_\delta+\nabla\tilde{P}.\nabla_{\tilde{x}}\Psi_\delta)\,dr\,d\tilde{r}\\
&+\int_{Q_h\times Q_h}\operatorname{sgn}(u_1-\tilde{u}_2)(g(t,x,u_1)-g(\tilde{t},\tilde{x},\tilde{u}_2))\Psi_\delta\,dr\,d\tilde{r}\\
&\leq\int_{Q_h}\int_{\Sigma_h\setminus\Sigma_{hp}}\operatorname{sgn}(\tilde{u}_2)f(\tilde{u}_2)\nabla P(\bar{\sigma}).\nu_h(\bar{\sigma})\Psi_\delta(\sigma,\tilde{r})\,d\mathcal{H}^n_\sigma\,d\tilde{r}\\
&+\int_{Q_h}\int_{\Sigma_h\setminus\Sigma_{hp}}\operatorname{sgn}(u_1)f(u_1)\nabla\tilde{P}(\bar{\tilde{\sigma}}).\nu_h(\bar{\tilde{\sigma}})\Psi_\delta(r,\tilde{\sigma})\,d\mathcal{H}^n_{\tilde{\sigma}}\,dr\\
&-\int_{Q_h}\operatorname*{ess\,lim}_{\tau\to0^-}\int_{\Sigma_h\setminus\Sigma_{hp}}\operatorname{sgn}(u_1(\sigma+\tau\nu_h))f(u_1(\sigma+\tau\nu_h))\nabla P(\bar{\sigma}).\nu_h(\bar{\sigma})\Psi_\delta(\sigma,\tilde{r})\,d\mathcal{H}^n_\sigma\,d\tilde{r}\\
&-\int_{Q_h}\operatorname*{ess\,lim}_{\tau\to0^-}\int_{\Sigma_h\setminus\Sigma_{hp}}\operatorname{sgn}(u_2(\tilde{\sigma}+\tau\nu_h))f(u_2(\tilde{\sigma}+\tau\nu_h))\nabla\tilde{P}(\bar{\tilde{\sigma}}).\nu_h(\bar{\tilde{\sigma}})\Psi_\delta(r,\tilde{\sigma})\,d\mathcal{H}^n_{\tilde{\sigma}}\,dr\\
&+\int_{Q_h}\int_{\Sigma_{hp}}\operatorname{sgn}(u_1^{hp}(\sigma)-\tilde{u}_2)(f(u_1^{hp}(\sigma))-f(\tilde{u}_2))\nabla P(\bar{\sigma}).\nu_h(\bar{\sigma})\Psi_\delta(\sigma,\tilde{r})\,d\mathcal{H}^n_\sigma\,d\tilde{r}\\
&+\int_{Q_h}\int_{\Sigma_{hp}}\operatorname{sgn}(u_2^{hp}(\tilde{\sigma})-u_1)(f(u_2^{hp}(\tilde{\sigma}))-f(u_1))\nabla\tilde{P}(\bar{\tilde{\sigma}}).\nu_h(\bar{\tilde{\sigma}})\Psi_\delta(r,\tilde{\sigma})\,d\mathcal{H}^n_{\tilde{\sigma}}\,dr.
\end{aligned}\tag{20}$$

Then through a classical reasoning we pass to the limit with δ on the left-hand side of (20). On right-hand side, we refer to Lemma 1. It comes:

$$\begin{aligned}
-\int_{Q_h}|u_1-u_2|\chi'(t)dxdt\;&\leq\;-\int_{Q_h}\operatorname{sgn}(u_1-u_2)(g(t,x,u_1)-g(t,x,u_2))\chi(t)\,dx\,dt\\
&+\;\int_{\Sigma_{hp}}\operatorname{sgn}(u_1^{hp}(\sigma)-u_2^{hp}(\sigma))(f(u_1^{hp}(\sigma))-f(u_2^{hp}(\sigma)))\nabla P(\sigma).\nu_h(\sigma)\chi(t)\,d\mathcal{H}^n.
\end{aligned}$$

The Lipschitz condition for g provides: $\forall\chi\in\mathcal{D}(0,T)$, $\chi\geq0$,

$$\begin{aligned}
-\int_{Q_h}|u_1-u_2|\chi'(t)dxdt\;&\leq\;\int_{\Sigma_{hp}}\operatorname{sgn}(u_1^{hp}-u_2^{hp})(f(u_1^{hp})-f(u_2^{hp}))\nabla P.\nu_h\chi(t)\,d\mathcal{H}^n\\
&\quad+M'_g\int_{Q_h}|u_1-u_2|\chi(t)\,dx\,dt.
\end{aligned}\tag{21}$$

