Monografías de la Real Academia de Ciencias de Zaragoza **35**, 27–39, (2011).

A note on some exact analytical solutions of the rotation of a rigid body with a external torque

M. Calvo, J.M. Franco, J.I. Montijano and L. Rández

Departamento Matemática Aplicada

Universidad de Zaragoza. 50009-Zaragoza, Spain.

Abstract

In two recent papers of M. Romano (Cel. Mech. 100: 181–189, 2008 and Cel. Mech. 101: 375–390, 2008) this author has derived new exact analytical solutions that describe both the dynamic and kinematic behavior a rigid body around a fixed point with spherical and symmetric ellipsoids of inertia self excited by some special torques in the fixed body reference frame. The aim of this note is to give an alternative simplified derivation of these solutions by using the symmetries of the system and suitable matrix transformations and to show that a wider class of analytical solutions may be derived.

1 Introduction and basic notations.

The motion of a rigid body around a fixed point is a classical problem of Mechanics that has been studied by many relevant mathematicians for more than two centuries. It is worth to mention the earlier contributions of Euler, during his stay in Berlin (1741–1766), in which he made a precise formulation of the differential equations of this problem and obtained some particular analytical solutions. They were published in several papers of the Royal Academie of Berlin (1751–1767). Later all these researches were published together in 1765 in Chapters 10th and 15th of the treatise: "Theoria Motus Corporum Solidorum seu Rigidorum". Many well known mathematicians like Poinsot, Lagrange, Jacobi, T. Levi-Civita, F. Klein, Kovalevskaya among others have studied for about two centuries some aspects and particular cases of the problem mainly concerned with the existence of first integrals, the integrability of these equations and the stability of some particular solutions.

Although the classical studies about the motion of the rigid body have been almost closed by more than a century, recent practical applications have open new problems in some fields such as robotics, spatial dynamics and molecular dynamics. Further actual computers have led to a substantial revision of classical algorithms used in the practical computation of solutions of these problems. Finally, since rigid body problems usually possess first integrals and sometimes periodic solutions (both stable and instable) their differential equations are excellent test problems for new numerical integrators of ODEs.

We will consider the rotational motion of a rigid body around a fixed point that will be taken as the origin O of two coordinates systems: A body fixed frame $\mathcal{B}$ with the axes directed along the principal axes of the ellipsoid of inertia of the body and an inertial fixed frame $\mathcal{I}$. In the remainder we will assume that the transformation from the inertial to the body frame is sufficiently smooth (of class $\mathcal{C}^p, p \geq 2$ in some interval of $\mathbb{R}$ but for simplicity we will take all $\mathbb{R}$) so that the coordinates $\mathbf{x}_{\mathcal{I}}$ and $\mathbf{x}_{\mathcal{B}}$ of a given vector in the corresponding systems satisfy

$$\mathbf{x}_{\mathcal{I}} = R(t)\, \mathbf{x}_{\mathcal{B}}, \tag{1}$$

where $R(t) = R_{\mathcal{I}\mathcal{B}}(t) \in \mathcal{C}^p$ is an orthogonal matrix of determinant $+1$. Clearly the columns of $R(t)$ give us the components of the unit vectors of the moving frame in the inertial frame.

Since $R(t)^T\, R(t) = I$ for all t, $R(t)^T\, R'(t) = \Sigma_R$ is a skew symmetric matrix that will be denoted by

$$R^{-1}R' = \begin{pmatrix} 0 & -w_3(t) & w_2(t) \\ w_3(t) & 0 & -w_1(t) \\ -w_2(t) & w_1(t) & 0 \end{pmatrix} = \Sigma(\mathbf{w}(t)). \tag{2}$$

It is important to note that $\mathbf{w} = (w_1(t), w_2(t), w_3(t))^T$ behaves as a vector under linear time independent transformations. In fact under a change of coordinates $\mathbf{x} \to P\mathbf{x}$ it is easy to show that $R \to PRP^{-1}$, $\Sigma \to P\Sigma P^{-1}$ and $\mathbf{w} \to P\mathbf{w}$.

The vector with components in the body frame $\mathbf{w}_{\mathcal{B}} = (w_1, w_2, w_3)^T$, is usually referred to as the instant angular velocity, and for all vector $\mathbf{v}_{\mathcal{B}}$ we have

$$\Sigma(\mathbf{w}_{\mathcal{B}})\, \mathbf{v}_{\mathcal{B}} = \mathbf{w}_{\mathcal{B}} \times \mathbf{v}_{\mathcal{B}}, \tag{3}$$

where $\times$ denotes the cross product and the signs in (2) have been chosen so that (3) holds.

