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Three families of isochrone-type canonical transformations

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Abstract

Hénon's isochrone Hamiltonian is formulated in extended phase space including a time transformation. Then, the Hamilton-Jacobi equation *a la Poincaré* is used to find suitable canonical transformations that reduces the original Hamiltonian to a function of only the momenta. We focus on three different time transformations, for each of which we build a family of canonical transformations where the new Hamiltonian remains unspecified. Materialization of particular transformations based on specific requirements lead to a partial differential equation which the new Hamiltonian ought to satisfy. Specifically, we show how different canonical transformations in the Literature may be recovered from our families.

Introduction

Hénon isochrone model [10] is a particular case of a central potential that has been successfully used in some fields of astronomy. Specifically, it is suitable for representing the mean potential of a stellar system, and may be used to understand the evolution of our galaxy [6]. The isochronal potential is studied in textbooks as an interesting example of an integrable problem depending on two parameters, which comprises the Keplerian potential as a limit case [3]. Besides, it may be used as a zero order to apply perturbation theory to systems that slightly depart from the spherical symmetry [8, 19]. To make tractable the perturbation approach it is necessary to find a suitable set of variables or elements, so that the integrable Hamiltonian in the new variables is usually expressed as a function of momenta only. The Hamilton-Jacobi equation provides a convenient way of finding the required transformation.

The Hamilton-Jacobi equation is commonly used for finding a single canonical transformation tailored to a specific problem; for instance, it is usually applied in the search for efficient numerical integrators or when dealing with perturbation methods. However, as far as the new Hamiltonian may remain formal in the procedure of computing the transformation [18], the Hamilton-Jacobi equation of a particular Hamiltonian may give rise to a whole family of transformations. General applications of this procedure have been discussed in Ref. [7]; we present here details on its application to finding canonical transformations useful in orbital problems.

We form the Hamilton-Jacobi equation of the isochronal potential in the extended phase space. Besides, we scale the Hamiltonian by a regularizing function, which further extends the powerful of this technique [15]. We deal with three specific regularizing functions for which we are able to compute the quadratures involved in the procedure. For each of them, the transformation equations are left as function of the (undefined) new Hamiltonian, thus providing three different families of canonical transformations.

Because of the super-integrability of the isochronal Hamiltonian, the possible solutions are constrained to planar, quasi periodic ellipses. Therefore, the closure of a trajectory is a two-torus and two actions (three in the extended phase space formulation) are required in the new Hamiltonian if one wants to retain the topology of the reduced problem. Resonances between the frequencies of the angle variables result in periodic motion that, of course, can be studied with a reduced Hamiltonian that may depend on less momenta.

With an aim on perturbation theory, we discuss several transformation equations derived from the three computed families of isochronal canonical transformations. We show that simple requirements on the transformation, as for instance "simplification", result in elementary partial differential equations from whose solution the new Hamiltonian may be determined. We further show that a variety of canonical transformations in the literature, ranging from historic ones like Delaunay's, Levi-Civita's or Hill's, to the recent transformation due to Yanguas [19], can be recovered from our family.

1 Hamilton-Jacobi reduction of the isochrone

Given the Hamiltonian

$$\mathcal{H} = \left[\frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{b + \sqrt{b^2 + r^2}} + T\right] \chi(r, R, \Theta, N, T) \tag{1}$$

where $(r, \theta, \nu, t, R, \Theta, N, T)$ are Hill or polar-nodal variables in the extended phase space, μ and b are parameters, and χ is a regularizing function that only depends on distance and momenta, we look for a canonical transformation

$$(r, \theta, \nu, t, R, \Theta, N, T) \xrightarrow{\mathcal{T}_\Phi} (f, g, h, u, F, G, H, U)$$

that converts Eq. (1) in a certain function $\Phi = \Phi(F, G, H, U)$ depending only on the momenta.