By adding inequalities (19) and (21) we get:

$$\begin{aligned}
-\int_Q|u_1-u_2|\chi'(t)\,dx\,dt\;&\leq\;M'_g\int_Q|u_1-u_2|\chi(t)\,dx\,dt\\
&\quad+\int_{\Sigma_{hp}}\operatorname{sgn}(u_1^{hp}-u_2^{hp})(f(u_1^{hp})-f(u_2^{hp}))\nabla P.\nu_h\chi(t)\,d\mathcal{H}^n\\
&\quad-\int_{\Sigma_{hp}}\operatorname{sgn}(u_1-u_2)(f(u_1^{hp})-f(u_2^{hp}))\nabla P.\nu_h\chi(t)\,d\mathcal{H}^n.
\end{aligned}$$

We set for $\mathcal{H}^n$-a.e. σ in Σ_{hp},

$$I=\Big(\operatorname{sgn}(u_1^{hp}(\sigma)-u_2^{hp}(\sigma))-\operatorname{sgn}(u_1(\sigma)-u_2(\sigma))\Big)\,(f(u_1^{hp}(\sigma))-f(u_2^{hp})(\sigma))\nabla P(\bar{\sigma}).\nu_h(\bar{\sigma}),$$

and we develop a pointwize reasoning to establish that $I\leq0$, $\mathcal{H}^n$-a.e. on Σ_{hp}. Indeed, for $\mathcal{H}^n$-a.e. σ in Σ_{hp}, if $\operatorname{sgn}(u_1^{hp}(\sigma)-u_2^{hp}(\sigma))=\operatorname{sgn}(u_1(\sigma)-u_2(\sigma))$ or $\operatorname{sgn}(u_1^{hp}(\sigma)-u_2^{hp}(\sigma))=0$ then $I=0$ while if $\operatorname{sgn}(u_1^{hp}(\sigma)-u_2^{hp}(\sigma))\neq\operatorname{sgn}(u_1(\sigma)-u_2(\sigma))$ and $\operatorname{sgn}(u_1^{hp}(\sigma)-u_2^{hp}(\sigma))\neq0$ then $\operatorname{sgn}(I)=\operatorname{sgn}(J)$ where,

$$J=\operatorname{sgn}(u_1^{hp}(\sigma)-u_2^{hp}(\sigma))(f(u_1^{hp}(\sigma))-f(u_2^{hp})(\sigma))\nabla P(\bar{\sigma}).\nu_h(\bar{\sigma}).$$

Assume for example that $u_1^{hp}(\sigma) < u_2^{hp}(\sigma)$, the study of the converse situation being similar. If $u_2(\sigma)$ belongs to $]u_1^{hp}(\sigma), u_2^{hp}(\sigma)[$ then we may write (12) for u_1 and for u_2 and with $k = u_2(\sigma)$. By adding and taking into account that $\text{sgn}(u_1(\sigma) - u_1^{hp}(\sigma)) = -\text{sgn}(u_2(\sigma) - u_2^{hp}(\sigma)) = 1$ we obtain that $J \leq 0$. If $u_2(\sigma)$ is not an element of $]u_1^{hp}(\sigma), u_2^{hp}(\sigma)[$ then we take $k = u_1^{hp}(\sigma)$ in (12) for u_2 if $u_2(\sigma) \leq u_1^{hp}(\sigma)$ and $k = u_2^{hp}(\sigma)$ in (12) for u_1 if $u_2(\sigma) \geq u_2^{hp}(\sigma)$. In each case $J \leq 0$.

Therefore (22) becomes

$$-\int_Q |u_1 - u_2|\chi'(t)dxdt \leq M'_g \int_Q |u_1 - u_2|\chi(t)dxdt,$$

for any nonnegative χ of $\mathcal{D}(0,T)$. The inequality of the Theorem 1 follows from the initial condition (5) for u_1 and u_2 and thanks to the Gronwall Lemma. That completes the proof.

4 Existence through the vanishing viscosity method

In [2] we have obtained an existence result of a weak entropy solution to (1)-(2) satisfying (11) when along the interface all the characteristics of the first-order operator set in Q_h have the same behavior: either there are all leaving the hyperbolic domain, either there are all entering in. However in the first situation, an existence property is also established in [1] by means of the vanishing viscosity method and thanks to the notion of process solution (note that for this special *outwards characteristics* framework, the uniqueness proof does not require (11), since data are living the hyperbolic zone along the interface). In this section we use the latter tools to provide an existence result whatever the behavior of characteristics along Σ_{hp} but we are not able to ensure that the weak entropy solution obtained this way fulfills (11).