Note that when R is a time dependent rotation around the x_3-axis, with angle $\phi = \phi(t)$

$$R(t) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{w} = (0, 0, \phi')^T,$$

and similarly for the rotations around the other axes.

Recall that in the inertial frame the basic equation of the dynamics of a rigid body with a fixed point O is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}_{\mathcal{I}} = \mathbf{M}_{\mathcal{I}}, \tag{4}$$

where the vectors $\mathbf{L}_{\mathcal{I}} = \mathbf{L}_{\mathcal{I}}(t)$ and $\mathbf{M}_{\mathcal{I}} = \mathbf{M}_{\mathcal{I}}(t)$ are the angular momentum and the total external torque with respect to O. By using (1), this basic equation with respect to the body frame becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}_{\mathcal{B}} + \mathbf{w}_{\mathcal{B}} \times \mathbf{L}_{\mathcal{B}} = \mathbf{M}_{\mathcal{B}}, \tag{5}$$

where $\mathbf{w}_{\mathcal{B}} = \mathbf{w}_{\mathcal{B}}(t)$ is the angular velocity of the body in the body frame with the components defined by (2).

Since the body frame has the axes directed along the principal axes of the ellipsoid of inertia $\mathbf{L}_{\mathcal{B}} = \mathbf{I}\ \mathbf{w}_{\mathcal{B}}(t) = (I_1 w_1, I_2 w_2, I_3 w_3)^T$ where $\mathbf{I} = \mathrm{diag}\ (I_i)$, and putting $\mathbf{M}_{\mathcal{B}} = (M_1, M_2, M_3)^T$, equation (5) can be written as

$$\begin{aligned} I_1 w_1' - (I_2 - I_3) w_2 w_3 &= M_1, \\ I_2 w_2' - (I_3 - I_1) w_3 w_1 &= M_2, \\ I_3 w_3' - (I_1 - I_2) w_1 w_2 &= M_3, \end{aligned} \tag{6}$$

which are the well known Euler's equations that describe the dynamics of the rigid body around O with respect to $\mathcal{B}$.

In addition to (6) we have the kinematic equations that describe the orientation of the body frame $\mathcal{B}$ with respect to the inertial frame $\mathcal{I}$ that according to (2) are

$$R'(t) = R(t)\ \Sigma(\mathbf{w}_{\mathcal{B}}(t)). \tag{7}$$

This is a linear matrix equation with the components of Σ given after solving (6) and the unknown matrix function $R(t)$ which is orthogonal for all t and therefore depends on three free parameters.

Several analytical solutions have been obtained for special mass distributions and/or torques. In the Euler-Poinsot case of a free body $\mathbf{M} = 0$ and equations (6) can be solved in terms of elliptic functions and when the kinematic equations are written in terms of Euler angles the integration can be reduced to quadratures. For a symmetric rigid body with fixed point which is different from its center of mass under the gravity force the Lagrange-Poisson and Kovalevskaya heavy top cases are two well known examples of integrable problems.

Concerning the integration of (7) it is worth to mention that some authors say that the solution of the linear kinematic equations (7) can be written in the matrix form

$$R(t) = R(0)\ \exp\left(\int_0^t \Sigma(\mathbf{w}_{\mathcal{B}}(s))\,\mathrm{d}s\right), \tag{8}$$

and therefore the solution reduces to quadratures. However, as follows from the theory of matrix functions, (8) is the solution of (7) only when

$$\Sigma(\mathbf{w}_{\mathcal{B}}(t)) \quad \text{and} \quad \int_0^t \Sigma(\mathbf{w}_{\mathcal{B}}(s))\,\mathrm{d}s \qquad \text{commute.} \tag{9}$$

In this note we will consider some cases of the motion of a rigid body around a fixed point under some prescribed torques studied recently by Romano in [2], [3]. We will show that taking into account the assumed symmetries of these problems it is possible to simplify the derivation of the analytical solution of the corresponding kinematic equations.

2 Rigid body with a spherical ellipsoid of inertia under a constant torque in $\mathcal{B}$

Suppose a rigid body with $I_1 = I_2 = I_3 = I$ under a constant torque $\mathbf{M}_{\mathcal{B}} = \mu\ \mathbf{u}_{\mathcal{B}}$ in the body frame $\mathcal{B}$ with $\mu = \|\mathbf{M}_{\mathcal{B}}\|_2$ and $\mathbf{u}_{\mathcal{B}}$ a unit vector in the direction of the torque. Note that we are considering bodies with spherical dynamic symmetry and this class of bodies contain the class of bodies with geometric axial symmetry.

