Monografías de la Real Academia de Ciencias de Zaragoza **35**, 13–20, (2011).

On the stability of the planar $n+1$ ring body problem with quasi-homogeneous potentials

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Abstract

In a previous work we analyzed the linear stability of the planar $n+1$ ring body problem where the potential of the central body is a Manev's type potential. By introducing a perturbation parameter (ϵ_0) to the Newtonian potential associated with the central primary, we showed that unstable cases for the unperturbed problem, as $n \leq 6$, may become stable for some values of the perturbation.

The purpose of this paper is to study the possibility of increasing the range of values of the mass parameter ($\mu = m/m_0$) and the parameter ϵ_0 that let stable the configuration. For that, we introduce a second perturbation term (with parameter ϵ_1) to the Newtonian potential of the bodies in the ring. We show some results for different values of the parameters.

1 Introduction

The n-ring configuration consists of n bodies of equal masses placed at the vertices of a regular n-gon which is rotating about its center of masses with constant angular velocity. Another body of mass m_0 is placed at the center of the n-gon.

For Newtonian forces, the stability of this configuration depends essentially of two parameters, the mass relation between the central body and the peripheral ones ($\mu = m_0/m$) and their number (n). It is known since Maxwell that the configuration is unstable for $n \leq 6$, whereas for $n \geq 7$ the configuration is stable when the mass ratio is within certain values [8, 10, 9]. Generally speaking, the greater the number of bodies the smaller the mass parameter to have stability.

Recently, the authors analyzed the stability when the central body is a spheroid or a radiating body [2](see also [7]), which may be considered as bodies attracted by a Manev's type force, and that belongs to a more general class of quasi-homogeneous potentials [5, 4, 1, 6]. In this case a new parameter ϵ representing the oblateness (prolateness) or the radiation coefficient must be considered in the force function, which takes the form:

$$U = k^2\, m_0 \sum_{1\le j\le n} m_j \left(\frac{1}{r_{j0}} + \frac{\epsilon}{r_{j0}^2}\right) + k^2 \sum_{1\le i<j\le n} \frac{m_i\, m_j}{r_{ij}}.$$

Note that the parameter ϵ may be positive, negative or null.

In the above mentioned work [2] we proved that the stability depends also on ϵ, in such a way that for values $n \le 6$, unstable for Newtonian forces, we found regions for $\epsilon < 0$ in which the configuration is stable. Besides, for $n > 6$ the influence of ϵ increases the stability bound.

Thus, the "non Newtonian" part of the potential due to the central primary modifies the region of stability. Hence, we decided to investigate the effect on the stability of the n bodies placed on the ring when these bodies are of the same type as the central one, i.e., spheroids or even radiating bodies. The procedure we will follow along the paper is analogous to the one described in [2, 3].

2 The problem

We assume that bodies on the ring are identical and when all bodies are spheroids or radiating sources, we have a new parameter ϵ_1, whereas ϵ_0 is the same as in [2]. Then, the force function is

$$U = k^2\, m_0 \sum_{1\le j\le n} m_j \left(\frac{1}{r_{j0}} + \frac{\epsilon_0}{r_{j0}^2}\right) + k^2 \sum_{1\le i<j\le n} m_i\, m_j \left(\frac{1}{r_{ij}} + \frac{\epsilon_1}{r_{ij}^2}\right), \tag{1}$$

Both parameters are independent each other and may take positive, negative or null values.