For any positive ϵ, we introduce $\phi_\epsilon = \phi + \epsilon \mathbb{I}_{\mathbb{R}}$ and we consider the next formal problem : find a bounded and measurable function u_ϵ on Q such that

$$\begin{aligned} \partial_t u_\epsilon - \sum_{i=1}^n \partial_{x_i}(f(u_\epsilon)\partial_{x_i}P) + g(t,x,u_\epsilon) &= \epsilon\Delta\phi_\epsilon(u_\epsilon) \quad \text{in } Q_h, \\ \partial_t u_\epsilon - \sum_{i=1}^n \partial_{x_i}(f(u_\epsilon)\partial_{x_i}P) + g(t,x,u_\epsilon) &= \Delta\phi_\epsilon(u_\epsilon) \quad \text{in } Q_p, \\ u_\epsilon &= 0 \quad \text{on } \Sigma, \\ u_\epsilon(0,.) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{22}$$

subject to the transmission conditions across the interface:

$$\begin{aligned} -(\epsilon\nabla\phi_\epsilon(u_\epsilon) + f(u_\epsilon)\nabla P).\nu_h &= (\nabla\phi_\epsilon(u_\epsilon) + f(u_\epsilon)\nabla P).\nu_p \quad \text{on } \Sigma_{hp}, \\ u_{\epsilon|Q_h} &= u_{\epsilon|Q_p} \quad \text{on } \Sigma_{hp}. \end{aligned} \tag{23}$$

4.1 The viscous problem

First of all, as stated in [1], we remind the next existence and uniqueness result:

Theorem 2 *For any positive ϵ, there exists a unique weak solution u_ϵ to* (22)-(23) *in $W(0,T) \cap L^\infty(Q)$ with $\partial_t u_\epsilon$ in $L^2_{loc}(0,T;L^2(\Omega))$. This solution fulfills*

$$\partial_t u_\epsilon - div(\lambda_\epsilon(x)\nabla\phi_\epsilon(u_\epsilon)) + f(u_\epsilon)\nabla P) + g(t,x,u_\epsilon) = 0 \ a.e. \ on \ Q, \tag{24}$$

$$u_\epsilon(0,.) = u_0 \ a.e. \ on \ \Omega,$$

where $\lambda_\epsilon(x) = \epsilon \mathbb{I}_{\Omega_h}(x) + \mathbb{I}_{\Omega_p}(x)$.

The lack of regularity of the initial data but also the fact that the diffusive term depends on the space variable through λ_ϵ only allow us to establish in [1] the following *a priori* estimates:

Proposition 4 *There exists a constant C independent of ϵ such that:*

$$\|u_\epsilon\|_{L^\infty(Q)} \leq M, \tag{25}$$

$$\|(\lambda_\epsilon)^{1/2}\nabla\widehat{\phi}(u_\epsilon)\|^2_{L^2(Q)^n} + \|(\epsilon\lambda_\epsilon)^{1/2}\nabla u_\epsilon\|^2_{L^2(Q)^n} \leq C, \tag{26}$$

$$\|\partial_t u_\epsilon\|_{L^2(0,T;H^{-1}(\Omega))} \leq C,$$

where M is defined in paragraph 1.1 and $\widehat{\phi}(\tau) = \int_0^\tau \sqrt{\phi'(s)}ds$.

4.2 The ϵ-limit

As a consequence of the proposition 4 and by using a reasoning highlighted in [7], chapter 2:

Proposition 5 *Assume that*

$$\phi^{-1} \text{ is Hölder continuous with an exponent } \theta \text{ in }]0,1[. \tag{27}$$

Then there exists a measurable function u of $L^\infty(Q)$ with $\phi(u)$ in $L^2(0,T;V)$ such that up to a subsequence when ϵ goes to 0^+, the sequence $(u_\epsilon)_{\epsilon>0}$ converges toward u in $L^\infty(Q)$ weak $*$, in $L^q(Q_p)$ for any finite q and a.e. on Q_p. Besides we also have

$$\nabla\phi_\epsilon(u_\epsilon) \rightharpoonup \nabla\phi(u) \text{ weakly in } L^2(Q_p)^n,\ \epsilon\nabla\phi_\epsilon(u_\epsilon) \to 0^+ \text{ strongly in } L^2(Q_h)^n.$$

To characterize the function u - that is formally to pass to the limit with respect to ϵ in (22)-(23) - on the hyperbolic zone we take advantage of (25) and we use that:

Claim 1 *(see [6]) - Let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^q$ $(q \geq 1)$ and $(u_n)_{n>0}$ a sequence of measurable functions on $\mathcal{O}$ such that:*

$$\exists M > 0, \forall n > 0, \|u_n\|_{L^\infty(\mathcal{O})} \leq M.$$

Then, there exist a subsequence $(u_{\varphi(n)})_{n>0}$ and a measurable function π in $L^\infty(]0,1[\times\mathcal{O})$ such that for all continuous and bounded functions ψ on $\mathcal{O}\times]-M,M[$,

$$\forall\xi \in L^1(\mathcal{O}),\ \lim_{n\to+\infty}\int_{\mathcal{O}}\psi(x,u_{\varphi(n)})\xi\,dx = \int_{]0,1[\times\mathcal{O}}\psi(x,\pi(\alpha,w))\,d\alpha\,\xi\,dx.$$

