Monografías de la Real Academia de Ciencias de Zaragoza **38:** 53–64, (2012)
ISSN 1132-6360

Estimates of the solutions of some asymmetric problem defined on $\mathbb{R}^N$, $N \geq 3$

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This paper is dedicated to Prof. Monique Madaune Tort

Abstract

We study here the following equations defined on $I\!R^N$, $N \geq 3$

$$(E_f) \qquad (-\Delta + q(x))u \;=\; \mu m(x)u + f\,,\; \lim_{|x|\to+\infty} u(x) = 0,$$

$$(Q_f) \qquad (-\Delta + q(x))u \;=\; m(x)(au^+ - bu^-) + f\,,\; \lim_{|x|\to+\infty} u(x) = 0.$$

We consider an "intermediate" case where $q \geq 0$ grows less than $|x|^2$ and m decreases less than $|x|^{-2}$ which are the cases usually studied. We study in particular the sign of the solutions for the parameters μ, a, b varying around the associated principal eigenvalue μ^* defined by $(-\Delta + q(x))u \;=\; \mu^* m(x)u$.

1 Introduction

We study here the sign of the solution to the following equation defined on $I\!R^N$, $N \geq 3$:

$$(E_f) \qquad (-\Delta + q(x))u \;=\; \mu m(x)u + f\,,\; \lim_{|x|\to+\infty} u(x) = 0,$$

and then the associated asymmetric problem defined on $I\!R^N$, $N \geq 3$:

$$(Q_f) \qquad (-\Delta + q(x))u \;=\; m(x)(au^+ - bu^-) + f\,,\; \lim_{|x|\to+\infty} u(x) = 0,$$

where the potential q and the weight m are not necessarily positive. μ, a, b are real parameters. As usual, for any function changing sign, we set

$$u = u^+ - u^- \text{ where } u^+(x) = max(u(x), 0).$$

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The data $f \in L^\infty_{loc}$.
There are several results on asymmetric problems, most of them being defined on a bounded domain (*e.g.* [9]), [1]).
Our problem is defined on $I\!R^N$ and it is well-known that weights must be introduced for having a discrete spectrum. Two cases for Equation (E_f) defined on $I\!R^N$ have been intensively studied: the case where the potential q grows fast enough to ∞ and $m \equiv 1$ ([3] [5]) and the case $q \equiv 0$ with a weight decreasing fast enough (even for the p-Laplacian) ([14], [9], [10], ...).
Here we consider an *"intermediate"* case (the potential grows "slowly" , the weight decreases "slowly") and we show that several of the above mentioned results are still valid.
We study also some "completely indefinite" problems that is q and m may change sign. These problems have a long history since it starts at the end of the 19th century with Hilb, Bocher, Boggio, ... and later by Atkinson, Mingarelli always for ODE.
It has been shown by Richardson (1918) that there is a pair of non real eigenvalues ($\simeq \pm 4, 36i$) for the following equation:

$$-y'' - \frac{9\pi^2}{4} y = \lambda g(x) y \quad 0 < x < 2;\ y(0) = y(2) = 0,$$

with $g(x) = 1$ if $0 < x < 1$ and $g(x) = -1$ for $1 < x < 2$.
The first paper on the multidimensional case is due to Holmgren in 1904 who studies the Dirichlet problem on a bounded domain in $I\!R^2$:

$$\Delta u + \lambda g(x, y) u = 0$$

when g changes sign.
He shows that there exists 2 countable (and infinite) set of eigenvalues: one of positive ones, the other of negative ones.

$$... \le \lambda_j^- \le \lambda_{j-1}^- \le ...\lambda_1^- < (0) < \lambda_1^+ < \lambda_2^+ \le ... \le \lambda_j^+ \le ...$$

This paper is organized as follows:
We study first Equation (E_f) for $m > 0$; we set some hypotheses on q and m, and state our results. We prove them first when q is non negative before considering the case where it changes sign. We derive then the asymmetric case.
The last section is devoted to the "completely indefinite" problem that is q and m are not necessarily positive. We consider only real eigenfunctions. We extend some of the results obtained for a positive weight to some "indefinite problems".

