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Cascade of n-round homoclinic orbits to a center×saddle

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Abstract

We consider a Hamiltonian system with two degrees of freedom depending on a parameter, having an equilibrium point, p, of center×saddle type. We assume there is a homoclinic orbit to p. We explore the phenomenon of cascade of n-round homoclinic orbits when varying the parameter. Explicit computations are shown for the collinear equilibrium point L_3 of the planar restricted three-body problem.
Key words and expressions: equilibrium point, homoclinic orbits

1 Preliminaries and setting of the problem

In this work, we consider the existence of homoclinic orbits to a center×saddle equilibrium point of a given Hamiltonian system. We recall that homoclinic and heteroclinic connections of hyperbolic objects (equilibrium points being the simplest ones) play an important role when studying a dynamical system from a global point of view. In particular they are relevant in applications to Celestial Mechanics and Astrodynamics, more particularly in the design of libration point missions (see [4], [5], [7], [14] and references therein).

The general setting considered from now on is the following: we assume that we have a real analytic Hamiltonian $H(x;\mu)$ of two degrees of freedom, depending on a parameter μ, such that for all values of the parameter the origin is an equilibrium point of center×saddle type, i.e., the eigenvalues of the Jacobian matrix of the Hamiltonian vector field, X_H, at the origin are $\pm i\omega$, $\pm\lambda$ with $\omega\lambda \neq 0$, $\omega, \lambda \in R$. Let us denote by $H_0 = H(0)$ and let us consider the one-dimensional manifolds, stable W^s and unstable

W^u, and the corresponding branches W^s_+, W^s_-, W^u_+, W^u_-, associated with the origin. Due to the Lyapunov theorem (see [12]), we know that for each value of H close to H_0, there is an unstable periodic orbit. When varying H, we obtain the so called Lyapunov family of periodic orbits associated with the equilibrium point.

Let us assume now that, for a particular value of the parameter, say μ_1, one branch of W^s coincides with one branch of W^u, giving rise to a homoclinic orbit, Γ, to the origin. Two natural questions appear in this context:

1. How is the dynamics close to the homoclinic orbit Γ?
2. What happens to the homoclinic orbit when we consider values of the parameter in a neighborhood of μ_1?

Several authors have studied question 1. Maybe the first one was Conley ([2]) in the context of the planar RTBP taking the mass parameter as a natural parameter. We also mention the paper by Llibre et al. (see [11]) where they consider the same problem and prove the existence of homoclinic orbits to the collinear equilibrium point L_2 as well as the transversal intersection of the stable and unstable manifolds of the Lyapunov periodic orbits.

In a general analytic Hamiltonian, in the paper by Koltsova and Lerman [8], the authors prove, under generic conditions, two important results for $\mu = \mu_1$:

- the existence of countable families of periodic orbits accumulating to the homoclinic orbit and lying on the same energy level as the center×saddle, and
- the existence of homoclinic orbits to each hyperbolic Lyapunov periodic orbit.

In the 3 degree-of-freedom Hamiltonian case, given an equilibrium point of center×-center×saddle type, we can regard a homoclinic orbit to the equilibrium point not only as the skeleton of homoclinic orbits to periodic orbits closeby, but also of 2d-invariant tori (see [10]).

In order to answer the second question, we must introduce n-round homoclinic orbits. We define a homoclinic orbit to the origin to be n-round if it enters and also leaves some small neighborhood of the origin n times; in each path outside this neighborhood, it more or less follows the homoclinic orbit Γ (see [6]). In this context, we consider the set of values of the parameter μ in a neighborhood of μ_1, and we define the set

$$\Lambda_n = \{\mu > 0 \,/\ \text{there exists an } n\text{-round homoclinic orbit to the origin}\}.$$

The most complete investigation of n-round homoclinic orbits to a center×saddle in a one-parameter unfolding of reversible two-degree-of-freedom Hamiltonian systems was accomplished in [6] and [13]. An alternative proof for 2-round and 3-round homoclinic orbits was carried out in [8], and revisited in [9] for n-round homoclinic orbits, for $n = 2, 3$, and $n = m2^k$, $m = 2, 3$ and $k \in N$. Roughly speaking, the main result may be stated as follows: given $\mu_1 \in \Lambda_1$, there exist values of $\mu \in \Lambda_n$, close enough to μ_1, for all $n > 1$ (see [13]).

The purpose of this work is to show numerical evidence of this result in the context of the restricted three-body problem. For the details of the computations done, the reader is referred to [1].