Such a result has first been applied to the approximation through the artificial viscosity method of the Cauchy problem in $\mathbb{R}^p$ for conservation laws, as one can establish a uniform L^∞-control of approximate solutions [5]. It has also been applied to the numerical analysis of transport equations since "Finite-Volume" schemes only give an L^∞-estimate uniformly with respect to the mesh length of the numerical solution (see [6]). Here the approximating sequence is the sequence of solutions to viscous problems (22)-(23).

Theorem 3 *If* (27) *holds and if*

$$f \circ \phi^{-1} \text{ is Lipschitz continuous on } [-M; M], \tag{28}$$

then when ϵ *goes to* 0^+ *the sequence of solutions to viscous problems* (22)-(23)$_{\epsilon>0}$ *strongly converges in* $L^1(Q)$ *toward a weak entropy solution to the coupled parabolic-hyperbolic problem* (1)-(2).

Proof. We consider the function u highlighted in the proposition 5. Since $(u_{\epsilon|\Omega_h})_{\epsilon>0}$ is uniformly bounded, there exist a subsequence - still labelled $(u_{\epsilon|\Omega_h})_{\epsilon>0}$ - and a measurable and bounded function π - called a *process* - on $]0,1[\times Q_h$ such that for any continuous bounded function ψ on $Q_h\times]-M,M[$ and for any ξ of $L^1(Q_h)$

$$\lim_{\epsilon\to 0^+} \int_{Q_h} \psi(t,x,u_\epsilon)\xi \, dx \, dt = \int_{]0,1[\times Q_h} \psi(t,x,\pi(\alpha,t,x))\xi \, d\alpha \, dx \, dt \tag{29}$$

Our aim is first to establish that on the hyperbolic zone, the *process* π is reduced to $u_{|\Omega_h}$, independently of α in $]0,1[$, and secondly to prove that u is a weak entropy solution to (1)-(2) for initial data u_0. To do so, for any positive μ, we take first the scalar product in $L^2(]\mu,T[\times\Omega)$ between (24) and the function $\partial_1 H_1(u_\epsilon,k)\zeta_1$, where ζ_1 belongs to $\mathcal{D}(]-\infty,T[\times\Omega)$, $\zeta_1 \geq 0$, and for any m in $\mathbb{N}^*$ and any real k through

$$H_{1,m}(u_\epsilon,k) = \left((u_\epsilon-k)^2 + \left(\frac{1}{m}\right)^2\right)^{1/2} - \frac{1}{m}.$$

By denoting

$$\begin{aligned} \mathcal{Q}_{1,m}(u_\epsilon,k) &= \int_k^{u_\epsilon} \partial_1 H_{1,m}(\tau,k) f'(\tau) d\tau, \\ G_{1,m}(u_\epsilon,k) &= g(t,x,u_\epsilon)\partial_1 H_{1,m}(u_\epsilon,k), \end{aligned}$$

it comes after some integrations by parts

$$\begin{aligned} &- \int_\mu^T \int_\Omega (H_{1,m}(u_\epsilon,k)\partial_t\zeta_1 - \mathcal{Q}_{1,m}(u_\epsilon,k)\nabla P.\nabla\zeta_1 - G_{1,m}(u_\epsilon,k)\zeta_1) \, dx \, dt \\ &- \int_\Omega H_{1,m}(u_\epsilon(\mu,x),k)\zeta_1(\mu,x) \, dx \\ &\leq -\int_\mu^T \int_\Omega \lambda_\epsilon \partial_1 H_{1,m}(u_\epsilon,k)\nabla\phi_\epsilon(u_\epsilon).\nabla\zeta_1 \, dx \, dt, \end{aligned}$$

the inequality resulting from the convexity of the function $\xi \to H_{1,m}(\xi,k)$ for any real k.

Let us take the limit when μ tends to 0^+ - remember that u_ϵ is an element of $L^\infty(Q) \cap \mathcal{C}([0,T];L^2(\Omega))$ - and then the ϵ-limit separately on the parabolic and hyperbolic zones by using (29) and Proposition 5. We obtain:

$$\begin{aligned} &- \int_{Q_p} (H_{1,m}(u,k)\partial_t\zeta_1 - \mathcal{Q}_{1,m}(u,k)\nabla P.\nabla\zeta_1 - G_{1,m}(u,k)\zeta_1) \, dxdt \\ &- \int_{]0,1[\times Q_h} (H_{1,m}(\pi,k)\partial_t\zeta_1 - \mathcal{Q}_{1,m}(\pi,k)\nabla P.\nabla\zeta_1 - G_{1,m}(\pi,k)\zeta_1) \, d\alpha \, dx \, dt \\ &\leq \int_\Omega H_{1,m}(u_0,k)\zeta_1(0,x)dx - \int_{Q_p} \partial_1 H_{1,m}(u,k)\nabla\phi(u).\nabla\zeta_1 \, dx \, dt. \end{aligned} \tag{30}$$