By the spherical symmetry, Euler equations (5) in the body frame are

$$I\ \frac{\mathrm{d}\mathbf{w}_{\mathcal{B}}}{\mathrm{d}t} = \mu\ \mathbf{u}_{\mathcal{B}},$$

with the solution

$$\mathbf{w}_{\mathcal{B}}(t) = \mathbf{w}_{\mathcal{B}}^0 + t\, I^{-1}\, \mu\ \mathbf{u}_{\mathcal{B}}, \tag{10}$$

where $\mathbf{w}_{\mathcal{B}}^0$ is the initial angular velocity of the body at the initial time $t = 0$ in the body frame $\mathcal{B}$.

To obtain a complete analytical solution we must solve the linear matrix system (7) with a given $R(0)$ where $\mathbf{w}_{\mathcal{B}}(t)$ is the affine function (10).

More generally, for given constant vectors $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^3$ we will obtain the solution of

$$R'(t) = R(t)\ \Sigma(\mathbf{a} + t\ \mathbf{b}), \qquad \text{with a given } R(0). \tag{11}$$

First of all if $\mathbf{a}$ and have the same direction $\mathbf{b} = \nu\ \mathbf{a}$ and this implies that $\Sigma(\mathbf{a} + t\ \mathbf{b}) = (1 + \nu\ t)\ \Sigma(\mathbf{a})$. Hence

$$\exp\left(\int_0^t \Sigma(\mathbf{a} + s\ \mathbf{b})\ \mathrm{d}s\right) \quad \text{and} \quad \Sigma(\mathbf{a} + t\ \mathbf{b}),$$

commute and the solution of (11) is

$$R(t) = R(0)\ \exp\left[(t + \nu t^2/2)\Sigma(\mathbf{a})\right]. \tag{12}$$

For the explicit computation of the exponential in (12) we may use the well known formula

$$\exp(S) = I + \left(\frac{\sin\delta}{\delta}\right)\ S + \left(\frac{1 - \cos\delta}{\delta^2}\right)\ S^2, \tag{13}$$

where

$$S=\begin{pmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{pmatrix}, \qquad \delta^2 = s_1^2+s_2^2+s_3^2. \tag{14}$$

If $\mathbf{a}$ and $\mathbf{b}$ do not have the same direction we will make some time independent transformations to simplify the vector $\mathbf{a}+t\mathbf{b}$ as much as possible. Let $\beta \in [-\pi,\pi]$ and $\alpha \in [0,2\pi)$ be the polar coordinates of the unit vector $\mathbf{b}$ so that

$$\mathbf{b} = \|\mathbf{b}\|\ (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta). \tag{15}$$

We will make the orthogonal transformation $\mathbf{x}\to S_1\mathbf{x}$ with

$$S_1=\begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

that maps

$$\mathbf{b} \quad\to\quad S_1\,\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} = \mu\,\mathbf{e}_3,$$

where $\mathbf{e}_3$ is the unit vector along the third axis and $\mu = \|\mathbf{b}\|$.

It must be observed that, by the assumed spherical ellipsoid of inertia, the principal axes of inertia of the rigid body could be chosen with arbitrary orientation, in particular with the x_3-axis in the direction of the torque, however we will consider here the kinematic equations independently of the rigid body because they will be used in other problems without such a spherical symmetry.

As remarked above the argument $\mathbf{w}$ of $\Sigma(\mathbf{w})$ behaves as a vector in changes of coordinates and therefore

$$\mathbf{a}+t\,\mathbf{b} \quad\to\quad S_1\,(\mathbf{a}+t\,\mathbf{b}) = S_1\,\mathbf{a} + \ \mu\,t\,\mathbf{e}_3 = \widetilde{\mathbf{a}} + t\mu\,\mathbf{e}_3. \tag{17}$$

This implies that in the transformed system the dependence on t only appears in the third component of the angular velocity and the corresponding $\Sigma(S_1(\mathbf{a}+t\,\mathbf{b}))$ matrix will be

$$\Sigma(\widetilde{\mathbf{a}}+t\mu\,\mathbf{e}_3) = \begin{pmatrix} 0 & -\widetilde{a}_3 - t\mu & \widetilde{a}_2 \\ \widetilde{a}_3 + t\mu & 0 & -\widetilde{a}_1 \\ -\widetilde{a}_2 & \widetilde{a}_1 & 0 \end{pmatrix}.$$