In order to perform a stability analysis, we need a synodic frame in wich all primaries remain in rest. With respect such a frame, the equations of motion are:

$$\frac{d^2\boldsymbol{r}_i}{dt^2} + 2\boldsymbol{\Omega}\times\frac{d\boldsymbol{r}_i}{dt} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}_i) = \frac{\partial U_i}{\partial \boldsymbol{r}_i} \qquad i = 1,\ldots,n$$

with $\boldsymbol{\Omega}$ a vector perpendicular to the plane containing the primaries, and its norm, $\tilde{\omega} = \|\boldsymbol{\Omega}\|$ is

$$\begin{aligned}\tilde{\omega}^2 &= 1 + \frac{\mu}{4}\sum_{k=1}^{n-1} |\csc k\theta| + \frac{2\epsilon_0}{\alpha} + \frac{\mu\epsilon_1}{4\alpha}\sum_{k=1}^{n-1}\frac{1}{\sin^2 k\theta} \\ &= \omega^2 + \frac{2\epsilon_0}{\alpha} + \frac{\mu\epsilon_1}{12\alpha}(n^2-1)\end{aligned}$$

where ω^2 is the angular velocity corresponding to the Newtonian atraction, α is the radius of the ring and $\theta = \pi/n$.

Because of the geometry of the problem, it is convenient the use of cylindric coordinates (r, λ, z). The correspondig equations are

$$\left.\begin{aligned} &\ddot{r}_j - r_j(\dot{\lambda}_j + \tilde{\omega})^2 = \frac{\partial U_j}{\partial r_j}, \\ &r_j\ddot{\lambda}_j + 2\dot{r}_j(\dot{\lambda}_j + \tilde{\omega}) = \frac{1}{r_j}\frac{\partial U_j}{\partial \lambda_j}, \\ &\ddot{z}_j = \frac{\partial U_j}{\partial z_j}, \end{aligned}\right\} \qquad j = 1, \ldots, n \tag{2}$$

It is easy to prove that

$$r_j = 1, \quad \lambda_j = 2\theta j, \quad z_j = 0, \tag{3}$$

is an equilibrium solution of equations (2).

3 Linear stability of the equilibrium solution

As usual, in order to determine the linear stability, we slightly perturb the equilibrium. Let us introduce a new set of variables $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$, $\boldsymbol{z} = (z_1, \ldots, z_n)$ in such way that

$$r_j = 1 + \rho_j, \quad \lambda_j = 2\theta j + \sigma_j, \quad z_j = z_j, \qquad j = 1, \ldots, n,$$

and the variational equations of (2) become

$$\begin{aligned} &\ddot{\boldsymbol{\rho}} - 2\tilde{\omega}\dot{\boldsymbol{\sigma}} = \tilde{\omega}^2\boldsymbol{\rho} + A\boldsymbol{\rho} + B\boldsymbol{\sigma}, \\ &\ddot{\boldsymbol{\sigma}} + 2\tilde{\omega}\dot{\boldsymbol{\rho}} = C\boldsymbol{\rho} + D\boldsymbol{\sigma}, \\ &\ddot{\boldsymbol{z}} = E\boldsymbol{z}, \end{aligned} \tag{4}$$

where A, B, C, D, E are the matrices which elements are the second partial derivatives of the force function evaluated at the equilibrium (3) (see [3]).

After some transformations, it is possible to reduce the complexity of the system (4). Indeed, the first $2n$ equations are linear in ρ_j, σ_j, but the system is coupled in those $2n$ variables. To uncouple the system, let us introduce the $n \times n$ complex matrix $\mathcal{F}$ with elements $\mathcal{F}_{lk} = \exp(2\theta lk\sqrt{-1})$. Its inverse matrix is simply $\mathcal{F}^{-1} = \bar{\mathcal{F}}/n$, with $\bar{\mathcal{F}}$ its conjugate matrix.

As proven by Pendse ([8]), this transformation uncouples the system (4) due to the fact that matrices A, B, C, D, and E are periodic of period n and, besides, matrices B and C are odd functions, whereas the remaining matrices A, D, and E are even.