2 Example: the planar RTBP. Homoclinic orbits to L_3.

Now we consider the planar circular RTBP, whose well known Hamiltonian function, depending on the mass parameter $\mu \in (0, 1/2]$, is

$$H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) - xp_y + yp_x - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu)$$

with $r_1 = \sqrt{(x-\mu)^2 + y^2}$ and $r_2 = \sqrt{(x-\mu+1)^2 + y^2}$. We may also consider the equations of motion in the rotating (non canonical) coordinates $x, y, x' = p_x + y, y' = p_y - x$ (see [15])

$$\begin{aligned} x'' - 2y' &= D_x\Omega(x, y), \\ y'' + 2x' &= D_y\Omega(x, y), \end{aligned} \tag{1}$$

where

$$\Omega(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu).$$

The system of equations (1) has a first integral, called the Jacobi integral, which is given by

$$\mathcal{C} = 2\Omega(x, y) - x'^2 - y'^2. \tag{2}$$

This C value is related to H by $C = -2H + \mu(1-\mu)$ Furthermore, we recall that equations (1) satisfy the well known symmetry

$$(t, x, y, x', y') \longrightarrow (-t, x, -y, -x', y'). \tag{3}$$

This implies that, for each solution of equations (1), there also exists another one, which is seen as symmetric with respect to $y = 0$ in configuration space.

We also recall that the RTBP has five equilibrium points: the collinear points, L_1, L_2 and L_3, situated on the line containing the primaries, and the equilateral ones, L_4 and L_5, both forming equilateral triangles with the two primaries. We will consider that $x_{L_2} \le \mu - 1 \le x_{L_1} \le \mu \le x_{L_3}$, that is, L_1 is between both primaries, L_2 is on the left hand side of the small one and L_3 is on the right hand side of the large one.

We will concentrate on the collinear equilibrium point L_3. It is well known that, if we write the differential equations (1) as

$$\mathbf{x}' = \mathbf{X}(\mathbf{x})$$

then Spec $DX(L_i) = \{\pm i\omega, \pm\lambda\}$, so the equilibrium point L_i, $i = 1, 2, 3$ is a center×saddle point. In this case W^u_+, W^u_- are the two branches of the unstable manifold of L_3, whose (x, y) projection lies (when $t \to -\infty$) on the $y > 0$ and $y < 0$ region respectively and, similarly, W^s_+ and W^s_-. If, for a given value of μ, the unstable and stable manifolds intersect, they give rise to a homoclinic connection to L_3.

So, our setting now is the RTBP with one parameter, μ, and L_3 (instead of the origin in the previous Section) being the associated equilibrium point for a given $\mu \in (0, \mu/2]$. We say that a value μ belongs to Λ_n if for that μ there exists an n-round *symmetric* homoclinic orbit (SHO from now on) to L_3.

Our aim in this Section is, on the one hand, to numerically illustrate the existence of values of $\mu \in \Lambda_1$; and, on the other hand, given a fixed value of $\mu = \mu_1 \in \Lambda_1$, to display sequences of values in Λ_n tending to μ_1, for any $n > 1$. Of course we cannot explore *all* the values of n; we will only take the cases $n = 2, 3, 4$.

An easy strategy to detect SHO is simply the following: we consider $\Sigma = \{y = 0\}$ as surface of section and, for a given μ, denote by $x'_j(\mu)$ the x' coordinate of the j–th intersection of a branch of a manifold of L_3 (we will take from now on W^u_-) with Σ. If this j–th cut is orthogonal, that is,

$$x'_j(\mu) = 0, \tag{4}$$

the application of symmetry (3) to a trajectory following W^u_- up to its j–th cut with $y = 0$ forward in time will give rise to a symmetric trajectory following W^s_+ backward in time and therefore becoming a SHO.

Let us start analyzing the set Λ_1. We vary the μ parameter and we consider the function $x'_1(\mu)$ given in Fig. 1 left. Its behavior provides numerical evidence of the existence of a decreasing sequence of values of $\mu^1_n \in \Lambda_1$, with $\mu^1_1 < 0.01$ and $\mu^1_n \to 0$ when $n \to \infty$ (see [3] for an expression of such values). For any given value of μ^1_n, the corresponding (x, y) projection of the SHO typically surrounds once L_4 and L_5 describing a horseshoe–shaped orbit. See Fig. 1 right.

We plot the functions $x'_2(\mu)$ in Fig. 2. We can see that there are sequences of values in Λ_2 tending (on each side) to each value in Λ_1, and therefore providing the values of μ for 2-round SHO.