The transformation will be defined through a generating function $\mathcal{S} = \mathcal{S}(r, \theta, \nu, t, F, G, H, U)$ in mixed variables such that

$$f = \mathcal{S}_F, \quad g = \mathcal{S}_G, \quad h = \mathcal{S}_H, \quad u = \mathcal{S}_U, \quad R = \mathcal{S}_r, \quad \Theta = \mathcal{S}_\theta, \quad N = \mathcal{S}_\nu, \quad T = \mathcal{S}_t, \quad (2)$$

where we use the notation $\mathcal{S}_x = \partial\mathcal{S}/\partial x$. Then, from Eq. (1) we set the Hamilton-Jacobi equation

$$\left[\frac{1}{2}\left(\mathcal{S}_r^2 + \frac{1}{r^2}\mathcal{S}_\theta^2\right) - \frac{\mu}{b + \sqrt{b^2 + r^2}} + \mathcal{S}_t\right]\chi(r, \mathcal{S}_r, \mathcal{S}_\theta, \mathcal{S}_\nu, \mathcal{S}_t) = \Phi(F, G, H, U). \quad (3)$$

Because t, θ, and ν are not present in Eq. (1), the generating function may be chosen in separate variables

$$\mathcal{S} = U\,t + H\,\nu + G\,\theta + \mathcal{W}(r, F, G, H, U). \quad (4)$$

Then,

$$\frac{1}{2}\left(\mathcal{W}_r^2 + \frac{G^2}{r^2}\right) - \frac{\mu}{b + \sqrt{b^2 + r^2}} + U = \frac{1}{\chi(r, \mathcal{W}_r, G, H, U)}\,\Phi(F, G, H, U). \quad (5)$$

We limit ourselves to the cases in which $\mathcal{W}_r$ can be solved from a quadratic equation. Specifically, we request that $\chi = \chi^\star(r)\,\mathcal{W}_r^{-n}\,\Xi(G, H, U)$ and $n = 0, 1, 2$. Because Ξ can be subsumed in Φ, there is no ambiguity in dropping the star from χ, and writing Eq. (5) like

$$\frac{1}{2}\left(\mathcal{W}_r^2 + \frac{G^2}{r^2}\right) - \frac{\mu}{b + \sqrt{b^2 + r^2}} + U = \frac{\mathcal{W}_r^n}{\chi(r)}\,\Phi(F, G, H, U). \quad (6)$$

Therefore,

$$n = 0 \;\Rightarrow\; \mathcal{W}_r = \sqrt{\frac{2\,\Phi}{\chi(r)} - \frac{G^2}{r^2} + \frac{2\mu}{b + \sqrt{b^2 + r^2}} - 2U}, \quad (7)$$

$$n = 1 \;\Rightarrow\; \mathcal{W}_r = \frac{\Phi}{\chi(r)} \pm \sqrt{\frac{\Phi^2}{\chi^2(r)} - \frac{G^2}{r^2} + \frac{2\mu}{b + \sqrt{b^2 + r^2}} - 2U} \quad (8)$$

$$n = 2 \;\Rightarrow\; \mathcal{W}_r = \sqrt{\left(-\frac{G^2}{r^2} + \frac{2\mu}{b + \sqrt{b^2 + r^2}} - 2U\right)\frac{\chi(r)}{\chi(r) - 2\Phi}}. \quad (9)$$

that can be solved for $\mathcal{W}$ by quadrature.

We only discuss here the case $n = 0$. Then, $\mathcal{W} = \int_{r_0}^{r}\sqrt{Q(r, F, G, H, U)}\,\mathrm{d}r$, where $Q \geq 0$ is

$$Q = \frac{2\,\Phi}{\chi(r)} + \frac{2\mu}{b + \sqrt{b^2 + r^2}} - 2U - \frac{G^2}{r^2} = \frac{s^2}{s^2 - b^2}\,\mathcal{Q}, \quad (10)$$

with $s = \sqrt{r^2 + b^2}$ and

$$\mathcal{Q} = 2\left(\frac{\Phi}{\chi(s)} - U\right)\frac{s^2 - b^2}{s^2} + 2\mu\,\frac{s - b}{s^2} - \frac{G^2}{s^2}. \quad (11)$$