So for any ζ_1 in $\mathcal{D}(]-\infty,T[\times\Omega_h)$, $\zeta_1\geq 0$,

$$\begin{aligned}
&-\int_{]0,1[\times Q_h}(H_{1,m}(\pi,k)\partial_t\zeta_1-\mathcal{Q}_{1,m}(\pi,k)\nabla P.\nabla\zeta_1-G_{1,m}(\pi,k)\zeta_1)\;d\alpha\,dx\,dt\\
&\leq\int_{\Omega_h}H_{1,m}(u_0,k)\zeta_1(0,x)\,dx.
\end{aligned}$$

where the limit with respect to m provides:

$$\begin{aligned}
&-\int_{]0,1[\times Q_h}(|\pi-k|\partial_t\zeta_1-\operatorname{sgn}(\pi-k)(f(\pi)-f(k))\nabla P.\nabla\zeta_1-\operatorname{sgn}(\pi-k)g(\pi,t,x)\zeta_1)\;d\alpha\,dx\,dt\\
&\leq\int_{\Omega_h}|u_0-k|\zeta_1(0,.)\,dx.
\end{aligned}\tag{31}$$

Let us come back to (24) and consider the $L^2(Q_h)$-scalar product with $\partial_1 H_{2,m}^{\phi(w)}(\phi(u_\epsilon),\phi(k))\zeta_\epsilon\xi$ where for any m in $\mathbb{N}^*$ and any real k and w in $[-M,M]$

$$H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))=\left((\operatorname{dist}(\phi_\epsilon(u_\epsilon),\mathcal{I}(\phi_\epsilon(w),\phi_\epsilon(k))))^2+\left(\frac{1}{m}\right)^2\right)^{1/2}-\frac{1}{m}.$$

Lastly ζ_ϵ belongs to $H_0^1(\Omega_h)$, $\zeta_\epsilon\geq 0$, while ξ is an element of $\mathcal{D}(]0,T[\times\overline{\Omega_h})$, $\xi\geq 0$. By taking into account the convexity of the function $z\to H_{2,m}^{\phi_\epsilon(w)}(z,\phi_\epsilon(k))$ it comes:

$$\begin{aligned}
&-\int_{Q_h}\Big\{(\int_k^{u_\epsilon}\partial_1 H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(\tau),\phi_\epsilon(k))d\tau)\partial_t\xi\zeta_\epsilon-\mathcal{Q}_{2,m}^w(u_\epsilon,k)\nabla P.\nabla\xi\zeta_\epsilon\\
&\qquad-\partial_1 H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))g(t,x,u_\epsilon)\zeta_\epsilon\xi\Big\}\,dx\,dt\\
\leq\;&-\int_{Q_h}\mathcal{Q}_{2,m}^w(u_\epsilon,k)\nabla P.\nabla\zeta_\epsilon\xi\,dx\,dt-\epsilon\int_{Q_h}\partial_1 H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\nabla\phi_\epsilon(u_\epsilon).\nabla(\zeta_\epsilon\xi)\,dx\,dt,
\end{aligned}$$

by denoting

$$\mathcal{Q}_{2,m}^w(u_\epsilon,k)=\int_k^{u_\epsilon}\partial_1 H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(\tau),\phi_\epsilon(k))f'(\tau)d\tau.$$

Now we write:

$$\begin{aligned}
\epsilon\int_{Q_h}\partial_1 H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\nabla\phi_\epsilon(u_\epsilon).\nabla(\zeta_\epsilon\xi)\,dx\,dt &= \epsilon\int_{Q_h}\nabla(H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))).\nabla(\zeta_\epsilon\xi)\,dx\,dt\\
&= \epsilon\int_{Q_h}\nabla(H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\xi).\nabla\zeta_\epsilon\,dx\,dt\\
&\quad+\epsilon\int_{Q_h}\nabla(H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\zeta_\epsilon).\nabla\xi\,dx\,dt\\
&\quad-2\epsilon\int_{Q_h}H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\nabla\zeta_\epsilon.\nabla\xi\,dx\,dt,
\end{aligned}$$

and through an integration by parts of the first line in the last right-hand side:

$$\begin{aligned}
\epsilon\int_{Q_h}\partial_1 H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\nabla\phi_\epsilon(u_\epsilon).\nabla(\zeta_\epsilon\xi)\,dx\,dt &= \epsilon\int_{Q_h}\nabla(H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\xi).\nabla\zeta_\epsilon\,dx\,dt\\
&\quad-\epsilon\int_{Q_h}H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\zeta_\epsilon\Delta\xi\,dx\,dt\\
&\quad-2\epsilon\int_{Q_h}H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\nabla\zeta_\epsilon.\nabla\xi\,dx\,dt.
\end{aligned}$$