2 Hypotheses and Results for $m > 0$

2.1 Hypotheses

(H_q) We assume that q is continuous and there exists $c' > 0$ such that $q^+ \ge c' p^\beta$, $\beta > 0$, with

$$p := (1 + |x|^2)^{1/2}.$$

(H_m) We assume that m is continuous, bounded and m tends to 0 at infinity. More precisely, there exists $c" > 0$, $C" > 0$ such that

$$0 < C"/p^\gamma \leq m \leq c"/p^\alpha \;,\; \gamma \geq \alpha > 0.$$

(H_0) When q changes sign we assume that there exists $c_1 > 0$ such that

$$0 \leq q^- \leq c_1 m \;\Leftrightarrow\; c_1 m - q^- \geq 0.$$

(H_f) Assume $f = mh$, $h \in L^\infty \cap L^2_m(I\!R^N)$; $f \geq 0, \not\equiv 0$.

Remark 1 *$m/q^+ \to 0$ as $|x| \to \infty$ since $\alpha + \beta > 0$.*

We consider the weak formulation of (E_f). We introduce the space

$$V_q = \{u \in \mathcal{D}^{1,2}(I\!R^N) : \int_{I\!R^N} q^+ u^2 < \infty\}, \tag{1}$$

equipped with norm $[\int_{I\!R^N} (|\nabla u|^2 + q^+ u^2)]^{1/2}$. We seek $u \in V_q$, satisfying

$$\int_{I\!R^N} (\nabla u . \nabla \chi + q u \chi) \;=\; \mu \int_{I\!R^N} m u \chi + \int_{I\!R^N} f \chi,\; \forall \chi \in V_q. \tag{2}$$

Remark 2 *If $\int_{I\!R^N} q^+ u^2 < \infty$ then $\int_{I\!R^N} q u^2 < \infty$, by Remark 1.*

Indeed our hypotheses imply that u is continuous and even $\mathcal{C}^1$.

2.2 Existence of a PEV for a homogeneous problem, $m \geq 0$

$$(-\Delta + q) u = \mu m(x) u. \tag{EV}$$

Proposition 1 *If (H_m) , (H_q) and (H_0) are satisfied, (EV) posseses a "principal eigenvalue" (PEV) denoted by μ^*:*

$$\mu^* \;=\; inf_{u \in V_q} \frac{\int_{I\!R^N} (|\nabla u|^2 + q u^2)}{\int_{I\!R^N} m u^2}. \tag{3}$$

This eigenvalue is simple. The equality is achieved iff u is proportional to the associated eigenfunction.

Notation: From now on let us denote by ϕ the associated eigenfunction (groundstate) which is positive and such that $\int_{I\!R^N} m\phi^2 = 1$.

2.3 Sign of the Solutions of (E_f), $m \geq 0$

We consider now the following equation defined on $I\!R^N$:

$$(E_f) \qquad (-\Delta + q)u = \mu m(x)u + f, \lim_{|x|\to+\infty} u(x) = 0.$$

We always assume in this section that

Hypothesis (H) **:** (H_q), (H_m), (H_0), (H_f) are satisfied.

First let us recall a classical result:

Proposition 2 (Maximum Principle (MP)) *If (H) is satisfied, any solution u to (E_f) with $\mu < \mu^*$. is positive; conversely $u > 0 \Rightarrow \mu < \mu^*$.*

We compare now locally u solving (E_f) to ϕ for $|\mu - \mu^*|$ small enough : following *e.g.* [3] we say thet u is "fundamentally positive (resp. negative)" if there exists a constant $C > 0$ such that $u > C\phi$ (resp. $u < -C\phi$), ϕ being the groundstate.