Therefore, the transformation is: $\Theta = G$, $N = H$, $T = U$, $R = \sqrt{Q}$, and

$$f = \Phi_F\,\mathcal{I}_3, \tag{12}$$

$$g = \theta + G\,\mathcal{I}_1 + \Phi_G\,\mathcal{I}_3, \tag{13}$$

$$h = \nu + \Phi_H\,\mathcal{I}_3, \tag{14}$$

$$u = t - \mathcal{I}_2 + \Phi_U\,\mathcal{I}_3, \tag{15}$$

where

$$\mathcal{I}_1 = \int_{s_0}^{s} \frac{s^2}{(s^2-b^2)\sqrt{\mathcal{Q}}}\,\mathrm{d}\!\left(\frac{1}{s}\right), \qquad \mathcal{I}_2 = \int_{s_0}^{s} \frac{\mathrm{d}s}{\sqrt{\mathcal{Q}}}, \qquad \mathcal{I}_3 = \int_{s_0}^{s} \frac{\mathrm{d}s}{\chi\sqrt{\mathcal{Q}}} = -\int_{r_0}^{r} \frac{s^2}{\chi\sqrt{\mathcal{Q}}}\,\mathrm{d}\!\left(\frac{1}{s}\right). \tag{16}$$

Note that $\mathcal{I}_3 = \mathcal{I}_2$ for $\chi = 1$, and $\mathcal{I}_3 = -\mathcal{I}_1$ for $\chi = r^2 = s^2 - b^2$.

In order to avoid dealing with elliptic integrals we require for $\mathcal{Q}$ to be at most quadratic in s, what limits the possible choices of the regularizing function to $\chi = 1$, $\chi = s \pm b$, and $\chi = s^2 - b^2$. Then, we find convenient to write Eq. (11) as

$$\mathcal{Q} = -\alpha\left(\frac{p}{s^2} - \frac{2}{s} + \frac{1}{a}\right), \tag{17}$$

where α, p, and a, are certain functions of the momenta and parameters that will be specified after χ has been chosen. Therefore,

$$\mathcal{Q} = \alpha\,p\left(\frac{1}{s} - \frac{1}{s_1}\right)\left(\frac{1}{s_2} - \frac{1}{s}\right), \tag{18}$$

where $s_1 \geq s \geq s_2$ are the two possible roots of the conic $\mathcal{Q} = 0$:

$$s_{1,2} = \frac{p}{1 \pm \sqrt{1 - p/a}} = \frac{a\,(1-e^2)}{1 \pm e} = a\,(1 \pm e), \qquad e^2 = 1 - \frac{p}{a} < 1. \tag{19}$$

These roots, the extreme values of s, make natural the introduction of the auxiliary variables ψ and ϕ, defined by

$$s = a\,(1 - e\cos\psi), \qquad \mathrm{d}s = a\,e\sin\psi\,\mathrm{d}\psi, \tag{20}$$

$$s = \frac{p}{1 + e\cos\phi}, \qquad \mathrm{d}\!\left(\frac{1}{s}\right) = -\frac{e}{p}\sin\phi\,\mathrm{d}\phi. \tag{21}$$

Then,

$$\mathcal{I}_1 = -\frac{1}{\sqrt{\alpha}}\left(\frac{1}{\sqrt{p - 2b + b^2/a}}\,\frac{\phi_1}{2} + \frac{1}{\sqrt{p + 2b + b^2/a}}\,\frac{\phi_2}{2}\right), \tag{22}$$

$$\mathcal{I}_2 = \sqrt{\frac{a^3}{\alpha}}\,(\psi - e\sin\psi)\,, \tag{23}$$

where the two new auxiliary variables ϕ_1, ϕ_2, are defined by means of the trigonometric relations (see [19] for details)

$$\tan\frac{\phi_1}{2} = \sqrt{\frac{1 + e - b/a}{1 - e - b/a}}\sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2} = \sqrt{\frac{1 + e - b/a}{1 - e - b/a}}\tan\frac{\psi}{2}, \tag{24}$$

$$\tan\frac{\phi_2}{2} = \sqrt{\frac{1 + e + b/a}{1 - e + b/a}}\sqrt{\frac{1-e}{1+e}}\tan\frac{\phi}{2} = \sqrt{\frac{1 + e + b/a}{1 - e + b/a}}\tan\frac{\psi}{2}. \tag{25}$$

The integration of $\mathcal{I}_3$ needs the previous specification of the regularizing function χ.

2 Discussion

Details on the families of canonical transformations generated by the cases $\chi = 1$, $\chi = b + \sqrt{b^2 + r^2}$, and $\chi = r^2$ are given below.