Eventually,

$$
\begin{aligned}
& - \int_{Q_h} \Big\{ \big(\int_k^{u_\epsilon} \partial_1 H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(\tau),\phi_\epsilon(k))d\tau\big)\partial_t\xi\zeta_\epsilon - \mathcal{Q}_{2,m}^w(u_\epsilon,k)\nabla P.\nabla\xi\zeta_\epsilon \\
& \qquad\qquad - \partial_1 H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))g(t,x,u_\epsilon)\zeta_\epsilon\xi \Big\}\, dx\, dt \\
\leq & - \int_{Q_h} \mathcal{Q}_{2,m}^w(u_\epsilon,k)\nabla P.\nabla\zeta_\epsilon\xi\, dx\, dt - \epsilon\int_{Q_h} \nabla(H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\xi).\nabla\zeta_\epsilon\, dx\, dt \\
& + \epsilon\int_{Q_h} H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\zeta_\epsilon\Delta\xi\, dx\, dt + 2\epsilon\int_{Q_h} H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\nabla\zeta_\epsilon.\nabla\xi\, dx\, dt.
\end{aligned}
\tag{32}
$$

We consider now the particular choice for the function ζ_ϵ,

$$
\zeta_\epsilon(x) = 1 - \exp\left(- \frac{M'_{f\circ\phi_\epsilon^{-1}}\|\nabla P\|_\infty + \epsilon L}{\epsilon} s(x) \right),\ \epsilon \geq 0,
\tag{33}
$$

where for any positive parameter μ small enough, $s(x) = \min(\text{dist}(x,\Gamma_h),\mu)$ for x in Ω_h, with $L = \sup_{0<s(x)<\mu} |\Delta s(x)|$. That way (see [9, 10]), for any φ of $W^{1,1}(\Omega_h)$, $\varphi \geq 0$,

$$
M'_{f\circ\phi_\epsilon^{-1}}\|\nabla P\|_\infty \int_{\Omega_h} |\nabla\zeta_\epsilon|\varphi dx \leq \epsilon\int_{\Omega_h} \nabla\zeta_\epsilon.\nabla\varphi dx + (M'_{f\circ\phi_\epsilon^{-1}}\|\nabla P\|_\infty + L\epsilon)\int_{\Gamma_h} \varphi d\mathcal{H}^{n-1}.
\tag{34}
$$

Therefore, considering that,

$$
|\mathcal{Q}_{2,m}^w(u_\epsilon,k)| = \left| \int_{\phi_\epsilon(k)}^{\phi_\epsilon(u_\epsilon)} \partial_1 H_{2,m}^{\phi_\epsilon(w)}(\tau,\phi_\epsilon(k))(f\circ\phi_\epsilon^{-1})'(\tau)d\tau \right| \leq M'_{f\circ\phi_\epsilon^{-1}} H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k)) \text{ a.e. on } Q,
$$

and using (34) with $\varphi = H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\xi$, we argue that the first line of the right-hand side of (32) is less or equal than:

$$
M'_{f\circ\phi_\epsilon^{-1}}\|\nabla P\|_\infty \int_{\Sigma_h} H_{2,m}^{\phi_\epsilon(w)}(T_\epsilon(u_\epsilon),\phi(k))\xi d\mathcal{H}^n + o(\epsilon),
$$

with $\lim_{\epsilon\to 0^+} o(\epsilon) = 0$ and for $\mathcal{H}^n$-a.e. σ in Σ_h,

$$
T_\epsilon(u_\epsilon(\sigma)) = \begin{cases} 0 \text{ on } \Sigma_h\backslash\Sigma_{hp}, \\ \phi_\epsilon(u_\epsilon(\sigma)) \text{ on } \Sigma_{hp}. \end{cases}
$$

Hence (32) has became:

$$
\begin{aligned}
& - \int_{Q_h} \Big\{ \big(\int_k^{u_\epsilon} \partial_1 H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(\tau),\phi_\epsilon(k))d\tau\big)\partial_t\xi\zeta_\epsilon - \mathcal{Q}_{2,m}^w(u_\epsilon,k)\nabla P.\nabla\xi\zeta_\epsilon \\
& \qquad\qquad - \partial_1 H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))g(t,x,u_\epsilon)\zeta_\epsilon\xi \Big\}\, dx\, dt \\
\leq & \ M'_{f\circ\phi_\epsilon^{-1}}\|\nabla P\|_\infty \int_{\Sigma_h} H_{2,m}^{\phi_\epsilon(w)}(T_\epsilon(u_\epsilon),\phi(k))\xi d\mathcal{H}^n + o(\epsilon) \\
& + \epsilon\int_{Q_h} H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\zeta_\epsilon\Delta\xi\, dx\, dt + 2\epsilon\int_{Q_h} H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\nabla\zeta_\epsilon.\nabla\xi\, dx\, dt.
\end{aligned}
$$