Theorem 1 (Local fundamental positivity or negativity (FPN$_{loc}$)) *Assume (H). Let $R > 0$ be given; there exists $\delta > 0$ and 2 positive constants K' and $K"$ depending on R, f, q and m such that*

* *for $\mu^* - \delta < \mu < \mu^*$, then $u > \frac{K'}{\mu^*-\mu}\phi > 0$ on the ball $B_R = \{x \in I\!R^N/|x| < R\}$.*
* *for $\mu^* < \mu < \mu^* + \delta$, then $u < -\frac{K"}{\mu-\mu^*}\phi < 0$ on the ball $B_R = \{x \in I\!R^N/|x| < R\}$.*

This result is shown in [14], Theorem 5.2 for $q = 0$ and $\alpha > 2$. It is shown in [3], [5] for $\beta > 2$, $\alpha = 0$ and the authors obtain a global result: $R = \infty$. These comparison results are an extension of local maximum and antimaximum principle. Recall that the antimaximum principle has been introduced by P.Clément and L.Peletier in [8].

For $\mu < \mu^*$, (H) being satisfied, we deduce from the maximum principle a "global" fundamental positivity result.

Theorem 2 (Fundamental Positivity) *Assume (H); assume moreover that there exists a positive number C_h such that*

$$(4) \qquad h > C_h\phi > 0.$$

Then, for any given $\mu < \mu^*$, u solving (E_f) is fundamentally positive

$$(5) \qquad u > \frac{C_h}{\mu^* - \mu}\phi.$$

Analogously $h < C'_h\phi \Rightarrow u < \frac{C'_h}{\mu^*-\mu}\phi$.

2.4 Asymmetric problem (Q_f)*,* $m \geq 0$

Consider now (Q_f)

$$(Q_f) \qquad (-\Delta + q(x))u \;=\; m(x)(au^+ - bu^-) + f\,,\ \lim_{|x|\to+\infty} u(x) = 0,$$

We simply follow [9] which considers the case of the p-Laplacian on a bounded domain. We derive from above simple results on (Q_f).

Proposition 3 (Non existence) *Assume* (H)*. Then* (Q_f) *has no solution in* V_q *for* $a = \mu^*$ *and* $b \leq \mu^*$*.*

Proposition 4 (Maximum Principle) *Assume* (H) *and* $b < \mu^*$*; then any (eventual) solution is positive; conversely if there exists a positive solution, then* $b < \mu^*$*.*

Proposition 5 (Local antimaximum) *Assume* (H)*; for any given* $R > 0$*, there exists* $\delta > 0$ *depending on* R *and* h*, such that, if* $\mu^* < a < \mu^* + \delta$ *and* $\mu^* < b < a < \mu^* + \delta$*, then* $u < 0$ *on the ball* B_R*.*

Corollary 2.1 *Assume* (H)*; for any given* $R > 0$*, there exists* $\delta > 0$ *depending on* R *and* h*, such that, if* $\mu^* < a < \mu^* + \delta$ *and* $b < \mu^*$*,* (Q_f) *has no solution on* B_R*.*

Corollary 2.2 *Assume* (H)*; for any given* $R > 0$*, there exists* $\delta > 0$ *depending on* R *and* h*, such that, if* $\mu^* < b < \mu^* + \delta$ *and* $a < \mu^*$*,* (Q_f) *has at least 2 solutions: one is positive everywhere and the other negative on a ball.*

3 Proofs of the results for a positive weight: $m > 0$

3.1 Proof of Proposition 1:

3.1.1 Case $q > 0$

We introduce the hermitian forms

$$(6) \qquad a(u,\chi) := \int (\nabla u.\nabla\chi + qu\chi)\ ,\ b(u,\chi) := \int mu\chi,\ \forall \chi \in V_q.$$

Consider (EV) in its variational form. We seek $u \in V_q$. Following *e.g.* [15], (EV) can be written as

$$a(u,\chi) = \mu b(u,\chi).$$

We define a linear operator $T : V_q \to V_q$ by

$$a(Tu,\chi) = b(u,\chi).$$

T is continuous and selfadjoint in V_q since a is coercive and b is continuous:

$$\|Tu\|^2_{V_q} = a(Tu,Tu) = b(u,Tu) \leq C"\|u\|_{V_q}\|Tu\|_{V_q}.$$

Formally $T = (-\Delta + q)^{-1}M$ where M denotes the multiplication by m. We show now that T **is compact**. We have

$$\|Tu\|_{V_q} \leq C"\|u\|_{V_q}.$$

We choose a bounded sequence u_k in V_q and show that Tu_k is a Cauchy sequence. Since u_k is bounded in V_q, it is bounded in $H^1(B_R)$ where $B_R := \{x \in I\!R^N : |x| < R\}$. By compactness of $H^1(B_R)$ into $L^2(B_R)$ there exists a converging subsequence. On $B'_R := \{x \in I\!R^N : |x| > R\}$, we have for any $u \in V_q$:

$$\int_{B'_R} mu^2 \leq \sup_{B'_R} \frac{m}{q} \int_{B'_R} qu^2 \to 0 \text{ as } R \to \infty.$$