2.1 $\chi = 1$

From Eq. (11) and Eq. (17) we obtain

$$\alpha = \mu, \qquad a = \frac{\mu}{2(U - \Phi)}, \qquad p = \frac{G^2}{\mu} + 2b - \frac{1}{a}\, b^2, \tag{26}$$

that replaced in Eqs. (22) and (23) give

$$\begin{aligned} \mathcal{I}_1 &= -\frac{\phi_1}{2G} - \frac{\phi_2}{2\sqrt{G^2 + 4b\,\mu}}, & (27)\\ \mathcal{I}_3 = \mathcal{I}_2 &= \frac{\mu}{\sqrt{8(U - \Phi)^3}}\,(\psi - e\sin\psi). & (28) \end{aligned}$$

Therefore, the transformation is ($\Theta = G$, $N = H$, $T = U$, $R = \sqrt{Q}$)

$$\begin{aligned} f &= \Phi_F \frac{\mu}{\sqrt{8(U - \Phi)^3}}\,(\psi - e\sin\psi), & (29)\\ g &= \theta - \frac{\phi_1}{2} - \frac{G}{\sqrt{G^2 + 4b\,\mu}}\,\frac{\phi_2}{2} + \frac{\Phi_G}{\Phi_F}\, f, & (30)\\ h &= \nu + \frac{\Phi_H}{\Phi_F}\, f, & (31)\\ u &= t + \frac{\Phi_U - 1}{\Phi_F}\, f, & (32) \end{aligned}$$

where ψ is an implicit function of f and the new momenta. Note that Φ remains undefined and different choices may be done depending on a variety of criteria. Thus, for instance, the topology of the problem is maintained if we choose $\Phi = U + \Psi(F, G)$. Then, $h = \nu$, $u = t$, and

$$\begin{aligned} f &= \Psi_F \frac{\mu}{\sqrt{-8\Psi^3}}\,(\psi - e\sin\psi), & (33)\\ g &= \theta - \frac{\phi_1}{2} - \frac{G}{\sqrt{G^2 + 4b\,\mu}}\,\frac{\phi_2}{2} + \frac{\Psi_G}{\Psi_F}\, f, & (34) \end{aligned}$$

Among the variety of possible choices of Ψ a simplifying option is to take $\Psi_F = \Psi_G = \mu^{-1}\,(-2\Psi)^{3/2}$, which may be solved for Ψ, to give

$$\Phi = U - \frac{\mu^2}{2(F + G)^2} \tag{35}$$

that maximally simplifies the remaining transformation equations:

$$f = \psi - e\sin\psi, \tag{36}$$

$$g = \theta - \frac{\phi_1}{2} - \frac{G}{\sqrt{G^2+4b\,\mu}}\,\frac{\phi_2}{2} + f. \tag{37}$$

Yanguas' selection

$$\Phi = U - \frac{\mu^2}{2F^2} \tag{38}$$

depends on fewer momenta and, therefore, constrains the topology of the original system to periodic solutions only —which may be adequate for a perturbation study like [19].

In the Keplerian case $b = 0$, the selection of the new Hamiltonian $\Phi = \Phi(U, F)$ does not constrain the range of solutions and the specific selection Eq. (38) provides the popular Delaunay transformation that, taking into account that $\phi_1 = \phi_2 = \phi$ and $\Phi_G = 0$ in Eq. (30), is

$$f = \psi - e\sin\psi, \qquad g = \theta - \phi, \qquad h = \nu, \qquad u = t, \tag{39}$$

where the most extended notation writes $L \equiv F$, $\ell \equiv f$.

2.2 $\chi = b + \sqrt{b^2 + r^2}$

Now, we write

$$\alpha = \Phi + \mu, \qquad a = \frac{\Phi+\mu}{2U}, \qquad p = \frac{G^2}{\Phi+\mu} + 2b - \frac{1}{a}\,b^2, \tag{40}$$

that are repalced in $\mathcal{I}_1$, Eq. (22), and $\mathcal{I}_2$, Eq. (23). To integrate $\mathcal{I}_3$ we use the change of Eq. (20). We get

$$\mathcal{I}_1 = -\frac{\phi_1}{2G} - \frac{\phi_2}{2\sqrt{G^2+4b\,(\mu+\Phi)}}, \tag{41}$$

$$\mathcal{I}_2 = \frac{\mu+\Phi}{\sqrt{8U^3}}\,(\psi - e\sin\psi), \tag{42}$$

$$\mathcal{I}_3 = \frac{\psi}{\sqrt{2U}} - \frac{b\,\phi_2}{\sqrt{G^2+4b\,(\mu+\Phi)}}. \tag{43}$$

where ϕ_1, ϕ_2, are the same auxiliary variables defined in (24) and (25) respectively.