If we use the matrix $\mathcal{F}$ to define the complex transformation

$$\boldsymbol{\rho} = \mathcal{F}\boldsymbol{\xi}, \quad \boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\eta}, \quad \boldsymbol{z} = \mathcal{F}\boldsymbol{\zeta},$$

the system (4) is transformed into

$$
\begin{aligned}
&\ddot{\boldsymbol{\xi}} - 2\tilde{\omega}\,\dot{\boldsymbol{\eta}} = \tilde{\omega}^2\boldsymbol{\xi} + \Lambda^A\,\boldsymbol{\xi} + \Lambda^B\,\boldsymbol{\eta},\\
&\ddot{\boldsymbol{\eta}} + 2\tilde{\omega}\dot{\boldsymbol{\xi}} = \Lambda^C\,\boldsymbol{\xi} + \Lambda^D\,\boldsymbol{\eta},\\
&\ddot{\boldsymbol{\zeta}} = \Lambda^E\,\boldsymbol{\zeta},
\end{aligned}
\tag{5}
$$

an uncoupled system with respect to their indices, where Λ^X is the diagonal matrix of eigenvalues of matrix X. Note also that the new complex variables satisfy $\bar{\xi}_j = \xi_{n-j}$, $\bar{\eta}_j = \eta_{n-j}$, $\bar{\zeta}_j = \zeta_{n-j}$, so we only deal with equations with scripts $j = 1, 2, \ldots, [n/2], n$, where $[a]$ denotes the integer part of a.

To compute the eigenvalues, we follow the procedure given by [9], based on the results provided by [8]. By proceeding in such way, we find

$$
\begin{aligned}
\Lambda^A_j &= 2\tilde{\omega}^2 + \mu(2J_j - \frac{1}{4}\,L_j) + 2\epsilon_0 + 3\mu\epsilon_0(2 + S^A_j) + \frac{\mu\epsilon_1}{4}(\frac{1}{3}(n^2-1) - (P_j + 2Q_j)),\\
\Lambda^B_j &= \mathrm{i}\;\mu\,\Big(J_j + \frac{1}{8}\,M_j + \epsilon_0\,S^B_j + \frac{\epsilon_1}{8}(P_{j+1} - P_{j-1})\Big),\\
\Lambda^C_j &= \mathrm{i}\;\;\mu\,\Big(2\,J_j - \frac{1}{8}\,M_j + 3\,\epsilon_0\,S^B_j - \frac{\epsilon_1}{8}(P_{j+1} - P_{j-1})\Big),\\
\Lambda^D_j &= \mu\,\Big(-J_j + \frac{1}{4}\,N_j - \epsilon_0\,(2 + S^A_j) + \frac{\epsilon_1}{4}(3P_j - 2Q_j)\Big),\\
\Lambda^E_j &= -1 - 2\,\epsilon_0 - \mu\,\Big(S_j + \frac{1}{12}\,(L_j + N_j) + \epsilon_0\,(2 + S^E_j) + \frac{\epsilon_1}{4}P_j\Big),
\end{aligned}
$$

where $L_j, M_j, N_j, J_j, S_j, P_j, Q_j, S^A_j, S^B_j, S^E_j$ are given in [3].

4 Stability analysis

The roots of the characteristic equation determine the linear stability of the system (5). In this system, the last n equations, those corresponding to variable $\boldsymbol{\zeta}$, depend only on $\boldsymbol{\zeta}$, whereas those corresponding to $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are coupled. As a result, we split our analysis in two parts, one for the out-of-plane motion ($\boldsymbol{\zeta}$) and the other for the on-plane motion ($\boldsymbol{\xi}$ and $\boldsymbol{\eta}$).

4.1 Out-of-plane stability

The out-of-plane variations are determined by the equation:

$$
\ddot{\boldsymbol{\zeta}} = \Lambda^E\,\boldsymbol{\zeta} = -\Big[1 + 2\,\epsilon_0 + \mu\,\Big(S + \frac{1}{12}\,(L + N) + \epsilon_0\,(2 + S^E) + \frac{\epsilon_1}{4}P_j\Big)\Big]\;\boldsymbol{\zeta},
$$

therefore, the out-of-plane motion is stable when

$$
1 + 2\,\epsilon_0 + \mu\,\Big(S_j + \frac{1}{12}\,(L_j + N_j) + \epsilon_0\,(2 + S^E_j) + \frac{\epsilon_1}{4}P_j\Big) > 0, \quad \forall\, j = 1, \ldots, [n/2], n \tag{6}
$$