Hence finally there exists a subsequence such that, for k and l large enough, $\|T(u_k - u_l)\|_{V_q} \leq \epsilon$.
T admits a sequence of positive eigenvalues $Tu = \frac{1}{\mu}u$ and μ is eigenvalue for (EV).
μ^* is the smallest eigenvalue of (EV); it is simple and positive; it is a "principal eigenvalue", that is an eigenvalue with associated eigenfunction ϕ which does not change sign.

3.1.2 Case $q = q^+ - q^-$ and $q^- \not\equiv 0$

We deduce from (H_0) that $c_1 m + q^- > 0$ and we write (EV) as

$$Lu := (-\Delta + q(x) + c_1 m)u \;=\; (c_1 + \mu)m(x)u.$$

We are lead to an equation with a positive potential. All the results from the previous section still hold with q replaced by $Q(x) = q(x) + c_1 m(x)$, μ by $c_1 + \mu$ and μ^* by $c_1 + \mu^*$. Also the principal eigenfunction ϕ is still the same and $\mu - \mu^*$ is invariant. For the asymmetric case, -later- a is replaced by $a + c_1$ and b by $b + c_1$-.

3.2 Proof of Theorem 1

Write

(7) $$u = u_1\phi + u^\perp \text{ where } \int_{I\!R^N} mu^\perp\phi = 0.$$

For $f = mh$, analogously set $h = h_1\phi + h^\perp$. It follows from (E_f) that

$$\mu^* u_1 m\phi + (-\Delta + q)u^\perp = \mu m u_1\phi + \mu m u^\perp + mh_1\phi + mh^\perp.$$

Therefore

(8) $$u_1 \;=\; \frac{h_1}{\mu^* - \mu},$$

(9) $$(-\Delta + q)u^\perp \;=\; \mu m u^\perp + mh^\perp.$$

We show now that $u^\perp/\phi$ is bounded on any ball. First let us remark that

$$\int_{I\!R^N} (|\nabla u^\perp|^2 + q(u^\perp)^2) \geq \mu_2 \int_{I\!R^N} m(u^\perp)^2,$$

where μ_2 denotes the second eigenvalue of (EV); we derive:

$$(\mu_2-\mu)\int_{I\!R^N} m(x)(u^\perp)^2 \le \int_{I\!R^N} m(x)h^\perp u^\perp \le$$

$$\Big[\int_{I\!R^N} m(x)(h^\perp)^2 \int_{I\!R^N} m(x)(u^\perp)^2\Big]^{1/2}.$$

Hence

$$\int_{I\!R^N} m(x)(u^\perp)^2 \le \Big(\frac{1}{\mu_2-\mu}\Big)^2 \int_{I\!R^N} m(x)(h^\perp)^2. \tag{10}$$

* When $\mu<\mu^*<\mu_2,\ \mu_2-\mu>\mu_2-\mu^*$. Hence there exists $K>0$, depending on m and $h^\perp$, such that

$$\int_{I\!R^N} m(x)(u^\perp)^2 \le K(\mu_2-\mu^*)^{-2};$$

this upperbound is independent of μ!

* When $\mu^*<\mu<\mu_2$, we choose $\mu_2-\mu>\frac{1}{4}(\mu_2-\mu^*)$. Again there exists $K'>0$ such that

$$\int_{I\!R^N} m(x)(u^\perp)^2 \le K'(\mu_2-\mu^*)^{-2};$$

and again the upperbound is independent of μ.