Therefore, the transformation is $\Theta = G$, $N = H$, $T = U$, $R = \sqrt{Q}$,

$$f = \Phi_F\left(\frac{\psi}{\sqrt{2U}} - \frac{b\,\phi_2}{\sqrt{G^2+4b\,(\mu+\Phi)}}\right), \tag{44}$$

$$g = \theta - \frac{\phi_1}{2} - \frac{G\,\phi_2}{2\sqrt{G^2+4b\,(\mu+\Phi)}} + \frac{\Phi_G}{\Phi_F}\,f, \tag{45}$$

$$h = \nu + \frac{\Phi_H}{\Phi_F}\,f, \tag{46}$$

$$u = t - \frac{\mu+\Phi}{\sqrt{8U^3}}\,(\psi - e\sin\psi) + \frac{\Phi_U}{\Phi_F}\,f, \tag{47}$$

2.2.1 Family 1

Assuming $b \neq 0$, Eq. (44) clearly simplifies if we require that $-2b\,\Phi_F = \sqrt{G^2 + 4b\,(\mu + \Phi)}$. Then,

$$\Phi = \frac{1}{4b}\left[F - 2b\,\Psi(G,H,U)\right]^2 - \frac{G^2}{4b} - \mu,$$

where Ψ is an arbitrary function. As the topology of the isochrone requires only three momenta (in the extended phase space formulation), we find great simplification by choosing $\Psi = \Psi(H)$. Thus,

$$f = \frac{F - 2b\,\Psi}{2b\,\sqrt{2U}}\,\psi - \frac{\phi_2}{2}, \tag{48}$$

$$g = \theta - \frac{\phi_1}{2} - \frac{G}{2b\,\sqrt{2U}}\,\psi, \tag{49}$$

$$h = \nu - \Psi_H \left(\frac{F - 2b\,\Psi}{\sqrt{2U}}\,\psi - b\,\phi_2\right), \tag{50}$$

$$u = t - \frac{(F - 2b\,\Psi)^2 - G^2}{4b\,\sqrt{8U^3}}\,(\psi - e\sin\psi), \tag{51}$$

Finally, if we select $\Psi = H/(2b)$, we get

$$\Phi = \frac{1}{4b}\left[(F - H)^2 - G^2\right] - \mu,$$

and

$$f = \frac{F - H}{2b\,\sqrt{2U}}\,\psi - \frac{\phi_2}{2}, \tag{52}$$

$$g = \theta - \frac{G}{2b\,\sqrt{2U}}\,\psi - \frac{\phi_1}{2}, \tag{53}$$

$$h = \nu - f, \tag{54}$$

$$u = t - \frac{(F - H)^2 - G^2}{4b\,\sqrt{8U^3}}\,(\psi - e\sin\psi), \tag{55}$$

2.2.2 Family 2:

A transformation that doest not need the non-vanishing of the parameter b is found as follows. First, in Eqs. (44)–(47) we require that $\Phi_F = \sqrt{2U}$; therefore, $\Phi = \sqrt{2U}\,(F + \Psi)$ with $\Psi = \Psi(G,H,U)$ an arbitrary function. Thus,

$$f = \psi - \frac{\sqrt{2U}\,b\,\phi_2}{\sqrt{G^2 + 4b\,(\mu + \sqrt{2U}\,(F + \Psi))}}, \tag{56}$$

$$g = \theta - \frac{\phi_1}{2} - \frac{G\,\phi_2}{2\sqrt{G^2 + 4b\,(\mu + \sqrt{2U}\,(F + \Psi))}} + \Psi_G\,f, \tag{57}$$

$$h = \nu + \Psi_H\,f, \tag{58}$$

$$u = t - \left(\frac{\mu}{\sqrt{8U^3}} + \frac{F + \Psi}{2U}\right)(\psi - e\sin\psi) + \left(\frac{F + \Psi}{2U} + \Psi_U\right) f. \tag{59}$$