Let us look at the ϵ-limit for this inequality: since $f\circ\phi^{-1}$ is Lipschitz continuous, $\lim_{\epsilon\to 0^+} M_{f\circ\phi_\epsilon^{-1}} = M_{f\circ\phi^{-1}}$ and stated in [9, 10], $(\zeta_\epsilon)_{\epsilon>0}$ goes to 1 in $L^1(\Omega)$ and $(\epsilon\nabla\zeta_\epsilon)_{\epsilon>0}$ goes to 0 in $L^1(\Omega)^n$. For the term over Σ_h we denote

$$
I_\epsilon = \int_{\Sigma_h} H_{2,m}^{\phi(w)}(T_\epsilon(u_\epsilon),\phi(k))\xi d\mathcal{H}^n.
$$

As soon as $H_{2,m}^{\phi(w)}(.,\phi(k))$ is nonlinear, the weak convergence of the traces of $\phi(u_\epsilon)$ on Σ_{hp} (observe that on $\Sigma_h\backslash\Sigma_{hp}$ there is no difficulty to pass to the limit on ϵ) is not sufficient. That is why we consider the sequence $(H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\xi)_{\epsilon>0}$. Thanks to the proposition 5 and since $H_{2,m}^{\phi_\epsilon(w)}(.,\phi_\epsilon(k))$ is Lipschitz, up to a subsequence $(H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\xi)_{\epsilon>0}$ strongly converges toward $H_{2,m}^{\phi(w)}(\phi(u),\phi(k))\xi$ in $L^q(Q_p)$, $1\le q<+\infty$. Besides, thanks to a chain rule argument and estimate (26), we argue that $(H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\xi)_{\epsilon>0}$ is uniformly bounded in $L^2(0,T;V)\cap L^\infty(Q_p)$ and so weakly converges (up to a subsequence) towards $H_{2,m}^{\phi(w)}(\phi(u),\phi(k))\xi$ in $L^2(0,T;V)$. The trace operator from $L^2(0,T;V)$ into $L^2(\Sigma_p)$ being linear and continuous, $(H_{2,m}^{\phi_\epsilon(w)}(\phi_\epsilon(u_\epsilon),\phi_\epsilon(k))\xi)_{\epsilon>0}$ weakly converges towards $H_{2,m}^{\phi(w)}(\phi(u),\phi(k))\xi$ in $L^2(\Sigma_p)$, and so in $L^2(\Sigma_{hp})$. It comes $\lim_{\epsilon\to0^+}I_\epsilon=I$ where

$$I=\int_{\Sigma_h}H_{2,m}^{\phi(w)}(T(u),\phi(k))\xi d\mathcal{H}^n,$$

where for $\mathcal{H}^n$-a.e. σ in Σ_h,

$$T(u(\sigma))=\begin{cases}0\text{ on }\Sigma_h\backslash\Sigma_{hp},\\ \phi(u(\sigma))\text{ on }\Sigma_{hp}.\end{cases}$$

In addition, thanks to (25) and Claim 1, we are able to pass to the ϵ-limit in the left-hand side of (32). To resume, it comes for any function ξ of $\mathcal{D}(]0,T[\times\overline{\Omega_h})$, $\xi\ge0$:

$$\int_{]0,1[\times Q_h}\left\{\int_k^\pi\partial_1H_{2,m}^{\phi(w)}(\phi(\tau),\phi(k))d\tau\partial_t\xi-\mathcal{Q}_{2,m}^w(\pi,k)\nabla P.\nabla\xi-\partial_1H_{2,m}^{\phi(w)}(\phi(\pi),\phi(k))g_h(t,x,\pi)\xi\right\}d\alpha dxdt$$
$$\ge-M'_{f\circ\phi^{-1}}\|\nabla P\|_\infty\int_{\Sigma_h}H_{2,m}^{\phi(w)}(T(u),\phi(k))\xi d\mathcal{H}^n.$$