In both cases, $u^\perp$ being bounded in $L^2_m(I\!R^N)$, is also bounded on $L^2_m(B_R)$ for any ball B_R and by (H_m) it is bounded in $L^2(B_R)$. By bootstrap method it is bounded on $\mathcal{C}^1(B_R)$. On B_R, the groundstate ϕ is bounded below by some constant C_R, and the weight by C'_R and hence it is possible to choose $\mu^*-\mu$ small enough so that

$$|u^\perp| \le K''\frac{h^\perp}{(\mu_2-\mu^*)}\phi. \tag{11}$$

Combining (7), to (11) we derive the results. •

3.3 Proof of Theorem 2

This follows simply by comparison. Denoting as above $L=-\Delta+q+c_1m$, we have $L(u-k\phi)=\mu m(u-k\phi)+H$ with $H=mh-(\mu^*-\mu)mk\phi$. For $k=\frac{C_h}{\mu^*-\mu}$, then $H>0$ and the Maximum Principle gives the result. •

3.4 Proof of Proposition 3

This is a classical consequence of Fredholm's alternative; (Q_f) can be written:

$$(-\Delta+q)u=\mu^*mu+m[h+(\mu^*-b)u^-]$$

and the term $m[h+(\mu^*-b)u^-]$ is positive.
Conversely, $b<\mu^*$ is a necessary condition; by contradiction choose $u=-\phi<0$. $Lu=bmu+F$ with $F=(\mu^*-b)mu>0$. •

3.5 Proof of Proposition 4

Assume u changes sign. Multiplying (Q_f) by u^- we derive

$$(\mu^* - b)\int_{I\!R^N} m(u^-)^2 < 0.$$

Hence $u > 0\, a.e$ and by regularity $u > 0$.
Conversely we exhibit a counterexample : $u = -\phi < 0$ solves (Q_f) with $f = (\mu^* - b)(-\phi) > 0$. •

3.6 Proof of Proposition 5

Since $a - b > 0$ write (Q_f) as

$$(-\Delta + q)u = amu + m(h + (a - b)u^-),$$

and apply Theorem 1. •

3.7 Proof of Corollary 2.1

By Proposition 4, $u > 0$ and by Proposition 5, locally $u < 0$. •

3.8 Proof of Corollary 2.2

By Proposition 4, the equation $(-\Delta+q)u = amu+mh$ has a solution which is > 0 so that $u^- \equiv 0$ and $u = u^+$. Hence

$$(-\Delta + q)u = amu + mh = amu^+ - bmu^- + mh.$$

Analogously $(-\Delta + q)u = bmu + mh$ has a solution which is < 0 on B_R and therefore $u^+ \equiv 0$ on B_R. Therefore

$$(-\Delta + q)u = bmu + mh = amu^+ - bmu^- + mh.$$

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4 Completely indefinite problem

4.1 Hypotheses (H')

We assume
(H_q) q is continuous and there exists $c' > 0$ such that $q^+ \geq c'p^\beta$, $\beta \geq 0$.
(H'_m) m is continuous, bounded and m^+ tends to 0 at infinity. More precisely, there exists $K" > 0$, $k" > 0$ such that

$$0 < g^+ := m^+ + k"/p^\gamma \leq K"/p^\alpha,\ \gamma \geq \alpha > 0.$$

$(H"_m)$ There exists $c_2 > 0$ such that $0 \leq m^- \leq c_2q^+$.

(H_0') There exists $c_1 > 0$ such that

$$0 \leq q^- \leq c_1 g^+ \;\Leftrightarrow\; c_1 g^+ - q^- \geq 0.$$

(H^+) $\tau^* > 0$ where τ^* is the principal eigenvalue of

$$(-\Delta + q(x))u = \tau g^+(x)u \text{ on } I\!R^N,$$

and ψ the associated positive principal eigenfunction such that $\int_{I\!R^N} g^+\psi^2 = 1$.