Note that $S, L, N, (2+S^E), P$ are non-negative quantities, so if ϵ_0 and ϵ_1 are positive, the stability condition (6) is always satisfied, hence vertical motions are linearly stable. For any values of ϵ_0 and ϵ_1 we have two different situations depending on the value of j:

a) $j = n$: $S_n = n$, $L_n = 0$, $N_n = 0$, $2 + S_n^E = 2\,n$, $P_n = 0$ and the stability condition is: $(1 + 2\,\epsilon_0)\,(1 + \mu\, n) > 0 \Longleftrightarrow 1 + 2\epsilon_0 > 0$.

b) $j \neq n$: $S_j = 0$, $2 + S_j^E = 0$, hence, the stability condition is

$$1 + 2\,\epsilon_0 + \mu\,\Big(\frac{1}{12}(L_j + N_j) + \frac{\epsilon_1 P_j}{4}\Big) > 0,$$

Now, for $\epsilon_0 > -1/2$ and $\epsilon_1 > 0$ all the conditions are satisfied. If $\epsilon_1 < 0$, the value is restricted to:

$$\epsilon_1 > -\frac{4\,(1+2\epsilon_0)}{\mu P_j} - \frac{N_j + L_j}{3P_j}$$

4.2 In-plane stability

Let us now consider the variational displacements on the plane containing the bodies. As it is known [3], the stability of this linear system is determined by the purely imaginary roots, $(\mathrm{i}\,\tilde{\omega}\,x)$, of its characteristic equation, where x must be a real root of the quartic equation

$$x^4 - q\,x^2 + r\,x + s = 0, \tag{7}$$

whose coefficients q, r, s (for each script j) are functions of the eigenvalues of the matrices A, B, C, D, and E (see [3]).

For $j = n$, we have

$$2 + S_n^A = 0,\ S_n^B = 0,\ S_n = n,\ L_n = M_n = N_n = C_n = J_n = 0,$$
$$P_n^- - P_n^+ = 0, \quad P_n = 0, \quad Q_n = 0$$

and the polynomial equation (7) is reduced to

$$x^4 - \Big(1 - \frac{2\epsilon_0}{\tilde{\omega}^2} - \frac{\mu\,\epsilon_1}{12\tilde{\omega}^2}(n^2-1)\Big)\,x^2 = 0.$$

Their four roots, namely

$$0, 0, +\sqrt{1 - 2\epsilon_0/\tilde{\omega}^2 - \frac{\mu\,\epsilon_1}{12\tilde{\omega}^2}(n^2-1)}, -\sqrt{1 - 2\epsilon_0/\tilde{\omega}^2 - \frac{\mu\,\epsilon_1}{12\tilde{\omega}^2}(n^2-1)}$$

are real because we recall $\tilde{\omega}^2 = \omega^2 + 2\epsilon_0 + \frac{\mu\,\epsilon_1}{12}(n^2-1)$. Consequently, we only need to analyze the cases for the scripts $j = 1, \ldots, [n/2]$. The following three conditions must be fulfilled (see [2]):

$$q > 0, \tag{8}$$

$$\Gamma = 2q(q^2 - 4s) - 9r^2 > 0, \tag{9}$$

$$\Delta = 4q^3r^2 - 27r^4 + 16q^4s - 144qr^2s - 128q^2s^2 + 256s^3 > 0. \tag{10}$$

5 Stability domains

In this section we analyze the stability for two cases, namely a) $\epsilon_0 = 0, \epsilon_1 \neq 0$ and b) $\epsilon_0 \neq 0, \epsilon_1 \neq 0$, in order to complete the work done in [2] where we considered the case c) $\epsilon_0 \neq 0, \epsilon_1 = 0$. The procedure we follow is the same that the one described in [2].