Then, we avoid secular terms in ψ in the u (time) transformation by requiring that $\Psi_U = \frac{\mu}{\sqrt{8U^3}}$; therefore, $\Psi = -\frac{\mu}{\sqrt{2U}} + \Psi'$, where $\Psi' = \Psi'(G, H)$, and

$$
\begin{aligned}
f &= \psi - \frac{\sqrt{2U}\, b\, \phi_2}{\sqrt{G^2 + 4b\sqrt{2U}\,(F + \Psi')}}, &(60)\\
g &= \theta - \frac{\phi_1}{2} - \frac{G\,\phi_2}{2\sqrt{G^2 + 4b\sqrt{2U}\,(F + \Psi')}} + \Psi'_G\, f, &(61)\\
h &= \nu + \Psi'_H\, f, &(62)\\
u &= t - \frac{F + \Psi'}{2U}\left[\frac{\sqrt{2U}\, b\, \phi_2}{\sqrt{G^2 + 4b\sqrt{2U}\,(F + \Psi')}} - e\sin\psi\right]. &(63)
\end{aligned}
$$

Finally, as it is enough for the new Hamiltonian to depend on three momenta, we may choose $\Psi' = G$ giving the transformation

$$
\begin{aligned}
f &= \psi - \frac{\sqrt{2U}\, b\, \phi_2}{\sqrt{G^2 + 4b\sqrt{2U}\,(F + G)}}, &(64)\\
g &= \theta - \frac{\phi_1}{2} - \frac{G\,\phi_2}{2\sqrt{G^2 + 4b\sqrt{2U}\,(F + G)}} + f, &(65)\\
h &= \nu, &(66)\\
u &= t + \frac{F + G}{2U}\left[e\sin\psi - \frac{\sqrt{2U}\, b\, \phi_2}{\sqrt{G^2 + 4b\sqrt{2U}\,(F + G)}}\right], &(67)
\end{aligned}
$$

that completely reduces the Hamiltonian, Eq. (1), to a function of only the momenta $\Phi = \sqrt{2U}\,(F + G) - \mu$.

In the Keplerian case $b = 0$ and $\phi_1 = \phi_2 = \phi$. Furthermore, it is enough for the new Hamiltonian to depend only on two momenta. Now, Eqs. (44)-(47), are written

$$f = \frac{\Phi_F}{\sqrt{2U}}\,\psi, \quad g = \theta - \phi + \frac{\Phi_G}{\sqrt{2U}}\,\psi, \quad h = \nu + \frac{\Phi_H}{\sqrt{2U}}\,\psi, \quad u = t - \frac{\mu + \Phi}{\sqrt{8U^3}}\,(\psi - e\sin\psi) + \frac{\Phi_U}{\sqrt{2U}}\,\psi.$$

The choice $\Phi = 2U\,\Phi_U$, gives $\Phi = \sqrt{2U}\,\Psi(F, G, H)$, that reduces the time transformation to

$$u = t - \frac{\mu}{\sqrt{8U^3}}\,(\psi - e\sin\psi) + \frac{\Phi}{\sqrt{8U^3}}\,e\sin\psi$$

The simple option $\Phi = \sqrt{2U}\,(F + \mu)$ produces $\Phi_F = \sqrt{2U}$, $\Phi_G = \Phi_H \equiv 0$, and leads to the well known "first" Levi-Civita transformation [12],

$$f = \psi, \qquad g = \theta - \phi, \qquad h = \nu, \qquad u = t - \frac{\mu}{\sqrt{8T^3}}\,(\psi - e\sin\psi). \tag{68}$$

where u plays the role of the epoch of pericenter passage in the Keplerian motion, where the mean anomaly $\phi = n\,(t - u) = \psi - e\sin\psi$, and $n = (2T)^{3/2}/\mu$, from the energy equation $-T = -\mu/(2a)$. Note that we make use of the fact that $\Phi = 0$ to drop the corresponding term from the time transformation.

Other choice is to make $\Phi = 2U\,\Phi_U - \mu$, that produces $\Phi = \sqrt{2U}\,\Psi(F,G,H) - \mu$, and reduces the time transformation to

$$u = t + \frac{\mu + \Phi}{\sqrt{8U^3}}\, e \sin\psi.$$

The option $\Phi = \sqrt{2U}\,F - \mu$: $\Phi_F = \sqrt{2U}$, $\Phi_G = \Phi_H \equiv 0$, leads to the famous "second" Levi-Civita [13] transformation

$$f = \psi, \qquad g = \theta - \phi, \qquad h = \nu, \qquad u = t + \frac{\mu}{\sqrt{8T^3}}\, e \sin\psi, \tag{69}$$

where, again, we make $\Phi = 0$.