4.2 Existence of a Principal Eigenpair

Theorem 3 *If (H') holds, (EV) has a principal eigenvalue μ^* which is positive.*

The existence of principal eigenvalues for completely indefinite PDE is studied *e.g.* in [12] or more recently in [11] but always for problems defined on bounded domains.
Proof of Theorem 3: We write

$$(EV) \qquad (-\Delta + q(x))u = \mu m(x)u \text{ on } I\!R^N \;\Leftrightarrow$$

$$(EV') \qquad (-\Delta + q)u + \mu g^- u = \mu g^+ u,$$

where $g^\pm := m^\pm + k"/p^\gamma$.
Consider now a one parameter eigenvalue problem derived from (EV')

$$(EV_s) \qquad (-\Delta + q + sg^-)u = \mu_s g^+ u, \; x \in I\!R^N.$$

The weight is positive, due to Hypothesis (H_m') and we can apply the results of the previous sections. For any s, there exists a principal eigenpair $(\mu_s > 0, u_s > 0)$:

$$(12) \qquad \mu_s = inf_{u\in W} \frac{\int_{I\!R^N}(|\nabla u|^2 + qu^2 + sg^- u^2)}{\int_{I\!R^N} g^+ u^2}.$$

Here

$$W = \{u \in V_{(q^+ + sg^-)} \; s.t. \int_{I\!R^N}(|\nabla u|^2 + qu^2 + sg^- u^2) < \infty\}.$$

By (H'), $V_{q^+ + sg^-} = W$.
We derive from (H^+) and (12) that

$$(13) \qquad \mu_s < \tau^* + s\frac{\int_{I\!R^N} g^-\psi^2}{\int_{I\!R^N} g^+\psi^2} = \tau^* + s\int_{I\!R^N} g^-\psi^2.$$

The curve (s, μ_s) is continuous and increases. $\mu_0 = \tau^* > 0$.
Finally we derive from the properties of eigenvalues that $\mu_s < \nu_s$ the principal eigenvalue of the same equation defined (with Dirichlet boundary conditions) on the smaller domain $\Omega_+ := \{x \in I\!R^N : m(x) > 0\}$. From $(-\Delta + q)u = \nu^* m^+ u, \; x \in \Omega_+$, we derive as before $(-\Delta + q + sg^-)u = \nu_s g^+ u, \; x \in \Omega_+$. The curve (s, ν_s) cuts the bisctrix at ν^* the fixed point of $\nu_s = s$. Hence the curve (s, μ_s) starting from $\tau^* > 0$, below (s, ν_s) cuts the bisectrix at μ^* which is the positive principal eigenvalue of (EV). •

4.3 Sign of the solutions

Assume that (H') is satisfied and

(H'_f) $\quad f \in L^\infty \cap L^2_{1/|m|}(I\!R^N)$; $f = m^+h \geq 0, \not\equiv 0$.

4.3.1 Maximum Principle

Theorem 4 (Maximum Principle (MP)) *If (H') and (H'_f) are satisfied, any solution u to (E_f) with $\mu < \mu^*$ is positive; conversely $u > 0 \Rightarrow \mu < \mu^*$.*

Proof of Theorem 4 (MP):
With the same notations as above , Equation (E_f) can be written

$$(E'_f) \qquad (-\Delta + q + \mu g^-)u = \mu g^+ u + f.$$

Consider the one parameter equation:

$$(E"_s) \qquad (-\Delta + q + s g^-)u = \mu g^+ u + f.$$

Considering the curve (s, μ_s) we have $s < \mu^* \Leftrightarrow \mu_s < \mu^*$. To any $\mu < \mu^*$, it corresponds $s < \mu^*$ and $\mu_s < \mu^*$ and we can apply Proposition 2 to $(E"_s)$. •

4.3.2 Local Antimaximum Principle

Theorem 5 (Local Antimaximum Principle) *Assume (H') and (H'_f). Let $R > 0$ be given; there exists $\delta > 0$ depending on R, f, q and m such that for $\mu^* < \mu < \mu^* + \delta$, then $u < 0$ on the ball $B_R = \{x \in I\!R^N / |x| < R\}$.*

Proof of Theorem 5
(E_f) can be written as before as $(E"_s)$. Always as before considering the curve (s, μ_s), $s > \mu^* \Leftrightarrow \mu_s > \mu^*$. We apply Theorem 1 to $(E"_s)$. •

4.4 Asymmetric Problems

We consider (Q_f).