In a similar way, we plot in Fig. 3 the functions $x'_3(\mu)$ and $x'_4(\mu)$ in the neighborhood of a fixed value of $\mu \in \Lambda_1$, denoted by μ_1. We see again the existence of sequences of values of $\mu \in \Lambda_3$ and Λ_4 tending to μ_1.

As a final remark, we note that the jumps observed in the curves $x'_j(\mu)$, $j = 1, 2, 3, 4$, in the different figures can also be analysed graphically in detail. This has been done in [1].

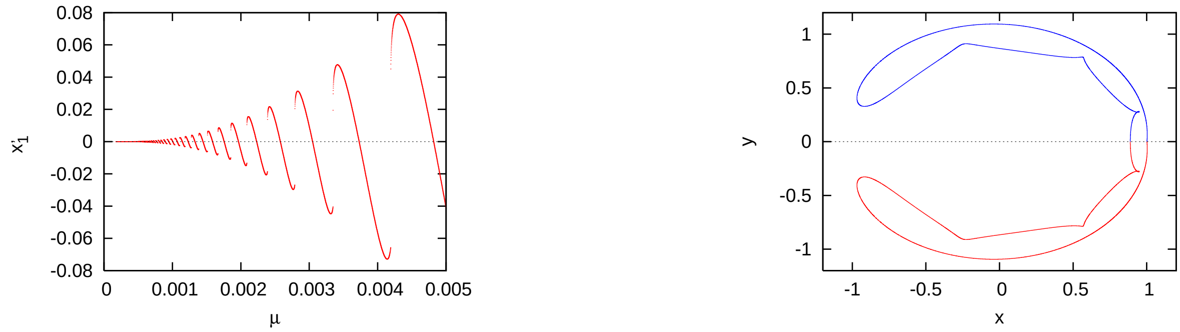

Figure 1.— Left. Function $x'_1(\mu)$. Right. Homoclinic invariant manifold $-(x, y)$ projection– for $\mu = 0.0037257851523$.

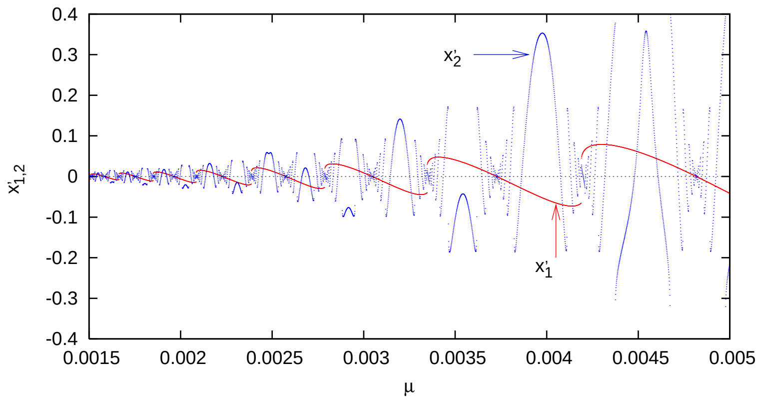

Figure 2.— Functions $x'_1(\mu)$ (in red) and $x'_2(\mu)$ (blue).

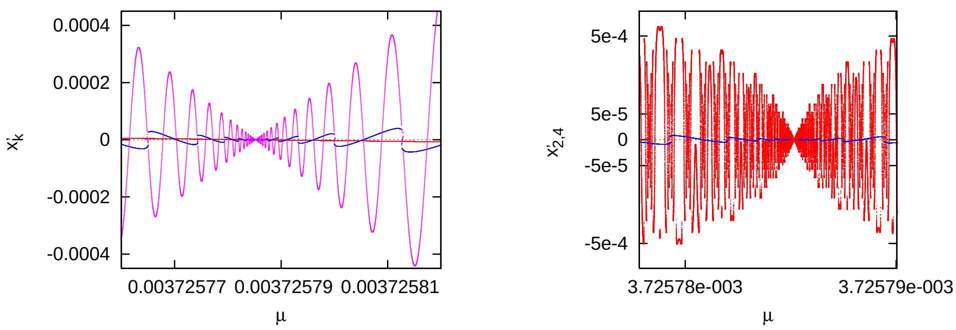

Figure 3.— Left: functions $x'_k(\mu)$, for $k = 1$ (in red), $k = 2$ (in blue), $k = 3$ (in magenta). Right. Functions $x'_2(\mu)$ and $x'_4(\mu)$. For display purposes, the function x'_4 has been rescaled using the arcsinh(x) function, and the y axis has been labeled accordingly.

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