2.3 $\chi = s - b = -b + \sqrt{b^2 + r^2}$

This case is analogous to the previous one, and provides similar transformations that embrace also those of Levi-Civita for the Keplerian case.

2.4 $\chi = r^2$

We get now

$$\alpha = \mu, \qquad a = \frac{\mu}{2U}, \qquad p = \frac{G^2 - 2\Phi}{\mu} + 2b - \frac{b^2}{a}, \tag{70}$$

that replaced in Eqs. (22) and (23) give

$$\begin{aligned} -\mathcal{I}_3 = \mathcal{I}_1 &= -\frac{\phi_1}{2\sqrt{G^2 - 2\Phi}} - \frac{\phi_2}{2\sqrt{G^2 - 2\Phi + 4b\,\mu}}, && (71)\\ \mathcal{I}_2 &= \frac{\mu}{\sqrt{8U^3}}\,(\psi - e\sin\psi), && (72) \end{aligned}$$

Therefore, the family of transformations is $\Theta = G$, $N = H$, $T = U$, $R = \sqrt{Q}$,

$$\begin{aligned} f &= \Phi_F\left(\frac{\phi_1}{2\sqrt{G^2 - 2\Phi}} + \frac{\phi_2}{2\sqrt{G^2 - 2\Phi + 4b\,\mu}}\right), && (73)\\ g &= \theta - \frac{G - \Phi_G}{\Phi_F}\, f, && (74)\\ h &= \nu + \frac{\Phi_H}{\Phi_F}\, f, && (75)\\ u &= t - \frac{\mu}{\sqrt{8U^3}}\,(\psi - e\sin\psi) + \frac{\Phi_U}{\Phi_F}\, f, && (76) \end{aligned}$$

In view of Eqs. (73)–(76), a straightforward simplification requirement is $\Phi_H = 0$ and $\Phi_F = \Phi_U = G - \Phi_G$. Then, we might choose

$$\Phi = \frac{1}{2}\left[G^2 - (G - F - U)^2\right]$$

and the transformation is

$$f = \frac{\phi_1}{2} + \frac{G-F-U}{\sqrt{(G-F-U)^2+4b\,\mu}}\,\frac{\phi_2}{2}, \tag{77}$$

$$g = \theta - f, \tag{78}$$

$$h = \nu, \tag{79}$$

$$u = t - \frac{\mu}{\sqrt{8U^3}}\,(\psi - e\sin\psi) + f. \tag{80}$$

In the Keplerian case $b = 0$, $\phi_1 = \phi_2 = \phi$, further simplifies to $f = \phi$; besides, the reduced Hamiltonian only needs to depend on two momenta (in the extended phase space formulation) and if we choose $\Phi = \frac{1}{2}\,[G^2 - (G-F)^2] = F\left(G - \frac{1}{2}F\right)$, then, taking into account that $\Phi_U = 0$ in Eq. (76), we recover the well known TR-mapping [15, 5].

$$f = \phi, \qquad g = \theta - f, \qquad h = \nu, \qquad u = t - \frac{\mu}{\sqrt{8U^3}}\,(\psi - e\sin\psi), \tag{81}$$

3 Conclusions

Normally, the Hamilton-Jacobi equation is used for dealing with a specific class of problems. That is why the new Hamiltonian is commonly chosen before solving the quadratures introduced by the method. The success in that pre-selection strongly depends on the intuition and experience of the user, and may require a tedious sequence of trials. However, the new Hamiltonian can be hold formal to a large extent in the solution of the Hamilton-Jacobi equation, a fact that may help in the selection procedure.

In the case of Henon's isochronal potential, we solve the Hamilton-Jacobi equation keeping formal the new Hamiltonian, thus obtaining whole families of canonical transformations, contrary to single ones, from which we recover classical transformations in the literature as well as come up with new ones. This way of proceeding provides a great insight that is conveniently used in the search for the new Hamiltonian that defines the canonical transformation.

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