5.1 Stability regions for $\epsilon_0 = 0, \epsilon_1 \neq 0$

In this case, the central body is a point whereas the surrounding bodies are spheroids or radiating bodies (i.e. under Manev's type force). In Fig. (1) we present the stability region on the parametric plane (ϵ_1, μ) for $n = 6$ (left) and $n = 7$ (right). Note that when $\epsilon_1 = 0$ we recover the classical result, that is, for $n = 6$ the system is unstable whereas for $n = 7$ we have stability for $\mu < 0.007150403074$. Besides, it is shown that there are values of ϵ_1 where we get stability, but the upper bound of μ for stability decreases when ϵ_1 increases.

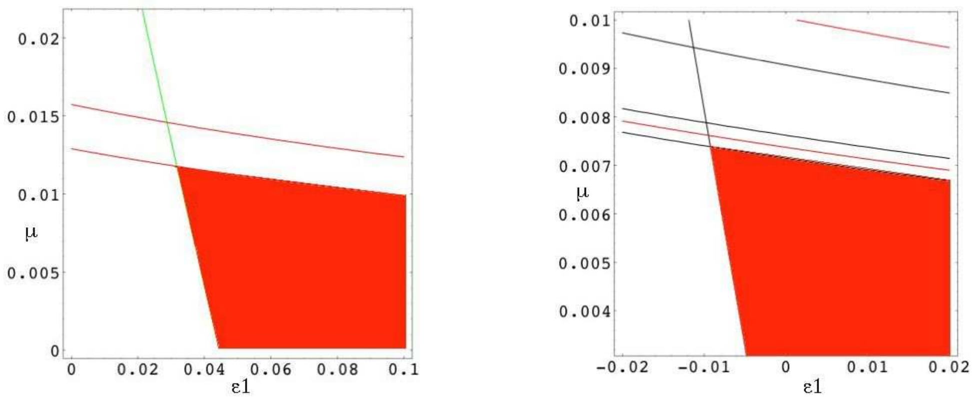

Figure 1.— Stability regions for $n = 6$ (left) and $n = 7$ (right) when $\epsilon_0 = 0$

5.2 Stability regions for $\epsilon_0 \neq 0, \epsilon_1 \neq 0$

The results presented in this section correspond to the case $n = 7$, since this is the first case of possible stability for the classical problem (Newtonian forces). For another different number of bodies the procedure is the same. Besides, as illustration of the behavior when every body acts Manev's forces, we take only three cases ($\epsilon_1 = -0.05,\ 0,\ 0.05$) for bodies on the circle and make the 2-D plot on the plane (ϵ_0, μ). The corresponding stability regions are represented in Fig. (2) and show a similar aspect, although we can conclude that the size of the stability area decreases with ϵ_1. Besides, for a certain value of ϵ_0 given, the stability value of the mass factor μ also decreases with ϵ_1.

The aim of this paper is to analize the possibility of increasing the interval of values of μ and ϵ_0 having a stable configuration. In fact, the upper bound for these two parameters