Next we will make another time independent transformation in the (1,2) plane

$$\mathbf{y}\to S_2\mathbf{y} \equiv \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{y}, \tag{18}$$

with the purpose of vanishing the second component of the $\widetilde{\mathbf{a}}$ vector. In fact, defining θ by

$$\widetilde{a}_1 = \sqrt{\widetilde{a}_1^2 + \widetilde{a}_2^2}\,\cos\theta, \quad \widetilde{a}_2 = \sqrt{\widetilde{a}_1^2 + \widetilde{a}_2^2}\,\sin\theta,$$

for the transformed angular velocity we have

$$S_2 S_1 \left(\widetilde{\mathbf{a}} + t\mu\, \mathbf{e}_3\right) = \begin{pmatrix} \sqrt{\widetilde{a}_1^2 + \widetilde{a}_2^2} \\ 0 \\ \widetilde{a}_3 + t\mu \end{pmatrix} \equiv \begin{pmatrix} \widetilde{w}_1 \\ 0 \\ \widetilde{a}_3 + t\mu \end{pmatrix} = \widetilde{\mathbf{w}}, \tag{19}$$

with $\widetilde{w}_3(t) = \widetilde{a}_3 + t\mu$, which is the simplest form of the angular velocity under time independent rotations.

In conclusion with the time independent transformation

$$\mathbf{x} \quad \to \quad S\mathbf{x}, \qquad S = S_2 S_1,$$

the kinematic equations are

$$\widetilde{R}'(t) = \widetilde{R}(t)\;\Sigma(\widetilde{\mathbf{w}}), \qquad \widetilde{\mathbf{w}} = (\widetilde{w}_1, 0, \widetilde{a}_3 + t\mu)^T. \tag{20}$$

with $\widetilde{R}(t) = S\;R(t)\;S^{-1}$, $\Sigma(\widetilde{\mathbf{w}}_{\mathcal{B}})$ the skew symmetric associated to (19) and the initial conditions

$$\widetilde{R}(0) = S\;R(0)\;S^{-1}. \tag{21}$$

After this simplification, to write the solution of (20) in terms of elementary functions we observe that for the orthogonality of $\widetilde{R}(t)$ each row $\mathbf{v}^T = \mathbf{v}_k^T = (\widetilde{r}_{k1}, \widetilde{r}_{k2}, \widetilde{r}_{k3})$ can be considered as a point of $\mathcal{S}^2 \subset \mathbb{R}^3$ that satisfies the linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^T = \mathbf{v}^T\;\Sigma(\widetilde{\mathbf{w}}_{\mathcal{B}}). \tag{22}$$

Now we will map

$$\begin{array}{ccc} \mathcal{S}^2 & \to & \mathcal{C}\bigcup\{\infty\} \\ \mathbf{v} = (v_1, v_2, v_3)^T & \mapsto & \omega \end{array}$$

by means of the stereographic projection from the south pole on the complex equatorial plane given by

$$\omega = \omega(\mathbf{v}) = \frac{v_2 - i v_1}{1 + v_3}, \qquad \omega(-1) = \infty, \tag{23}$$

with the inverse

$$v_1 = \frac{i(\omega - \overline{\omega})}{1 + |\omega|^2}, \quad v_2 = \frac{\omega + \overline{\omega}}{1 + |\omega|^2}, \quad v_3 = \frac{1 - |\omega|^2}{1 + |\omega|^2}. \tag{24}$$

With the transformation (23),(24), the linear equation (22) in the new stereographic variables is transformed into the Riccati equation

$$\omega' = \left(\frac{\widetilde{w}_1}{2}\right)\,\omega^2 - i\widetilde{w}_3(t)\,\omega + \left(\frac{\widetilde{w}_1}{2}\right). \tag{25}$$

Next we introduce the standard change of variables

$$\omega \;\to\; \zeta, \qquad \omega = -\frac{2\zeta'}{\zeta \widetilde{w}_1},$$

that transforms a Riccati equation into a linear equation and (25) becomes

$$\zeta'' + i\widetilde{w}_3(t)\,\zeta' + \frac{1}{4}\widetilde{w}_1^2\,\zeta = 0, \tag{26}$$

with $\widehat{w}_1$ a real constant and $\widehat{w}_3$ an affine function of t defined by (19)

Finally with the linear change of time

$$t \;\to\; z, \qquad t = (z - z_0)/\gamma,$$

with

$$z_0 = \frac{\widetilde{a}_3\gamma}{\mu}, \quad \gamma^2 = \frac{\mu}{2i}, \quad \widetilde{w}_3(t) = \widetilde{a}_3 + \mu t,$$

we arrive to the Hermite equation for the unknown function $\zeta = \zeta(z)$

$$\frac{d^2\zeta}{dz^2} - 2z\,\frac{d\zeta}{dz} + 2\nu\,\zeta = 0, \quad \text{with the complex constant } \nu = \frac{i\widetilde{w}_1^2}{4\mu}. \tag{27}$$

It must be remarked that this transformed equation was obtained by Romano in ([2]).