When one refers to F.Otto's reasoning in [9, 10], p. 115, lemma 7.34, this inequality implies that for any ζ of $L^\infty(\Sigma_h)$ and ξ of $L^1(\Sigma_h)$, $\xi\ge0$,

$$\operatorname*{ess\,lim}_{\tau\to0^-}\int_{]0,1[\times\Sigma_h}\mathcal{Q}_{2,m}^\zeta(\pi(\alpha,\sigma+\tau\boldsymbol{\nu}),k)\nabla P(\bar\sigma).\nu_h\xi d\alpha d\mathcal{H}^n\le M'_{f\circ\phi^{-1}}\|\nabla P\|_\infty\int_{\Sigma_h}H_{2,m}^{\phi(\zeta)}(T(u),\phi(k))\xi d\mathcal{H}^n.$$

By taking for $\mathcal{H}^n$-a.e. σ in Σ_h

$$\zeta(\sigma)=\phi^{-1}(T(u(\sigma)))=\begin{cases}0\text{ on }\Sigma_h\backslash\Sigma_{hp},\\ u(\sigma)\text{ on }\Sigma_{hp},\end{cases}$$

where $u(\sigma)$ is defined as $\phi^{-1}(\phi(u(\sigma)))$ and belongs to $L^\infty(\Sigma_{hp})$, and then the limit with respect to m we get the boundary conditions for π:

$$\operatorname*{ess\,lim}_{\tau\to0^-}\int_{]0,1[\times\Sigma_h}\mathcal{F}(\pi(\alpha,\sigma+\tau\boldsymbol{\nu}),\phi^{-1}(T(u(\sigma))),k)\nabla P(\bar\sigma).\nu_h\xi d\alpha d\mathcal{H}^n\le0.\tag{35}$$

To conclude, the process π fulfills (31) and (35), that means π is an *entropy process solution* to the quasilinear first-order hyperbolic problem set on Q_h: find a measurable and bounded function w such that formally

$$\begin{cases}\partial_tw-\sum_{i=1}^n\partial_{x_i}(f(w)\partial_{x_i}P)+g(t,x,w)=0\text{ in }Q_h,\\ w=u_{\Gamma_h}\text{ on }\Gamma_h,\\ w(0,.)=u_0\text{ on }\Omega_h,\end{cases}$$

where u_{Γ_h} is the element of $L^\infty(\Gamma_h)$ given, for $\mathcal{H}^n$-a.e. σ in Σ_h, by:

$$u_{\Gamma_h}(\sigma) = \begin{cases} 0 \text{ on } \Sigma_h \backslash \Sigma_{hp}, \\ u(\sigma) \text{ on } \Sigma_{hp}, \end{cases}$$

the trace $u(\sigma)$ being defined as above. Due to [9, 10] we know that this problem has a unique solution. Namely we may rewrite in the context of entropy process the proof provided in [9, 10]. It follows that if π_1 and π_2 are two process solutions for initial data $u_{0,1}$ and $u_{0,2}$ respectively, then for a.e. t in $]0, T[$,

$$\int_{]0,1[\times\Omega_h} |\pi_1(\alpha, t, x) - \pi_2(\beta, t, x)| \, d\alpha \, d\beta dx \, dt \leq \int_{\Omega_h} |u_{0,1} - u_{0,2}| \, dx \, e^{M'_g t}.$$

Classically we first deduce that when $u_{0,1} = u_{0,2}$ on Ω_h, there exists a measurable function u_h on Q_h such that a.e. on Q_h, $u_h(.) = \pi_1(\alpha, .) = \pi_2(\beta, .)$ for a.e. α and β of $]0, 1[$. Another consequence of the uniqueness property is that the whole sequence $(u_\epsilon)_{\epsilon>0}$ strongly converges to u_h in $L^q(Q_h)$, $1 \leq q < +\infty$. Thus $u_h = u_{|\Omega_h}$ a.e. on Q_h and in (30) the integrals over $]0, 1[$ are performed. By taking the limit with respect to m, it follows that u fulfills (3) and (5). Boundary conditions (4) are written in (35) due to the definition of $\phi^{-1}(T(u))$ that completes the proof of Theorem 3. ■

4.3 Conclusions

As a conclusion, let us collect the existence result stated in section 3 and the existence property of section 4. The key point of the uniqueness theorem 1 being the existence of a strong trace of a weak entropy solution to (1)-(2), while the existence property just requires additional assumptions on f and ϕ. So we claim:

Corollary 1 *Suppose that* (27) *and* (28) *hold. Then the coupling problem* (1)-(2) *has a weak entropy solution that is the* $L^1(Q)$*-limit of a sequence of solutions to viscous problems* (22)-(23)$_{\epsilon>0}$ *when* ϵ *goes to* 0^+.

In addition if u *fulfills* (11) *then* u *is the only weak entropy solution satisfying* (11).

Observe that in [2], when the interface is included either in the set of outwards characteristics for the first order operator set in Q_h, either in set of inward characteristics for the first order operator set in Q_h, we highlight in each situation a weak entropy solution to (1) that fulfills (11). So that, in this special framework, the coupling problem (1) has a unique weak entropy solution.

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