4.4.1 Results

Theorem 6 *Assume (H') and (H'_f). If $b < \mu^*$, then the maximum principle is valid.*

Theorem 7 *Assume (H') and (H'_f). Then conclusion of Corollary 2.2 is still valid.*

Proof of Theorem 6
We assume $u = u^+ - u^-$ for $b < \mu^*$. Multiply (Q_f) by u^- and integrate. We obtain

$$\mu^* \int m^+(u^-)^2 \leq -(Lu, u^-) = b \int m(u^-)^2 - \int f u^- \leq$$

$$b\int m^+(u^-)^2 - b\int m^-(u^-)^2 \leq b\int m^+(u^-)^2.$$

Hence $(\mu^* - b)\int m^+(u^-)^2 \leq 0$ and $u^- = 0\, a.e.$ Therefore $u > 0$. $\bullet$.

Proof of Theorem 7

This is exactly as in Section 3. This follows simply from Theorems 4 and 5.

Acknowledgement. The author thanks Conseil Régional Midi-Pyrénées -CTP project

References

[1] M.Arias, J.Campos, M.Cuesta, J.P. Gossez *Asymmetric Elliptic Problems with Indefinite Weights*

[2] W. Allegretto, *Principal eigenvalues for indefinite weight elliptic problems in* $\mathbb{R}^n$. **Proc. Amer. Math. Soc.**, 116, 1992, p.701-706.

[3] B. Alziary, J. Fleckinger, P. Takáč, *Positivity and negativity of solutions to a Schrödinger equation in* $\mathbb{R}^N$. **Positivity**, 5, 4, 2001, p.359-382.

[4] B. Alziary, J. Fleckinger, P. Takáč, *An extension of maximum and anti-maximum principles to a Schrödinger equation in* $\mathbb{R}^2$. **J. Diff. Equ.**, 156, 1999, p.122-152.

[5] B. Alziary, P. Takáč, *A pointwise lower bound for positive solutions of a Schrödinger equation in* $\mathbb{R}^N$. **J. Diff. Equ.**, 133, 2, 1997, p.280-295.

[6] K. J. Brown, C. Cosner, J. Fleckinger, *Principal eigenvalues for problems with indefinite weight function on* $\mathbb{R}^N$. **Proc. Amer. Math. Soc.**, 109, 1, 1990, p.147-155.

[7] K. J. Brown, A. Tertikas, *The existence of principal eigenvalues for problems with indefinite weight function in* $\mathbb{R}^n$. **Proc. Royal Soc. Edinburgh**, 123A, 1993, p.561-569.

[8] P. Clément, L. Peletier, *An anti-maximum principle for second order elliptic operators.* **J. Diff. Equ.**, 34, 1979, p.218-229.

[9] J. Fleckinger, J. P. Gossez, P. Takáč, F. de Thélin, *Existence, nonexistence et principe de l'antimaximum pour le p-laplacien.* **C. R. Acad. Sci. Paris 321**, 1995, p.731-734.

[10] J. Fleckinger, J. P. Gossez, F. de Thélin, *Antimaximum principle in* $\mathbb{R}^N$*: local versus global.* **J. Diff. Equ.**, 8, 7B, 2004, p.159-188.

[11] J. Fleckinger, J. Hernandez, F. de Thélin, *Existence of multiple principal eigenvalues for some indefinite linear eigenvalue problems.*, **Boll. U.M.I.**, 8, 7B, 2004, p.159-188.

[12] J. Fleckinger, A. B. Mingarelli, *On the eigenvalues of non-definite elliptic operators*, **Differential Equations**, North Holland, 1984, p. 219-222.

[13] G.V. Richardson *Contribution to the study of oscillation properties of linear differential equations of second order,* **Amer. Jal. Math***, 40, (1918), p.283-316* Amer. Jal. Math, 40, (1918), p.283-316

[14] N. Stavrakakis, F. de Thélin, *Principal eigenvalues and anti-maximum principle for some quasilinear elliptic equations on* $\mathbb{R}^n$. **Math. Nachr.**, 212, 1, 2000, p.155 - 171.

[15] H.F. Weinberger *Variational methods for eigenvalue approximation*, chap III, Regional Conf. Series in Appl. Math., vol 15, Siam, Philadelphia, Penn., 1974.