Let $\langle\varphi_1(z), \varphi_2(z)\rangle$ be a basis of solutions of Hermite's equation (27), then any solution of this equation can be written as $\zeta(z) = c_1\,\varphi_1(z) + c_2\,\varphi_2(z)$, with arbitrary constants c_1, c_2. Since $z = z_0 + \gamma t$

$$\zeta(\gamma t + z_0) = c_1\,\varphi_1(\gamma t + z_0) + c_2\,\varphi_2(\gamma t + z_0),$$

and in the complex variable of the stereographic projection the solution becomes

$$\omega(t) = -\frac{2\gamma(c_1\,\varphi_1'(\gamma t + z_0) + c_2\,\varphi_2'(\gamma t + z_0))}{\widehat{w}_1\,(c_1\,\varphi_1(\gamma t + z_0) + c_2\,\varphi_2(\gamma t + z_0))}. \tag{28}$$

For the choice of the arbitrary constants c_1, c_2 observe that for a given unit vector $\mathbf{v}$ at $t = 0$, we have a unique $\omega(0) \in \mathbb{C}$ and we must select these constants so that

$$\omega(0) = -\frac{2\gamma(c_1\,\varphi_1'(z_0) + c_2\,\varphi_2'(z_0))}{\widetilde{w}_1\,(c_1\,\varphi_1(z_0) + c_2\,\varphi_2(z_0))}. \tag{29}$$

This equation shows that c_1, c_2 are not uniquely determined because for homogeneity if c_1, c_2 is a solution Kc_1, Kc_2 is also solution for any $K \in \mathbb{C}$. Nevertheless the same homogeneity appears in (28) and therefore we can take any solution $K\zeta(t)$ because it leads to the same $\omega(t)$.

Concerning the choice of the basis solutions, for Hermite's equation there are several possibilities: In many classical references on the subject it is remarked that if $\zeta(z) = \zeta_\nu(z)$ is a solution of (27), $\zeta_\nu(-z)$ is also solution and denoting by $W(z) = W(\zeta_\nu(z), \zeta_\nu(-z))$

the Wronskian of these solutions, it is easy to show that $W'(z) = 2zW(z)$ which implies $W(z) = W(0)\exp(z^2)$. Hence taking a solution with $W(0) \neq 0$ we have two independent solutions. The standard choice for φ_1 is the so called Hermite's function $H_\nu(z)$ given by

$$H_\nu(z) = \frac{1}{2\Gamma(-\nu)} \sum_{m\geq 0} \frac{(-1)^m \Gamma((m-\nu)/2)}{m!} (2z)^m.$$

Hermite's symmetric basis $\langle H_\nu(z), H_\nu(-z)\rangle$ is relevant for theoretical studies because it has an integral representation in the complex plane and allows further developments.

Next we will present another basis essentially equivalent to te given by Romano in [2] using hypergeometric functions. Observe that

$$\zeta(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!},$$

is a solution of (27) if the coefficients satisfy the two-term recurrence $a_{n+2} = 2(n-\nu)a_n$. Hence for $a_0 = 1, a_1 = 0$ we have the even solution

$$\varphi_1(z) = \sum_{j\geq 0} a_{2j} \frac{z^{2j}}{2j!}, \qquad (a_0 = 1, a_{2j} = 2(2j-2-\nu)a_{2j-2}).$$

For $a_0 = 0, a_1 = 1$ the odd solution

$$\varphi_2(z) = \sum_{j\geq 0} a_{2j+1} \frac{z^{2j+1}}{2j+1!}, \qquad (a_1 = 1, a_{2j+1} = 2(2j-1-\nu)a_{2j-1}).$$

In the general solution $\zeta(z) = c_1\varphi_1(z) + c_2\varphi_2(z)$ for the computation of c_1, c_2 we can proceed as above.