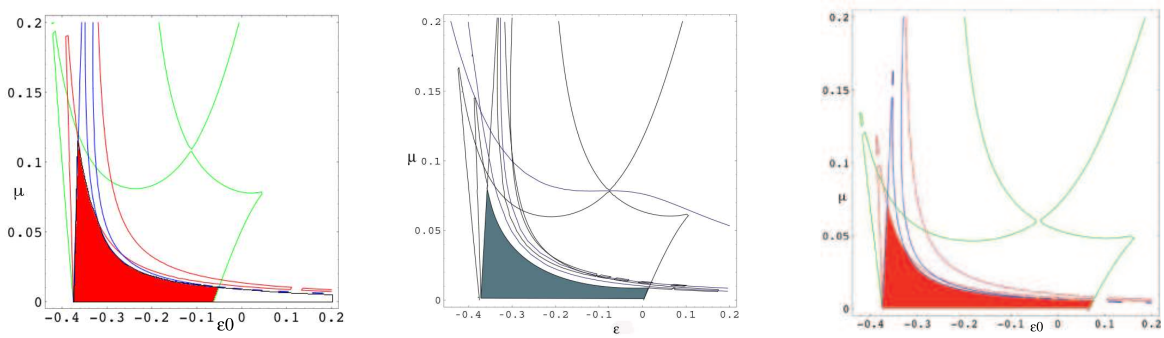

Figure 2.— Stability region: $n = 7$, $\epsilon_1 = -0.05$(l), $\epsilon_1 = 0$(c) and $\epsilon_1 = 0.05$(r)

can be increased as it can see in Fig.(2): in the graphic on the left the bound for μ is increased having negative values of ϵ_1 and in the graphic on the right the interval of possible values of ϵ_0 is bigger when we take positive values for ϵ_1 than if we do not take into account this parameter.

In Fig.(3) we show the stability region for two values of ϵ_0: one that corresponds to stability (l) and another that corresponds to unstability (r) when $\epsilon_1 = 0$. We get stability for values of $\epsilon_1 > 0$ but the value of μ has to decrease when ϵ_1 increases. The value of μ is increased only if we consider negative values of the parameters.

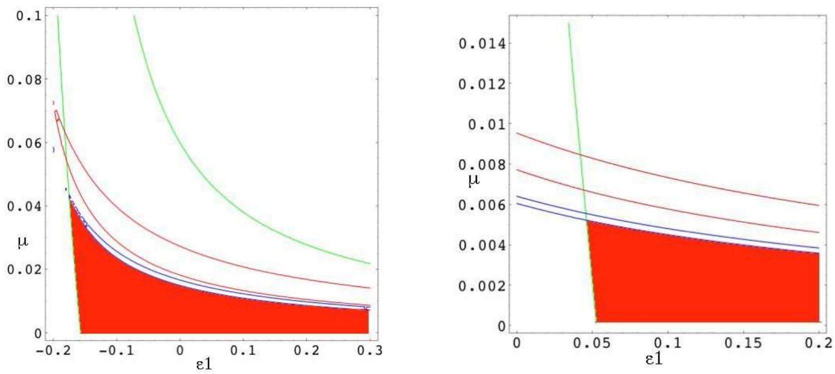

Figure 3.— Stability region for $\epsilon_0 = -0.2$ (l) and $\epsilon_0 = 0.07$ (r)

Finally, we may have values of μ for which the system is unstable, however, we could adjust the values either of the parameter ϵ_0 or ϵ_1 or both in order to have stability. For instance, in the Newtonian case ($\epsilon_0 = \epsilon_1 = 0$, and again $n = 7$) the system is stable for $0 < \mu < 0.007150403074$. Is it possible to find values of ϵ_0 and ϵ_1 in order to have stability for a grater value of μ, let say $\mu = 0.04$. The answer is positive as we can see in Fig. (4), where we plot for $\mu = 0.04$ the stability region on the plane (ϵ_0, ϵ_1).

Acknowledgments

Supported by the Spanish Ministry of Science and Technology (Projects # MTM2009-10767 and # AYA2008-05572).

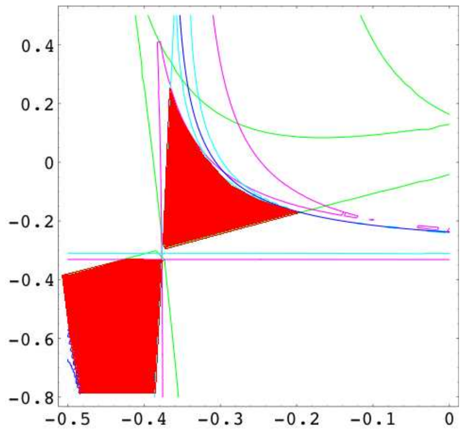

Figure 4.— Stability region in the plane (ϵ_0, ϵ_1) for $\mu = 0.04$

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