Finally, another alternative is to transform Hermite's equation

$$\zeta \quad \to \quad \eta, \qquad \zeta(z) = e^{z^2/2}\,\eta(z),$$

arriving to the normal form

$$\eta''(z) + (1 - 2\nu - z^2)\eta(z) = 0.$$

This is (with a constant scale of independent variable) Weber's equation (see e.g. [6]) that has been extensively used because it appears in the solution of some wave equations by separation of variables. Taking the basis of Weber-Hermite functions (also called the parabolic cylinder functions) $D_\nu(z)$ and $D_{-\nu}(z)$ for the last equation we obtain immediately the corresponding basis in the original equation.

Remarks.

- The above analytical solution of a body with a spherical ellipsoid of inertia excited by a constant torque can be easily extended to the case of a piecewise constant torque. In fact, if the torque $\mathbf{M}$ is given by $\mu_0\mathbf{u}_{\mathcal{B},0}$ for $t \in [t_0, t_1)$, $\mu_1\mathbf{u}_{\mathcal{B},1}$ for $t \in [t_1, t_2)$, ... then it enough to match the corresponding solutions in the intervals $[t_0, t_1], [t_1, t_2], \ldots$.

- In this derivation, for any time dependent skew symmetric matrix $\Sigma(\mathbf{w}_{\mathcal{B}}(t))$ with an affine function $\mathbf{w}_{\mathcal{B}}(t)$, each row of the kinematic equations $R' = R\,\Sigma(\mathbf{w}_{\mathcal{B}}(t))$ is transformed into the linear second order equation (26) and conversely. This shows that any second order equation with time dependent coefficients that admit a basis analytical solutions and can be transformed into the form (26) leads to some torque with a integrable problem.

3 Axially symmetric rigid body under a constant torque in the direction of the symmetry axis

Here we consider the first problem of Romano in [3] of a rigid body with $I_1 = I_2 = I \neq I_3$ under a constant torque $\mathbf{M} = \mu \mathbf{e}_{\mathcal{B},3}$ ($\mu > 0$ constant) along the third axis of the body frame.

Now Euler's equations can be written as

$$\begin{aligned} I\,\frac{\mathrm{d}w_1}{\mathrm{d}t} &= (I - I_3) w_2 w_3, \\ I\,\frac{\mathrm{d}w_2}{\mathrm{d}t} &= (I_3 - I) w_3 w_1, \\ I_3\,\frac{\mathrm{d}w_3}{\mathrm{d}t} &= \mu. \end{aligned} \tag{30}$$

¿From the last equation $w_3(t) = w_3^0 + (\mu/I_3)t$ and putting

$$\alpha(t) = \left(\frac{I - I_3}{I}\right)\, w_3(t),$$

the first two equations of (33) are

$$\frac{\mathrm{d}w_1}{\mathrm{d}t} = \alpha(t)\; w_2, \qquad \frac{\mathrm{d}w_2}{\mathrm{d}t} = -\alpha(t)\; w_1.$$

By introducing the complex function $\zeta(t) = w_1(t) + i w_2(t)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta = -i\alpha(t)\;\zeta, \quad \Rightarrow \quad \zeta(t) = \zeta(0)\; e^{-i\widehat{\alpha}(t)}$$

with $\widehat{\alpha}(t) = \int_0^t \alpha(s)\mathrm{d}s$. Hence with the initial conditions $w_1(0) = w_1^0, w_2(0) = w_2^0$ the solution is

$$\begin{aligned} w_1 &= \cos(\widehat{\alpha}(t))\; w_1^0 + \sin(\widehat{\alpha}(t))\; w_2^0, \\ w_2 &= -\sin(\widehat{\alpha}(t))\; w_1^0 + \cos(\widehat{\alpha}(t))\; w_2^0. \end{aligned}$$

Thus the solution of the dynamic equations (33) is

$$\mathbf{w}(t) = \begin{pmatrix} \cos(\widehat{\alpha}(t))\; w_1^0 + \sin(\widehat{\alpha}(t))\; w_2^0 \\ -\sin(\widehat{\alpha}(t))\; w_1^0 + \cos(\widehat{\alpha}(t))\; w_2^0 \\ w_3^0 + (\mu/I_3)t \end{pmatrix}, \tag{31}$$

with

$$\widehat{\alpha}(t) = \int_0^t \alpha(s)\, \mathrm{d}s = \left(\frac{I - I_3}{I}\right) \left[w_3^0 t + \left(\frac{\mu}{2I_3}\right) t^2\right].$$

For the solution of the kinematic equations

$$R'(t) = R(t)\, \Sigma(\mathbf{w}(t)), \tag{32}$$

observe that under a non singular time dependent change of variable

$$R \longrightarrow \widetilde{R}, \qquad R = \widetilde{R}\, S(t),$$

we have,

$$R'(t) = R(t)\, \Sigma_R \quad \longrightarrow \quad \widetilde{R}'(t) = \widetilde{R}(t)\, \Sigma_{\widetilde{R}},$$

with

$$\Sigma_{\widetilde{R}} = -S'\, S^{-1} + S\, \Sigma_R\, S^{-1}.$$

In particular taking for S a rotation around the third axis

$$S = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \phi = \phi(t),$$

we get

$$S'\, S^{-1} = \begin{pmatrix} 0 & -\phi' & 0 \\ \phi' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$S\, \Sigma_R\, S^{-1} = \begin{pmatrix} 0 & -w_3 & (w_1 \sin\phi + w_2 \cos\phi) \\ w_3 & 0 & -(w_1 \cos\phi - w_2 \sin\phi) \\ -(w_1 \sin\phi + w_2 \cos\phi) & (w_1 \cos\phi - w_2 \sin\phi) & 0 \end{pmatrix},$$

and then the components of the angular velocity $\widetilde{\mathbf{w}} = (\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)$ associated to $\Sigma_{\widetilde{R}}$ are

$$\begin{aligned} \widetilde{w}_1 &= (w_1 \cos\phi - w_2 \sin\phi), \\ \widetilde{w}_2 &= (w_1 \sin\phi + w_2 \cos\phi), \\ \widetilde{w}_3 &= w_3 - \phi', \end{aligned}$$

Hence taking into account (34) and choosing $\phi(t) = \widehat{\alpha}(t) + \pi/2$ we have

$$\widetilde{w}(t) = \left(w_1^0, w_2^0, \left(\frac{2I - I_3}{I}\right) w_3^0 + \left(\frac{\mu(I - I_3)}{I I_3}\right) t\right).$$

These are the same kinematic equations studied in section 2 with a different set of constants in the angular velocity. Hence we may apply the transformations used there to get the analytical solution.

4 Axially symmetric rigid body under a rotating external torque constant in magnitude and perpendicular to the symmetry axis

This is the second case considered by Romano in [3]. Now the components of the torque in the body frame are

$$\mathbf{M}_{\mathcal{B}} = (\beta_0 I_1 \cos(\alpha t), \beta_0 I_1 \sin(\alpha t), 0)\,, \tag{33}$$

with non zero constants β_0 and α. Hence Euler's equations are

$$\begin{aligned} I_1 w_1' &= (I_1 - I_3) w_2 w_3 + \beta_0 I_1 \cos(\alpha t), \\ I_1 w_2' &= (I_3 - I_1) w_3 w_1 + \beta_0 I_1 \sin(\alpha t), \\ w_3' &= 0. \end{aligned} \tag{34}$$

For the derivation of analytic solutions Romano assumes that the angular velocity of rotation of the torque is a function of the initial condition given by

$$\alpha = \frac{I_3 - I_1}{I_1}\, w_3^0, \tag{35}$$

and then equations (34) can be written as

$$w_1' = -\alpha w_2 + \beta_0 \cos(\alpha t), \quad w_2' = \alpha w_1 + \beta_0 \sin(\alpha t), \quad w_3' = 0.$$

These dynamic equations have the general solution

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} \cos(\alpha t) & -\sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{pmatrix} \begin{pmatrix} w_1^0 + t\beta_0 \\ w_2^0 \end{pmatrix}, \qquad w_3(t) = w_3^0. \tag{36}$$

As in the previous case for solving the kinematic equations

$$R'(t) = R(t)\ \Sigma(\mathbf{w}(t)) \tag{37}$$

we introduce the time dependent transformation

$$R = \widetilde{R}\, S(t),$$

where $S = S(t)$ is a rotation around the third axis with angle $-\alpha t$. Now the transformed system is

$$\widetilde{R}'(t) = \widetilde{R}(t)\ \Sigma(\widetilde{\mathbf{w}}(t)), \tag{38}$$

where

$$\begin{pmatrix} \widetilde{w}_1(t) \\ \widetilde{w}_2(t) \end{pmatrix} = \begin{pmatrix} \cos(-\alpha t) & -\sin(-\alpha t) \\ \sin(-\alpha t) & \cos(-\alpha t) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1^0 + t\beta_0 \\ w_2^0 \end{pmatrix},$$

and $\widetilde{w}_3(t) = w_3^0 + \alpha$. Since the transformed angular velocity $\widetilde{\mathbf{w}}(t)$ is an affine function the analytical solution of (38) can be written in terms of Hermite's functions as in the section 2 and we have the desired analytical solution.

Finally, we consider the motion of an axially symmetric rigid body ($I_1 = I_2 \neq I_3$) where the initial angular velocity in the body frame $\mathbf{w}(0) = (w_1^0, w_2^0, w_3^0)$ is contained in the 1-2 plane subjected to a fixed torque perpendicular to its axis of symmetry. By the assumed rotational symmetry we can take the torque $\mathbf{M} = (I_1\mu, 0, 0)$ where μ an arbitrary constant directed along the first axis. In this case the dynamic equations have the solution

$$\mathbf{w}(t) = (w_1^0 + \mu t, w_2^0, 0), \tag{39}$$

and taking into account that (39) is an affine function, the solution of the kinematic equation

$$R'(t) = R(t)\,\Sigma(\mathbf{w}(t)), \tag{40}$$

for arbitrary $R(0)$ can be written in terms of Hermite's functions.

More generally, as it has been proved in section 2, for all affine angular velocity function $\mathbf{w}(t) = \mathbf{a} + t\ \mathbf{b}$ ($\mathbf{a}$, $\mathbf{b}$ constant vectors) after suitable change of variables the general solution of the kinematic equations (40) can be expressed in terms of Hermite's functions. In view of this, for all external torque $\mathbf{M}$ that satisfies Euler's equations

$$\mathbf{I}\,\frac{d\mathbf{w}}{dt} + \mathbf{w} \times \mathbf{I}\,\mathbf{w} = \mathbf{M}, \quad \text{with} \quad \mathbf{w}(t) = \mathbf{a} + t\,\mathbf{b}, \tag{41}$$

the corresponding rigid body problem is completely integrable because both (40) and (41) are completely integrable.

In particular with the values

$$\mathbf{a} = (w_1^0, w_2^0, 0)^T, \quad \mathbf{b} = (\mu, 0, 0)^T, \quad I_1 = I_2 \neq I_3,$$

by substituting in the left hand side of (41) we get

$$M_1 = I_1\mu, \quad M_2 = 0, \quad M_3 = 0,$$

and we have the above particular solutions derived by Romano in [3].

5 Final Remarks

It has been stated that with the present possibilities of machine computation the derivation of analytic solutions in problems of rigid body motion only possess an academic interest. However it must be noticed that numerical methods enable us to obtain very accurate solutions in short time intervals of integration but cannot capture the long time behavior of solutions. Thus, even in the simplest case of the torque free motion of a general rigid body with a well known analytical solution in terms of Jacobian elliptic functions many numerical integrators do not preserve the existing invariants of the problem.

Recent applications such as the attitude evolution of a spinning spacecraft have open new requirements such as computer algorithms for onboard computations. Now the main task is not the accuracy but the simplicity and reliability of the algorithms for usual spinning-up and spinning-down maneuvers. It must be noticed that very often thruster misalignment or thruster mismatch do not allow an exact knowledge of the torque.

Although Euler's angles, typically the 3-1-2 angle sequence, are used to describe the orientation of the body-fixed reference frame, in practical computations other (complex) variables are more convenient and sometimes avoid singularities. Analytical solution are very frequently a source of inspiration for this purpose.

Acknowledgements

This work was supported by project MTM2007-67530-C02-01.

References

[1] E. Leimanis, *The General Problem of the Motion of Coupled Rigid Bodies about a Fixed Point*, Springer Tracts in Natural Philosphy, vol 7, Springer-Verlag, Berlin (1965).

[2] M. Romano, *Exact analytical solution for the rotation of a rigid body having spherical ellipsoid of inertia and subjected to a constant torque*, Celest. Mech. Dyn. Astron. **100**, 181–189 (2008).

[3] M. Romano, *Exact analytical solution for the rotation of an axially symmetric rigid body subjected to a constant torque*, Celest. Mech. Dyn. Astron. **101**, 375–390 (2008).

[4] P. Tsiotras and J.M. Longuski, *A complex analytical solution of a near symmetric rigid body under body fixed torques*, Celest. Mech. Dyn. Astron. **51**, 281–301 (1991).

[5] A. Wintner, *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, Princeton (1941).

[6] E. T. Whittaker and G.N. Watson, *A course in modern analysis*, Cambridge University Press, Cambridge (1962).